Some Upper Bounds for the Domination Number

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Abstract

Let G be a graph of order $n(G)$, minimum degree $\delta(G)$, diameter $\mathrm{dm}(G)$, and let $\bar{G}$ be the complement of the graph G. A vertex set D is called a dominating set of G, if each vertex not in D has at least one neighbor in D. The domination number $\gamma(G)$ equals the minimum cardinality of a dominating set of G.

In this article we show the inequalities

- $\gamma(G) \leq \lfloor n(G)/3 \rfloor$, if $\delta(G) \geq 7$,
- $\gamma(G) + \gamma(\bar{G}) \leq \lfloor n(G)/3 \rfloor + 2$, if $\delta(G), \delta(\bar{G}) \geq 7$ and
- $\gamma(G) \leq \left\lfloor \frac{n(G)}{4} \right\rfloor + 1$, if $\mathrm{dm}(G) = 2$.

Using the concept of connectivity, we present some related upper bounds for the domination number of graphs with $\mathrm{dm}(G) = 2$ and $\mathrm{dm}(G) = 3$.

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1 Terminology and introduction

1.1 General introduction and notations

For graph-theoretical terminology and notation not defined here we follow Chartrand and Lesniak [7]. We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For each vertex

$v \in V(G)$, the *open neighborhood* $N(v) = N_G(v)$ of v is defined as the set of all vertices adjacent to v, $N[v] = N_G[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v, and $d(v) = |N(v)|$ is the *degree* of v. We denote by $\delta(G)$ the *minimum degree*, by $\Delta(G)$ the *maximum degree* and by $n(G) = |V(G)|$ the *order* of G.

If G is a connected graph, then the *distance* $d(u,v)$ between two vertices u and v is defined as the length of a shortest path from u to v, and the *diameter* is the number $\mathrm{dm}(G) = \max\{d(u,v) : u,v \in V(G)\}$. We define $N_p(u) = \{x \in V(G) | d(u,x) = p\}$. For two vertex sets X and Y let $[X,Y]$ be the set of edges with one endpoint in X and the other one in Y, and $|[X,Y]|$ denotes the cardinality of $[X,Y]$. If $X \subseteq V(G)$, then let $\bar{X} = V(G) \setminus X$, and let $G[X]$ be the subgraph induced by X. Furthermore, we write K_p for the *complete graph* of order p. The $K_p \oplus K_p$ is the graph with $2p$ vertices obtained by the vertex-disjoint union of two K_p such that each vertex in $K_p \oplus K_p$ has degree p.

1.2 Domination

A vertex set D is called a *dominating set* of G, if $N[D] = V(G)$. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of a dominating set over all dominating sets of G. A dominating set D satisfying $|D| = \gamma(G)$ is called a *γ-set* of G. If $U \subseteq N[a]$ for a vertex a and a vertex set U, then we say that *a dominates U* and write $a \to U$.

In the following we list some upper bounds for the domination number of a graph G, depending on the order, the minimum degree and the diameter of G and $\bar{G}$. A general summary over the domination number can be found in the textbook of Haynes, Hedetniemi and Slater [13].

1.2.1 Bounds in terms of order

Theorem 1.1 (Ore [19] 1962) *If a graph G has no isolated vertices, then*

$$\gamma(G) \leq n(G)/2.$$

Theorem 1.2 (Reed [22] 1996) *If G is a connected graph with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n(G)/8$.*

In 1989, McCuaig and Sheperd [17] showed that $\gamma(G) \leq 2n(G)/5$ for any graph G with $\delta(G) \geq 2$, except a class of seven graphs.
Since $\gamma(G) \leq \delta(G)n(G)/(3\delta(G)-1)$ is valid for any graph G with $\delta(G) \geq 7$ by Theorem 1.6, Haynes, Hedetniemi and Slater [13], p. 48 conjecture that $\gamma(G) \leq \delta(G)n(G)/(3\delta(G)-1)$ for any graph G with $4 \leq \delta(G) \leq 6$.
In Section 1.2.2 we show that for $\delta(G) = 6$ this conjecture follows directly

from Theorem 1.7. In addition, in the diploma thesis by S. Nünning in 2000, supervised by D. Rautenbach and L. Volkmann, this conjecture was proved for $\delta(G) = 4$. Hence, the question remains open for graphs G having $\delta(G) = 5$.

Theorem 1.3 (Cockayne, Ko, Shepherd [10] 1985) *If a connected graph G is claw-free and net-free, then $\gamma(G) \leq \lceil n(G)/3 \rceil$.*

In Section 2 we show that $\gamma(G) \leq \lfloor n(G)/3 \rfloor$, if $\delta(G) \geq 7$.

1.2.2 Bounds in terms of order and minimum degree

We only list a few of the known results in this area.

Theorem 1.4 (Arnautov [2] 1974 and Payan [20] 1975) *If a graph G has no isolated vertices, then*

$$\gamma(G) \leq \left(\frac{1 + \ln(\delta(G) + 1)}{\delta(G) + 1} \right) n(G).$$

Theorem 1.5 (Arnautov [2] 1974) *If a graph G has no isolated vertices, then*

$$\gamma(G) \leq \left(\frac{1}{\delta(G) + 1} \sum_{k=1}^{\delta(G)+1} \frac{1}{k} \right) n(G).$$

Theorem 1.6 (Caro, Roditty [5], [6] 1985, 1990) *If G is a graph of order $n(G)$ and minimum degree $\delta(G)$, then*

$$\gamma(G) \leq \left[1 - \delta(G) \left(\frac{1}{\delta(G) + 1} \right)^{1+1/\delta(G)} \right] n(G).$$

Theorem 1.7 (Clark, Shekhtman, Suen, Fisher [9] 1998) *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \left(1 - \prod_{k=1}^{\delta(G)+1} \frac{k}{k + \frac{1}{\delta(G)}} \right) n(G).$$

As a direct consequence of Theorem 1.7 we obtain the following corollary, which shows that the conjecture $\gamma(G) \leq \delta(G)n(G)/(3\delta(G) - 1)$ by Haynes, Hedetniemi and Slater [13] is valid for $\delta(G) = 6$.

Corollary 1.1 *If G is a connected graph with $\delta(G) = 6$, then*

$$\gamma(G) \leq \frac{6}{17} n(G).$$

Proof: Applying Theorem 1.7 for $\delta(G) = 6$ leads to

$$\begin{aligned}\gamma(G) &\leq \left(1 - \prod_{k=1}^{7} \frac{k}{k+\frac{1}{6}}\right) n(G) \\ &\leq (1 - 0.661)\, n(G) = 0.339\, n(G) < 0.352\, n(G) \\ &\leq \frac{6}{17} n(G).\end{aligned}$$

□

Theorem 1.8 (Clark, Dunning [8] 1997) *The following table contains upper bounds for $\gamma(G)$, if G is an arbitrary graph with n vertices and minimum degree δ.*

$n \setminus \delta$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1														
3	1	1													
4	2	2	1												
5	2	2	1	1											
6	3	2	2	2	1										
7	3	3	2	2	1	1									
8	4	4	3	2	2	2	1								
9	4	4	3	3	2	2	1	1							
10	5	4	3	3	2	2	2	2	1						
11	5	5	4	3	3	3	2	2	1	1					
12	6	6	4	4	3	3	2	2	2	2	1				
13	6	6	4	4	3	3	3	2	2	2	1	1			
14	7	6	5	4	4	3	3	3	2	2	2	2	1		
15	7	7	5	5	4	3-4	3	3	2-3	2	2	2	1	1	
16	8	8	6	5	4-5	4	3-4	3	2-3	2-3	2	2	2	2	1

1.2.3 Nordhaus-Gaddum type results

Results on the sum (product) of a parameter of G and the same parameter of its complement $\bar{G}$ are called Nordhaus-Gaddum type results (see [18]).

Theorem 1.9 (Jaeger, Payan [14] 1972) *For any graph G,*

$$\gamma(G)\gamma(\bar{G}) \leq n(G).$$

Theorem 1.10 (Jaeger, Payan [14] 1972) *For any graph G,*

$$\gamma(G) + \gamma(\bar{G}) \leq n(G) + 1.$$

Theorem 1.11 (Joseph, Arumugam [15] 1995) *Let G be an arbitrary graph. If $\delta(G)$, $\delta(\bar{G}) \geq 1$, then $\gamma(G) + \gamma(\bar{G}) \leq \lfloor n(G)/2 \rfloor + 2$.*

Theorem 1.12 (Dunbar, Haynes, Hedetniemi [11]) *Let G be an arbitrary graph. If $\delta(G), \delta(\bar{G}) \geq 2$, then $\gamma(G) + \gamma(\bar{G}) \leq \lfloor 2n(G)/5 \rfloor + 3$.*

Theorem 1.13 (Dunbar, Haynes, Hedetniemi [11]) *If G and $\bar{G}$ are connected graphs of order $n(G) \neq 10$ with $\delta(G), \delta(\bar{G}) \geq 3$, and $G \neq K_3 \times K_3$ (where $K_3 \times K_3$ is the cartesian product), then $\gamma(G)+\gamma(\bar{G}) \leq \lfloor 3n(G)/8 \rfloor + 2$.*

Theorem 1.14 (Payan [20] 1975) *Let G be an arbitrary graph. If $\gamma(\bar{G}) \geq 3$, then $\gamma(G) + \gamma(\bar{G}) \leq \delta(G) + 3$.*

In Section 2 we prove the inequality $\gamma(G) + \gamma(\bar{G}) \leq \lfloor n(G)/3 \rfloor + 2$, if $\delta(G), \delta(\bar{G}) \geq 7$.

1.2.4 Bounds in terms of diameter

In a graph of diameter 2, the open neighborhood of any vertex $v \in V(G)$ dominates G and the next upper bound is immediate.

Theorem 1.15 (Haynes, Hedetniemi, Slater [13], p. 55) *If a graph G has* $\mathrm{dm}(G) = 2$, *then* $\gamma(G) \leq \delta(G)$.

Theorem 1.16 (Brigham, Chinn, Dutton [4] 1988) *If G is a graph with* $\gamma(G) \geq 3$, *then* $\mathrm{dm}(\bar{G}) \leq 2$.

Theorem 1.17 (Haynes, Hedetniemi, Slater [13]) *If G is a graph without isolated vertices and* $\mathrm{dm}(G) \geq 3$, *then* $\gamma(\bar{G}) = 2$.

Theorem 1.18 (MacGillivray, Seyffarth [16] 1996) *If G is a planar graph with* $\mathrm{dm}(G) = 2$, *then* $\gamma(G) \leq 3$.

Theorem 1.19 (MacGillivray, Seyffarth [16] 1996) *If G is a planar graph with* $\mathrm{dm}(G) = 3$, then $\gamma(G) \leq 10$.

In Section 3 we present some new results on the domination number for graphs with diameter 2 and 3, respectively. Hereby, we will use the terms of edge- and vertex-connectivity. Therefore we give a short introduction into the concept of connectivity.

1.3 Connectivity

An *edge-cut (vertex-cut)* of a connected graph G is a set of edges (vertices) whose removal disconnects G. The *edge-connectivity* $\lambda(G)$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of G and if G is non-complete, then the *vertex-connectivity* $\kappa(G)$ is defined as the minimum cardinality of a vertex-cut over all vertex-cuts of G. For the complete graph K_n we define $\kappa(K_n) = n - 1$. In 1932, Whitney [24] proved that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for every connected graph G.

Each edge-cut (vertex-cut) S satisfying $|S| = \lambda(G)$ ($|S| = \kappa(G)$) is called a *minimum edge-cut (minimum vertex-cut)*. We call a graph G *maximally edge-connected*, if $\lambda(G) = \delta(G)$, and a *trivial edge-cut* is an edge-cut which consists only of edges adjacent to a vertex of minimum degree. We call a graph G *super-edge-connected*, for short *super-λ*, if every minimum edge-cut is trivial. Hence, every super-edge-connected graph is also maximally edge-connected.

The following is known about edge-connectivity and super-edge-connectivity in graphs of diameter 2.

Theorem 1.20 (Plesník [21] 1975) *If G is a connected graph with* $\mathrm{dm}(G) \leq 2$, *then* $\lambda(G) = \delta(G)$.

The super-edge-connected graphs of diameter 2 are characterized by Wang and Li.

Theorem 1.21 (Wang, Li [23] 1999) *A connected graph G with* $\mathrm{dm}(G) = 2$ *is super-λ, if and only if G contains no induced complete graph $K_{\delta(G)}$ with all its vertices of degree $\delta(G)$.*

In 1992, Fiol [12] already showed that the condition in the characterization above is sufficient for graphs of diameter 2 to be super-λ.

2 A Nordhaus-Gaddum type result

Theorem 2.1 *If G is a connected graph with $\delta(G) \geq 7$, then*

$$\gamma(G) \leq \lfloor n(G)/3 \rfloor.$$

Proof: Let G be a connected graph with $\delta(G) \geq 7$. If $\delta(G) = 7$, then Theorem 1.7 leads to

$$\gamma(G) \leq \left(1 - \prod_{k=1}^{8} \frac{k}{k+\frac{1}{7}}\right) n(G) \leq 0.312\, n(G) < \frac{n(G)}{3}.$$

Since $\gamma(G)$ and $n(G)$ are integers, we obtain the desired result $\gamma(G) \le \lfloor n(G)/3 \rfloor$.

Now we assume that $\delta(G) \ge 8$. Let $f(x) = \frac{1}{x+1}\sum_{k=1}^{x+1}\frac{1}{k}$ be a function in $x \in \mathbb{N}$, where $x \ge 1$. We will show that $f(x)$ is monotonically decreasing. Therefore we consider the difference $f(x+1) - f(x)$ and show that it is negative.

$$\begin{aligned}
f(x+1) - f(x) &= \frac{1}{x+2}\sum_{k=1}^{x+2}\frac{1}{k} - \frac{1}{x+1}\sum_{k=1}^{x+1}\frac{1}{k} \\
&= \frac{1}{(x+2)^2} + \left(\frac{1}{x+2} - \frac{1}{x+1}\right)\sum_{k=1}^{x+1}\frac{1}{k} \\
&= \frac{1}{(x+2)^2} - \frac{1}{(x+2)(x+1)}\sum_{k=1}^{x+1}\frac{1}{k} \\
&= \frac{1}{(x+2)^2} - \frac{x+2}{(x+2)^2(x+1)}\sum_{k=1}^{x+1}\frac{1}{k} \\
&= \frac{1}{(x+2)^2}\left(1 - \frac{x+2}{x+1}\sum_{k=1}^{x+1}\frac{1}{k}\right) \\
&< 0
\end{aligned}$$

This observation and Theorem 1.5 leads to

$$\begin{aligned}
\gamma(G) &\le \left(\frac{1}{\delta(G)+1}\sum_{k=1}^{\delta(G)+1}\frac{1}{k}\right) n(G) \\
&\le \left(\frac{1}{9}\sum_{k=1}^{9}\frac{1}{k}\right) n(G) < 0.32\, n(G) \\
&< \frac{n(G)}{3}.
\end{aligned}$$

Again, since $\gamma(G)$ and $n(G)$ are integers, we obtain $\gamma(G) \le \lfloor n(G)/3 \rfloor$.□

Theorem 2.2 *If G is a graph with $\delta(G), \delta(\bar{G}) \ge 7$, then*

$$\gamma(G) + \gamma(\bar{G}) \le \left\lfloor \frac{n(G)}{3} \right\rfloor + 2.$$

Proof: Let G be a graph with $\delta(G), \delta(\bar{G}) \ge 7$. If $\mathrm{dm}(G) \ge 3$, then by Theorem 1.17 $\gamma(\bar{G}) \le 2$. Theorem 2.1 leads to $\gamma(G) \le \lfloor n(G)/3 \rfloor$ and thus $\gamma(G) + \gamma(\bar{G}) \le \left\lfloor \frac{n(G)}{3} \right\rfloor + 2$.

Analogously, if $\text{dm}(\bar{G}) \geq 3$, then $\gamma(G) \leq 2$, and thus $\gamma(G) + \gamma(\bar{G}) \leq 2 + \lfloor n(G)/3 \rfloor$. Since $\delta(G), \delta(\bar{G}) \geq 7$ the case $\text{dm}(G) = 1$ or $\text{dm}(\bar{G}) = 1$ cannot occur.

Now we consider the remaining case that $\text{dm}(G) = \text{dm}(\bar{G}) = 2$.

1: Let $\gamma(G), \gamma(\bar{G}) \geq 6$.

Theorem 1.9 leads to

$$\gamma(G) \leq \left\lfloor \frac{n(G)}{\gamma(\bar{G})} \right\rfloor \leq \left\lfloor \frac{n(G)}{6} \right\rfloor \text{ and } \gamma(\bar{G}) \leq \left\lfloor \frac{n(G)}{\gamma(G)} \right\rfloor \leq \left\lfloor \frac{n(G)}{6} \right\rfloor,$$

and thus

$$\gamma(G) + \gamma(\bar{G}) \leq 2\lfloor n(G)/6 \rfloor \leq \lfloor n(G)/3 \rfloor + 2.$$

2: Let $\gamma(G) = 5, \gamma(\bar{G}) \leq 5$.

2.1: Let $n(G) \geq 24$.

In this case $\gamma(G) + \gamma(\bar{G}) \leq 10 \leq \lfloor n(G)/3 \rfloor + 2$, and thus we are done.

2.2: Let $n(G) \leq 23$.

2.2.1: Let $\gamma(G) = 5, \gamma(\bar{G}) = 5$.

Applying Theorem 1.9, we deduce that the assumption $\gamma(G) = \gamma(\bar{G}) = 5$ leads to a contradiction.

2.2.2: Let $\gamma(G) = 5, \gamma(\bar{G}) = 4$.

2.2.2.1: Let $n(G) \geq 21$.

If $n(G) \geq 21$, then we are done, since $\gamma(G) + \gamma(\bar{G}) \leq 9 = 7 + 2 \leq \lfloor n(G)/3 \rfloor + 2$.

2.2.2.2: Let $n(G) \leq 19$.

Theorem 1.9 implies that this case cannot occur.

2.2.2.3: Let $n(G) = 20$ and $\delta(G) \leq 8$.

Let u be an arbitrary vertex of minimum degree in G. Then u dominates the vertex set $V(G) - N_G(u)$ in $\bar{G}$, and hence at most 8 vertices in $\bar{G}$ are not dominated by u. Since $\gamma(G) = 5$, each set of four vertices in $\bar{G}$ can be dominated by one vertex. That implies $\gamma(\bar{G}) \leq 3$, a contradiction.

2.2.2.4: Let $n(G) = 20$ and $\delta(G) = 9$.

Let u be an arbitrary vertex of minimum degree in G and let $R = V(G) - N[u]$. Since $\gamma(\bar{G}) = 4$, each set of three vertices in G can be dominated by one vertex. If there exists a vertex $v \in R$, such that $|N(v) \cap R| \geq 3$, then $\gamma(G) \leq 4$, since u and v dominate 14 vertices and the remaining 6 vertices can be dominated by two vertices. Hence, this assumption leads to a contradiction. Now we assume that no vertex in R has three or more neighbors in R. Consequently, $|N(v) \cap N(u)| \geq 7$ for each vertex $v \in R$, and thus the set $\{u, a, b, c\}$ is a dominating set for each three different vertices a, b, c in $N(u)$. This contradicts $\gamma(G) = 5$.

2.2.2.5: Let $n(G) = 20$ and $\delta(G) \geq 10$.

This case cannot occur, because $n(G) = 20$, $\delta(G) \geq 10$ and $\gamma(\bar{G}) = 4$ implies $\gamma(G) \leq 4$, since a vertex of minimum degree dominates 11 vertices and the remaining 9 vertices can be dominated by three vertices.

2.2.3: Let $\gamma(G) = 5, \gamma(\bar{G}) = 3$.

If $n(G) \geq 18$, then we are done, because in this case $\gamma(G) + \gamma(\bar{G}) \leq 6 + 2 \leq \lfloor n(G)/3 \rfloor + 2$. The assumption $n(G) \leq 14$ leads to a contradiction to $\delta(G), \delta(\bar{G}) \geq 7$.

In the case $15 \leq n(G) \leq 16$, we obtain a contradiction to the results in Theorem 1.8.

It remains the case $n(G) = 17$. We define u and R as in Case 2.2.2.4. It is easy to see that $|R| \leq 9$. If there exists a vertex $v \in R$, such that $|N(v) \cap R| \geq 4$, then the vertices u and v dominate at least 13 vertices. Since $\mathrm{dm}(G) = 2$, the remaining vertices can be dominated by two vertices. Hence $\gamma(G) \leq 4$, a contradiction.

Now we assume that $|N(v) \cap R| \leq 3$ for each vertex $v \in R$. Since $\delta(G) \geq 7$, we observe that $|N(v) \cap N(u)| \geq \delta(G) - 3 \geq 4$ for each vertex $v \in R$.

If there exists an arbitrary set $\{a, b, c\}$ of three different vertices in $N(u)$ such that $\{a, b, c\} \to R$, then $\gamma(G) \leq 4$, a contradiction.

Now we assume that for each set $\{a, b, c\}$ of three different vertices in $N(u)$, there exists a vertex $v \in R$ such that $v \notin N(\{a, b, c\})$. This implies that for each set $\{x_1, x_2, ..., x_{\delta(G)-3}\}$ of $\delta(G) - 3$ different vertices in $N(u)$, there exists a vertex $v \in R$, such that $N(v) \cap N(u) = \{x_1, x_2, ..., x_{\delta(G)-3}\}$. Since, there exist at least

$$\binom{\delta(G)}{\delta(G)-3} \geq \binom{7}{7-3} = 35 > 9$$

sets of $\delta(G) - 3$ different vertices in $N(u)$, we obtain a contradiction to $|R| \leq 9$.

2.2.4: Let $\gamma(G) = 5, \gamma(\bar{G}) \leq 2$.

The desired result follows directly by Theorem 2.1.

3: Let $\gamma(\bar{G}) = 5, \gamma(G) \leq 5$.

This case can be treated in a similiar manner as Case 2.

4: Let $\gamma(G), \gamma(\bar{G}) \leq 4$.

4.1: Let $\gamma(G), \gamma(\bar{G}) \leq 3$.

If $n(G) \geq 15$, then we observe $\gamma(G) + \gamma(\bar{G}) \leq 6 = 4 + 2 \leq \lfloor n(G)/3 \rfloor + 2$. The case $n(G) \leq 14$ contradicts the assumption $\delta(G), \delta(\bar{G}) \geq 7$, and thus we are done.

4.2: Let $\gamma(G) = \gamma(\bar{G}) = 4$.

4.2.1: Let $n(G) \neq 17$ or $n(G) = 17$ and $\delta(G) = 7$.

If $n(G) \leq 15$, then $\gamma(G)\gamma(\bar{G}) = 16 > n(G)$, a contradiction to Theorem 1.9. If $n(G) \geq 18$, then $\gamma(G) + \gamma(\bar{G}) = 6 + 2 \leq \lfloor n(G)/3 \rfloor + 2$, and thus we are done.

Now we consider the remaining case that $16 \leq n(G) \leq 17$.

By using the results in Theorem 1.8 for $n(G) = 16$, we only have to consider the case that $\delta(G) = 7$, if $n(G) = 16$.

We define R as above. We assume that there exists a vertex $v \in R$ such that $|N_G(v) \cap N_G(u)| \leq 3$, and thus $|N_{\bar{G}}(v) \cap N_G(u)| \geq 4$. Hence $|N_G(u) \setminus N_{\bar{G}}(v)| \leq 3$ in $\bar{G}$, and thus $V(\bar{G})$ can be dominated by u, v and a further vertex, since $\gamma(G) = 4$. This contradicts $\gamma(\bar{G}) = 4$, and thus $|N(v) \cap N(u)| \geq 4$, for each $v \in R$. Hence, we conclude that $|[R, N(u)]| \geq 4|R|$.

If there exists a vertex $x \in N(u)$, such that $|N(x) \cap R| \geq |R| - 3$, then G can be dominated by the vertices v and x and a further vertex, since $\gamma(\bar{G}) = 4$. This contradicts $\gamma(G) = 4$, and thus $|N(x) \cap R| \leq |R| - 4$, for each vertex $x \in N(u)$. This implies $|[R, N(u)]| \leq \delta(G)(|R| - 4)$.

Hence, we deduce that $4|R| \leq \delta(G)(|R|-4) = 7|R|-28$, a contradiction to $8 \leq |R| \leq 9$.

4.2.2: Let $n(G) = 17$ and $\delta(G) \in \{8, 9\}$.

By the proof to Case 4.2.1, we observe that $|N(x) \cap R| \leq |R| - 4$ for each vertex $x \in N(u)$, and thus $|[R, N(u)]| \leq \delta(G)(|R| - 4)$. If there exists a vertex $v \in R$, such that $|N(v) \cap R| \geq |R| - 4$, then u and v dominate at least $n(G) - 3$ vertices. The remaining three vertices can be dominated by one vertex, since $\gamma(\bar{G}) = 4$, a contradiction to $\gamma(G) = 4$.

Hence, $|N(v) \cap R| \leq |R| - 5$, which implies $|N(v) \cap N(u)| \geq \delta(G) - |R| + 5 \geq 5$ for each $v \in R$ and thus $|[R, N(u)]| \geq 5|R|$.

Since $|[R, N(u)]| \leq \delta(G)(|R| - 4)$, we obtain $5|R| \leq \delta(G)(|R| - 4) = (16 - |R|)(|R| - 4)$, which is equivalent to $64 + |R|^2 \leq 15|R|$, a contradiction for $7 \leq |R| \leq 8$.

4.2.3: Let $n(G) = 17$ and $\delta(G) \geq 10$.

This case cannot occur, since $\delta(\bar{G}) \geq 7$.

4.3: Let $\gamma(G) = 4, \gamma(\bar{G}) \leq 3$ or let $\gamma(G) \leq 3, \gamma(\bar{G}) = 4$.

If $n(G) \geq 15$, then $\gamma(G) + \gamma(\bar{G}) \leq 7 = 5 + 2 \leq \lfloor n(G)/3 \rfloor$, our desired result. Since $\delta(G), \delta(\bar{G}) \geq 7$, the case $n(G) \leq 14$ cannot occur.

5: Let $\gamma(G) \geq 6, \gamma(\bar{G}) \leq 5$.

5.1: Let $\gamma(\bar{G}) = 5$.

By using Theorem 1.9, we obtain $\gamma(G) \leq \lfloor n(G)/\gamma(\bar{G}) \rfloor \leq \lfloor n(G)/5 \rfloor$ and thus $\gamma(G) + \gamma(\bar{G}) \leq \lfloor n(G)/5 \rfloor + 5 \leq \lfloor n(G)/3 \rfloor + 2$, if $n(G) \geq 30$

If $n(G) \leq 29$, then $\gamma(G)\gamma(\bar{G}) \geq 6 \cdot 5 = 30 > 29 \geq n(G)$, a contradiction to Theorem 1.9.

5.2: Let $\gamma(\bar{G}) = 4$.

Analogously to the proof of Case 5.1, we obtain $\gamma(G) \leq \lfloor n(G)/\gamma(\bar{G}) \rfloor \leq \lfloor n(G)/4 \rfloor$ and thus $\gamma(G) + \gamma(\bar{G}) \leq \lfloor n(G)/4 \rfloor + 4 \leq \lfloor n(G)/3 \rfloor + 2$, if $n(G) \geq 24$. In the case $n(G) \leq 23$, we obtain the contradiction $\gamma(G)\gamma(\bar{G}) \geq 6 \cdot 4 = 24 > 23 \geq n(G)$.

5.3: Let $\gamma(\bar{G}) = 3$.

In this case we observe, by Theorem 1.9 that $n(G) \geq 18$, since $\gamma(G)\gamma(\bar{G})$

$\geq 3 \cdot 6 = 18$.
By Theorem 3.4, we have $\gamma(G) \leq \lfloor n(G)/4 \rfloor + 1$, and thus $\gamma(G) + \gamma(\bar{G}) \leq \lfloor n(G)/4 \rfloor + 1 + 3 \leq \lfloor n(G)/3 \rfloor + 2$, if $n(G) \geq 21$ or $n(G) = 19$ or $n(G) = 18$. Now we consider the remaining case that $n(G) = 20$. Let u and R be defined as above. Clearly, we obtain $|R| \leq 12$.

We assume that there exists a vertex $v \in R$, such that $|N(v) \cap R| \geq |R| - 7$. Then $V(G)$ can be dominated by u, v and three additional vertices, since $\mathrm{dm}(G) = 2$. This contradicts $\gamma(G) \geq 6$, and so $|N(v) \cap R| \leq |R| - 8$ for each $v \in R$. Thus, $|N(v) \cap N(u)| \geq \delta(G) - |R| + 8$ for each $v \in R$. Hence we obtain that

$$|[N(u), R]| \geq |R| \cdot (\delta(G) - |R| + 8) = |R|(20 - |R| - 1 - |R| + 8) = 27 \cdot |R| - 2|R|^2.$$

If there exists a vertex $x \in N(u)$ such that $|N(x) \cap R| \geq |R| - 6$, then $V(G)$ can be dominated by the vertices v and x and further three vertices, since $\mathrm{dm}(G) = 2$. This contradicts $\gamma(G) \geq 6$, and thus $|N(x) \cap R| \leq |R| - 7$ for each vertex $x \in N(u)$. This implies

$$|[R, N(u)]| \leq \delta(G)(|R| - 7) = (20 - 1 - |R|)(|R| - 7) = 26 \cdot |R| - 133 - |R|^2,$$

and thus

$$\begin{aligned} 27|R| - 2|R|^2 &\leq 26|R| - 133 - |R|^2 \\ \Leftrightarrow \quad |R| - |R|^2 &\leq -133 \\ \Leftrightarrow \quad 0 &\leq |R|^2 - |R| - 133, \end{aligned}$$

a contradiction to $|R| \leq 12$.
5.4: Let $\gamma(\bar{G}) = 2$.
The desired result follows directly from Theorem 2.1.
6: Let $\gamma(G) \leq 5, \gamma(\bar{G}) \geq 6$.
This case can be proved in an analogous way to Case 5.
Since we have discussed all possible cases, the proof is complete. □

3 Domination number and diameter

We start with some simple connections between the domination number and the diameter of the complementary graph. These results are supplements to Theorem 1.16 and Theorem 1.17.

A dominating edge is an adjacent pair of vertices that dominate $V(G)$.

Theorem 3.1 *A graph G has $\gamma(G) = 1$ or a dominating edge if and only if* $\mathrm{dm}(\bar{G}) \geq 3$.

Proof: Let G be a graph such that $\gamma(G) \leq 2$ and let D be a minimum dominating set. Firstly, we assume $|D| = 1$, say $a \in D$. Then, $d(a, G) = n(G) - 1$, which implies that a is a isolated vertex in $\bar{G}$, and thus $\text{dm}(\bar{G}) = \infty \geq 3$. Now let $|D| = 2$, say $a, b \in D$ with $ab \in E(G)$. Since $\{a, b\}$ is a dominating set, $N(a, \bar{G}) \cap N(b, \bar{G}) = \emptyset$ and $ab \in E(G)$ implies $ab \notin E(\bar{G})$. Hence, $d(a, b) \geq 3$ in $\bar{G}$, and thus $\text{dm}(\bar{G}) \geq 3$.

Now let G be a graph satisfying $\text{dm}(\bar{G}) \geq 3$. Firstly, let $\text{dm}(\bar{G}) = \infty$, which means that $\bar{G}$ is not connected. If there exists an isolated vertex in $\bar{G}$, then $\gamma(G) = 1$, and we are done. If there exist no isolated vertices in $\bar{G}$, then for each pair of two vertices a, b, which are in different components in $\bar{G}$, we observe that $\{a, b\} \to V(G)$ in G and $ab \in E(G)$, our desired result. Secondly, let $3 \leq \text{dm}(\bar{G}) < \infty$, and thus $\bar{G}$ is a connected graph. Let a, b be two vertices of distance 3 in $\bar{G}$. Then, $a \to N_{\bar{G}}[b], b \to N_{\bar{G}}[a]$ and $a, b \to V(G) \setminus (N_{\bar{G}}[b] \cup N_{\bar{G}}[a])$ in G and $ab \in E(G)$. If $\Delta(G) \leq n(G) - 2$, then $\gamma(G) \geq 2$, and thus $\{a, b\}$ is a dominating edge of $V(G)$, which implies $\gamma(G) = 2$. The case that $\Delta(G) = n(G) - 1$ cannot occur, since $\text{dm}(\bar{G}) < \infty$.
□

Lemma 3.1 *If G is a graph, then $\gamma(G) \geq 3$ if and only if $N_{\bar{G}}(a) \cap N_{\bar{G}}(b) \neq \emptyset$ for each pair of different vertices $a, b \in V(\bar{G})$.*

Proof: Let G be a graph satisfying $\gamma(G) \geq 3$. Since $\gamma(G) \geq 3$, for each pair of vertices $a, b \in V(G), a \neq b$, we have $N_G(a) \cup N_G(b) \neq V(G)$. Hence, for each pair of vertices a, b in $\bar{G}$, we observe $N_{\bar{G}}(a) \cap N_{\bar{G}}(b) \neq \emptyset$.

Now let G be a graph satisfying $N_{\bar{G}}(a) \cap N_{\bar{G}}(b) \neq \emptyset$ for each pair of vertices $a, b \in V(\bar{G}), a \neq b$. Hence, in G we have $N_G[a] \cup N_G[b] \neq V(G)$ for each pair of vertices a, b, and thus $\gamma(G) \geq 3$. □

Corollary 3.1 *Let G be graph. If $\gamma(G) \geq 3$, then each edge lies on a triangle in $\bar{G}$.*

Corollary 3.2 *Let G be graph. If there exists at least one edge in $E(G)$, which does not lie on a triangle, then $\gamma(\bar{G}) \leq 2$.*

3.1 The domination number in graphs of diameter 2

The following theorem improves Theorem 1.15, if $\kappa(G) < \delta(G)$.

Theorem 3.2 *If G is a graph of diameter 2, then $\gamma(G) \leq \kappa(G)$.*

Proof: Let G be a graph of diameter 2 and let S be a minimum vertex-cut. Furthermore, let X denote the vertex set of an arbitrary component of $G - S$ and let $Y = V(G) \setminus (X \cup S)$. Since $G \neq K_{n(G)}$, we conclude that $Y \neq \emptyset$.

If each vertex not in S has at least one neighbor in S, then S is a dominating set of G, and thus $\gamma(G) \leq |S| = \kappa(G)$.

Now we assume that there exists at least one vertex $u \in V(G) \setminus S$ with $N(u) \cap S = \emptyset$. Without loss of generality $u \in X$. Then $d(u, v) \geq 2$ for all $v \in S$ and thus $d(u, w) \geq 3$ for all $w \in Y$, a contradiction. □

Theorem 3.3 *Let G be a graph of diameter 2. If G is not super-λ, then $\gamma(G) \leq 3$.*

Proof: Because of Theorem 1.21 there exists an induced $K_{\delta(G)}$ in G with vertex set X such that $d(x) = \delta(G)$ for each vertex $x \in X$. It is easy to see that $[X, \bar{X}]$ is a minimum edge-cut. By $\bar{X}_1 \subseteq \bar{X}$ we denote the set of vertices in $\bar{X} = V(G) \setminus X$ with at least one neighbor in X. Furthermore, let $\bar{X}_0 = \bar{X} \setminus \bar{X}_1$.

If $\bar{X}_0 = \emptyset$, then G is isomorphic to $K_{\delta(G)} \oplus K_{\delta(G)}$, and thus $\gamma(G) \leq 2$.

Now let $\bar{X}_0 \neq \emptyset$ and let v be an arbitrary vertex in $\bar{X}_0$. If we assume that there exists a vertex $u \in \bar{X}_1$ such that $vu \notin E(G)$, then $d(v, u') \geq 3$, where $u' \in N(u) \cap X$. Hence $uv \in E(G)$ for all $v \in \bar{X}_0$ and $u \in \bar{X}_1$. By using this fact, it is easy to prove that $\{u, v, w\}$ is a dominating set for each $v \in \bar{X}_0, u \in \bar{X}_1$ and $w \in X$, and thus $\gamma(G) \leq 3$. □

By using Theorem 1.21, we can formulate the above theorem in the following way.

Corollary 3.3 *Let G be a graph of diameter 2. If G contains an induced complete graph $K_{\delta(G)}$ with all its vertices of degree $\delta(G)$, then $\gamma(G) \leq 3$.*

Theorem 3.4 *Let G be a graph. If $\mathrm{dm}(G) = 2$, then*

$$\gamma(G) \leq \lfloor n(G)/4 \rfloor + 1.$$

Proof: If $\delta(G) \leq \lfloor n(G)/4 \rfloor + 1$, then we obtain the desired result by Theorem 1.15.

Now let $\delta(G) = \lfloor n(G)/4 \rfloor + 2 + k$, where $k \geq 0$ and let u be an arbitrary vertex of minimum degree in $V(G)$. Furthermore, we define $R = V(G) - N[u]$. If $|N(v) \cap N(u)| \geq k + 3$ for each vertex $v \in R$, then

$$(N(u) \setminus \{a_1, a_2, ..., a_{k+2}\}) \to (V(G) - N[u])$$

for each $k+2$ different vertices $a_1, a_2, .., a_{k+2} \in N(u)$, and thus $\{u\} \cup (N(u) \setminus \{a_1, a_2, ..., a_{k+2}\})$ is a dominating set. Since

$$|\{u\} \cup (N(u) \setminus \{a_1, a_2, ..., a_{k+2}\})| \leq 1 + \delta(G) - (k + 2) = 1 + \lfloor n(G)/4 \rfloor,$$

we obtain our desired result. Now we have to investigate the case that there exists a vertex $v \in R$, such that $|N(v) \cap N(u)| \leq k+2$, and thus $|N(v) \cap R| \geq$

$\delta(G)-(k+2) = \lfloor n(G)/4 \rfloor$. Consequently, v dominates $\lfloor n(G)/4 \rfloor+1$ vertices in R, and thus there exist at most $|R| - \lfloor n(G)/4 \rfloor - 1$ vertices, which are not dominated by $\{u, v\}$. Since $\mathrm{dm}(G) = 2$, they can be dominated by at most $\left\lceil \frac{|R|-\lfloor n(G)/4 \rfloor -1}{2} \right\rceil$ vertices. Hence,

$$\begin{aligned}\gamma(G) &\le 2+\left\lceil \frac{|R| - \lfloor n(G)/4 \rfloor - 1}{2} \right\rceil \\ &= 2+\left\lceil \frac{n(G) - \delta(G) - 1 - \lfloor n(G)/4 \rfloor - 1}{2} \right\rceil \\ &= 2+\left\lceil \frac{n(G) - 2\lfloor n(G)/4 \rfloor - k - 4}{2} \right\rceil,\end{aligned}$$

and thus

$$\gamma(G) \le \begin{cases} 2+\left\lceil \frac{4p-2p-k-4}{2} \right\rceil \le 2+\left\lceil \frac{2p-k-4}{2} \right\rceil < p+1 \\ 2+\left\lceil \frac{4p+1-2p-k-4}{2} \right\rceil \le 2+\left\lceil \frac{2p-k-3}{2} \right\rceil \le 2+p-1 \\ 2+\left\lceil \frac{4p+2-2p-k-4}{2} \right\rceil \le 2+\left\lceil \frac{2p-k-2}{2} \right\rceil \le 2+p-1 \\ 2+\left\lceil \frac{4p+3-2p-k-4}{2} \right\rceil \le 2+\left\lceil \frac{2p-k-1}{2} \right\rceil \le 2+p-1 \end{cases}$$

$$\begin{aligned} &= \lfloor n(G)/4 \rfloor + 1, \quad \text{if } n(G) = 4p \\ &= \lfloor n(G)/4 \rfloor + 1, \quad \text{if } n(G) = 4p+1 \\ &= \lfloor n(G)/4 \rfloor + 1, \quad \text{if } n(G) = 4p+2 \\ &= \lfloor n(G)/4 \rfloor + 1, \quad \text{if } n(G) = 4p+3 \text{ and } k \ge 1. \end{aligned}$$

Now we consider the remaining case that $n(G) = 4p+3$ and $k = 0$, which implies $\delta(G) = p+2$ and the existence of a vertex $v \in R$, such that $|N(v) \cap N(u)| \le 2$.

Case 1: There exists a vertex $w \in R$ such that $|N(w) \cap N(u)| = 1$.

In this case we see that $|N(w) \cap R| \ge p+1$ and thus $|R - N[w]| \le 2p-2 = 2(p-1)$. Since $\mathrm{dm}(G) = 2$, the vertices in $R - N[w]$ can be dominated by $p-1$ vertices. This leads to $\gamma(G) \le 2+p-1 = p+1 = \lfloor n(G)/4 \rfloor + 1$.

Case 2: For each vertex $w \in R$, we have $|N(w) \cap N(u)| \ge 2$.

If there exists at least one edge xy in $N(u)$, then $\{x, a_1, a_2, a_3, ..., a_p\}$ is a dominating set of G, where $a_1, a_2, a_3, ..., a_p$ are p different vertices in $N(u)$ and $y \ne a_i, i = 1, 2, 3..., p+1$. It remains the case $E(G[N(u)]) = \emptyset$. If there exists a vertex $w \in N(u)$ such that $|N(w) \cap R| \ge p+2$, then $|V(G) \setminus (N[u] \cup N[w])| \le 2p-2$. Since $\mathrm{dm}(G) = 2$, the vertices in $V(G) \setminus (N[u] \cup N[w])$ can be dominated by $p-1$ vertices. This leads to $\gamma(G) \le 2+p-1 = p+1 = \lfloor n(G)/4 \rfloor + 1$, our desired result.

Now let $|N(w) \cap R| = p+1$ for each vertex $w \in N(u)$. Let a be an arbitrary vertex in $N(u)$ and $R' = R \setminus N(a)$. If there exists a vertex $x \in N(a) \cup N(u)$ with $|N(a) \cap R'| \ge 3$, or a vertex $x \in R'$ with $|N(x) \cap R'| \ge 2$, then $V(G)$ can be dominated by $\{u, w, x\}$ and further $p-2$ vertices, and thus $\gamma(G) \leq p+1$.

In the remaining case we have $|N(x) \cap R'| \leq 2$ for each vertex $x \in N(u) \cup N(w)$ and $|N(x) \cap R'| \leq 1$ for each vertex $x \in R'$, and thus $|[R', N(w) \cup N(u)]| \leq 2(p+1) + 2(p+2)$. Since $|N(x) \cap R'| \leq 1$ for each vertex $x \in R'$, we obtain $[R', N(w) \cup N(u)]| \geq (p+1)|R'| = (p+1)(2p-1)$. This leads to $(p+1)(2p-1) \leq 2(p+1)+2(p+2)$, a contradiction, if $p \geq 3$. By using the results of Theorem 1.8, we obtain the desired result for $p \leq 2$. Since we have discussed all possible cases, the proof is complete. □

The following example shows some graphs satisfying $\gamma(G) = \lfloor n(G)/4 \rfloor + 1$.

Example 3.1 *All graphs with $n(G) = 8$ and $G \in \mathcal{B}$ (see Figure 1) satisfy $\gamma(G) = \lfloor n(G)/4 \rfloor + 1$. Note that $\delta(G) = 3$ for all these graphs.*
For all 4-regular graphs G with $\mathrm{dm}(G) = 2$ *and $n(G) = 10$, we have $\gamma(G) \geq 3$, and thus $\gamma(G) = \lfloor n(G)/4 \rfloor + 1$.*

The following observation shows that there are a lot of graphs of diameter 2 with $\gamma(G) \leq \delta(G) - 1$.

Observation 3.1 *Let G be a graph of diameter 2.*
If $\kappa(G) < \delta(G)$, then $\gamma(G) \leq \kappa(G) \leq \delta(G) - 1$.
If $\delta(G) \geq 3$ and $\mathrm{dm}(\bar{G}) \geq 3$, *then $\gamma(G) \leq 2 \leq \delta(G) - 1$.*
If $\gamma(\bar{G}) \geq 4$, then $\gamma(G) \leq \delta(G) - 1$.
If $\delta(G) \geq \lfloor n(G)/4 \rfloor + 2$, then $\gamma(G) \leq \delta(G) - 1$.
If $\delta(G) \geq 4$ and G is not super-λ, then $\gamma(G) \leq 3 \leq \delta(G) - 1$.

Proof: The observations are direct consequences of Theorems 3.2, 3.1, 1.14, 3.4 and 3.3. □

Now we present some further sufficient conditions for $\gamma(G) \leq \delta(G) - 1$ in graphs G with diameter 2.

Theorem 3.5 *Let G be a graph of diameter 2. If there exists at least one vertex u of minimum degree such that $|N(u) \cap N(v)| \geq 3$ for all $v \in N_2(u)$, then $\gamma(G) \leq \delta(G) - 1$.*

Proof: Firstly, remark that the hypothesis implies $\delta(G) \geq 3$.

Let u be a vertex of minimum degree satisfying the conditions of the theorem and let X be an arbitrary subset of $N(u)$ of cardinality $\delta(G) - 2$. By our hypothesis, each vertex in $V(G) \setminus N[u]$ has at least one neighbor in X and thus $X \cup \{u\}$ is a dominating set, which leads to $\gamma(G) \leq \delta(G) - 1$. □

Corollary 3.4 *Let G be a graph of diameter 2. If $|N(u) \cap N(v)| \geq 3$ for all $u \in V(G)$ and $v \in N_2(u)$, then $\gamma(G) \leq \delta(G) - 1$.*

Theorem 3.6 *Let G be a graph of diameter 2. If there exists at least one vertex v of minimum degree such that there exists at least one edge $ab \in N(v)$ with $N(x) \cap N(v) \neq \{a\}$, for each $x \in N(a) \setminus N[v]$, then $\gamma(G) \leq \delta(G) - 1$.*

Proof: Let u be a vertex of minimum degree and ab be an edge in the neighborhood of u satisfying the conditions of the theorem. We define $X = N(u) \setminus \{a\}$. By $\text{dm}(G) = 2$ and our hypothesis for the vertex a, each vertex in $V(G) \setminus N[u]$ has at least one neighbor in X and thus X is a dominating set, where the vertex a is dominated by b. Thus, $\gamma(G) \leq \delta(G) - 1$.

Corollary 3.5 *Let G be a graph of diameter 2. If there exists at least one vertex u of minimum degree such that $|N(u) \cap N(v)| \geq 2$ for all $v \in N_2(u)$, and $N(u)$ is not independent, then $\gamma(G) \leq \delta(G) - 1$.*

The cardinality of a maximum independent vertex set in a graph G is called the *independence number of G* and is denoted by $\alpha(G)$.

Corollary 3.6 *Let G be a graph of diameter 2. If $|N(u) \cap N(v)| \geq 2$ for all $v \in V(G)$ and $u \in N_2(v)$ and $\alpha(G) \leq \delta(G) - 1$, then $\gamma(G) \leq \delta(G) - 1$.*

Corollary 3.7 *Let G be a connected graph of diameter 2 and $\delta(G) \geq 3$. If G is claw-free and paw-free, then $\gamma(G) \leq \delta(G) - 1$.*

The next observation describes some properties of graphs G of diameter 2 satisfying $\gamma(G) = \delta(G)$.

Observation 3.2 *Let G be a graph of diameter 2.*
If $\delta(G) \geq 4$ and $\gamma(G) = \delta(G)$, then

- *G is super-λ,*
- $\text{dm}(\bar{G}) \leq 2$,

If $\gamma(G) = \delta(G)$, then

- $\kappa(G) = \delta(G)$
- $\delta(G) \leq \lfloor n(G)/4 \rfloor + 1$
- *for each vertex u of minimum degree, there exist at least one vertex v where $v \in N_2(u)$ such that $|N(u) \cap N(v)| \leq 2$.*

If $\gamma(G) = \delta(G)$ and there exists a vertex u of minimum degree such that $|N(u) \cap N(v)| \geq 2$ for all $u, v \in V(G), v \in N_2(u)$, then $N(u)$ is independent.

If $\gamma(G) = \delta(G)$ and for each vertex u of minimum degree there exists at least one vertex $v \in N_2(u)$ such that $|N(u) \cap N(v)| = 1$, and $N(u)$ is not independent, then there exist vertices $a', b' \in N_2(u)$ such that $|N(a') \cap N(u)| = \{a\}$ and $|N(b') \cap N(u)| = \{b\}$, for each vertices $a, b \in N(u)$ where $ab \in E(G)$.

Problem 3.1 *Characterize the graphs G of diameter 2 with $\gamma(G) = \delta(G)$.*

Theorem 2.1 is only shown for graphs with minimum degree greater or equal 7. With the help of Theorem 3.4 and the above observation, we can show that: For all graphs of diameter 2, except the graphs in $\mathcal{B}$ (see Figure 1), we have $\gamma(G) \leq \lfloor n(G)/3 \rfloor$.

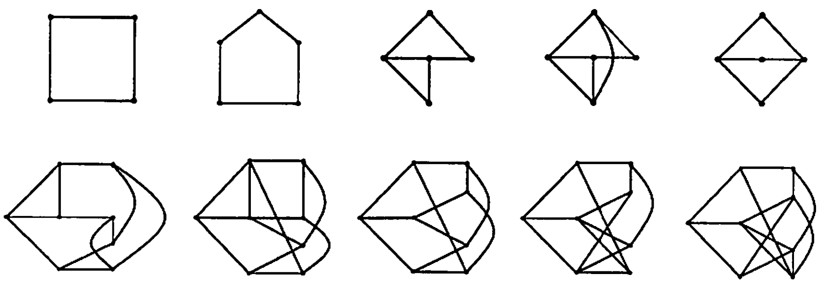

Figure 1: Graphs in family $\mathcal{B}$

Observation 3.3 *Let G be a graph in $\mathcal{B}$ with 8 vertices. For each graph $G' = G + e$, where $e = ab$, $a, b \in V(G), ab \notin E(G)$, we have $\gamma(G') = 2$ or $G' \in \mathcal{B}$.*

Proof: If we add an edge, such that there exists a vertex v of degree 5 in G', then v dominates 6 vertices and the remaining 2 vertices can be dominated by one vertex, since $\mathrm{dm}(G) = 2$. Hence $\gamma(G') = 2$. In the remaining case, one can check the result by trying. □

Theorem 3.7 *Let G be a graph with $\mathrm{dm}(G) = 2$ and $n(G) \geq 3$. If G is not isomorhic to a graph of the family $\mathcal{B}$, then*

$$\gamma(G) \leq \lfloor n(G)/3 \rfloor.$$

Proof: If $n(G) \geq 9$ or $6 \leq n(G) \leq 7$, then $\lfloor n(G)/4 \rfloor + 1 \leq \lfloor n(G)/3 \rfloor$ and thus, by Theorem 3.4, we have $\gamma(G) \leq \lfloor n(G)/3 \rfloor$.
1: Let $n(G) = 3$.

It is easy to prove that $\gamma(G) = 1$, and thus $\gamma(G) \leq \lfloor n(G)/3 \rfloor$.

2: Let $n(G) = 4$.

If $G \notin \mathcal{B}$, then it is easy to prove that $\gamma(G) = 1$, our desired result.

3: Let $n(G) = 5$.

3.1: Let $n(G) = 5, \delta(G) = 1$ or $n(G) = 5, \Delta(G) = 4$.

If $\delta(G) = 1$, then $\gamma(G) = 1 = \lfloor n(G)/3 \rfloor$. If $\Delta(G) = 4$, then one vertex of maximum degree dominates all vertices in G and thus $\gamma(G) = 1 = \lfloor n(G)/3 \rfloor$.

3.2: Let $n(G) = 5, \delta(G) = 2$ and $\Delta(G) \leq 3$. For all the graphs G satisfying the condition of this case we observe that $G \in \mathcal{B}$.

4: Let $n(G) = 8$.

4.1: Let $n(G) = 8$ and $\delta(G) \leq 2$.

Theorem 1.15 leads directly to the desired result.

4.2: Let $n(G) = 8$ and $\delta(G) = 3$.

In the following cases, let u be a vertex of minimum degree such that $|E(G[N(u)])|$ is maximal. Furthermore, let $N(u) = \{x, y, z\}$, $R = V(G) \setminus N[u]$ and $R = \{a, b, c, d\}$.

4.2.1: Let $|N(u) \cap N(v)| \geq 2$ for each $v \in R$.

If $N(u)$ is not independent, then we are done by using Corollary 3.5. Now let $N(u)$ be independent. Since $|N(u) \cap N(v)| \geq 2$ for each $v \in R$, we have $|[N(u), R]| \geq 8$. Hence, there exists a vertex in $N(u)$ with at least 3 neighbors in R, say $a, b, c \in N(x)$. If $xd \in E(G)$, then $\{u, x\}$ is a dominating set, and thus $\gamma(G) = 2$.

If $xd \notin E(G)$, then $dy, dz \in E(G)$, and thus $\{d, x\}$ is a dominating set, which implies $\gamma(G) = 2$.

4.2.2: There exists a vertex $w \in R$, such that $|N(u) \cap N(w)| = 1$.

4.2.2.1: Let $|E(G[N(u)])| \geq 2$.

Let without loss of generality $xy, yz \in E(G)$. If we assume that there exists a vertex $v \in N(u)$ such that for each $v' \in N(v) \cap R$ then we obtain $|N(v') \cap N(u)| \geq 2$. Hence $N(u) \setminus \{v\}$ is a dominating set and thus $\gamma(G) = 2$.

Now we assume that for each vertex $v \in N(u)$, there exists a vertex $v' \in N(v) \cap R$ such that $N(v') \cap N(u) = \{v\}$ and, without loss of generality, $N(a) \cap N(u) = \{x\}, N(b) \cap N(u) = \{y\}, N(c) \cap N(u) = \{z\}$.

If $ab \notin E(G)$, then $ac, ad \in E(G)$, and hence $\{a, y\}$ is a dominating set of $V(G)$. Analogously, if $ac \notin E(G)$, then $ab, ad \in E(G)$, and hence $\{a, z\}$ is a dominating set of $V(G)$. It remains the case that $ab, ac \in E(G)$. Clearly, if $ad \in E(G)$, then we are done. Now let $ad \notin E(G)$.

We consider the cases $cd \notin E(G)$ or $cd \in E(G)$. If $cd \notin E(G)$, then $cb \in E(G)$, because $N(c) \cap N(u) = \{z\}$ and $d(c) \geq 3$. Since $d(d) \geq 3$, we observe that d has at least one neighbor in N(u), say v. Thus there exists a dominating set $\{v, v'\}$, where $v' \in \{a, b, c\}$, and thus $\gamma(G) \leq 2$.

If $cd \in E(G)$, then $\{c, u\}$ is a dominating set and thus $\gamma(G) \leq 2$.

4.2.2.2: Let $|E(G[N(u)])| = 1$.

Let, without loss of generality, $xy \in E(G)$. By Theorem 3.6, we know that there must exist two vertices $v, v' \in R, v \neq v'$ such that $N(v) \cap N(u) = \{x\}$ and $N(v') \cap N(u) = \{y\}$. Otherwise, $V(G)$ can be dominated by two vertices in $N(u)$, our desired result.

Without loss of generality, $N(a) \cap N(u) = \{x\}$, $N(b) \cap N(u) = \{y\}$. Then, since $d(z) \geq 3$, we observe $zc, zd \in E(G)$. If $ab \notin E(G)$, then $ac, ad \in E(G)$ and $bc, bd \in E(G)$. This graph is isomorphic to a graph in $\mathcal{B}$, a contradiction. If we add a further edge, then we obtain by Observation 3.3 a graph with domination number 2 or a graph in $\mathcal{B}$.

Now let $ab \in E(G)$. Then, without loss of generality, $ac \in E(G)$. If $cd \in E(G)$, then $\{c, y\}$ is a dominating set and thus $\gamma(G) \leq 2$.cc

If $cd \notin E(G)$ and $ad \in E(G)$, then $a \to R$, and thus we are done. If $cd \notin E(G)$ and $ad \notin E(G)$, then we have to discuss the cases $dx \in E(G)$ or $dy \in E(G)$. If $dy \in E(G)$, then $\{y, c\}$ is a dominating set, and thus $\gamma(G) \leq 2$.

If $dx \in E(G), dy \notin E(G)$, then $bd \in E(G)$, since $d(d) \geq 3$. If $bc \in E(G)$, then $\{u, b\}$ is a dominating set. If $bc \notin E(G)$, then $cx \in E(G)$ or $cy \in E(G)$, since $d(c) \geq 3$. In the case $cx \in E(G)$, we see that $d(x) = 5$ and thus $V(G)$ can be dominated by x and a further vertex, since $\mathrm{dm}(G) = 2$. If $cx \notin E(G)$, but $cy \in E(G)$, then $G \in \mathcal{B}$, a contradiction.

4.2.2.3: Let $E(G[N(u)]) = \emptyset$.

Let, without loss of generality, $N(a) \cap N(u) = \{x\}$ and $yb, yc, zc \in E(G)$.

4.2.2.3.1: Let $zd \in E(G)$.

4.2.2.3.1.1: Let $ac \in E(G)$.

Since $d(a) \geq 3$, we have $ab \in E(G)$ or $ad \in E(G)$, say $ab \in E(G)$. Furthermore, $d(x) \geq 3$ implies $xb \in E(G)$ or $xc \in E(G)$ or $xd \in E(G)$.

If $ad \in E(G)$, then $a \to R$, and thus we are done. Now let $ad \notin E(G)$ and thus $d(a) = 3$. By our choice of the vertex u, we have $E(G[N(a)]) = \emptyset$, and thus the cases $xb \in E(G), xc \in E(G)$ or $bc \in E(G)$ cannot occur.

Hence $xd \in E(G)$. Since $d(d) \geq 3$, we have to discuss the three cases $db \in E(G)$ or $dc \in E(G)$ or $dy \in E(G)$. By our choice of u, the assumption $dc \in E(G)$ implies $db \in E(G)$ or $dy \in E(G)$. Hence, it is sufficient to consider the cases $db \in E(G)$ or $dy \in E(G)$. Firstly, let $db \in E(G)$. Then $G \in \mathcal{B}$, a contradiction.

Secondly, let $db \notin E(G)$, but $dy \in E(G)$. Since $d(b) \geq 3$, we have $bz \in E(G)$. Again, we deduce that $G \in \mathcal{B}$, a contradiction.

4.2.2.3.1.2: Let $ac \notin E(G)$.

In this case $ab, ad \in E(G)$ and $xb, xd, bd \notin E(G)$. Furthermore, since $d(x) \geq 3$, we conclude that $xc \in E(G)$. Let us consider the cases $dy \in E(G)$ or $dy \notin E(G)$.

If $dy \notin E(G)$, then $dc \in E(G)$, because of $d(d) \geq 3$. Since $dx, dy, db \notin E(G)$, we deduce that $d(d) = 3$, a contradiction to the choice of u.

If $dy \in E(G)$, we have to consider the cases $bz \in E(G)$ or $bz \notin E(G)$.

If $bz \in E(G)$, then G is isomorph to a graph in $\mathcal{B}$, a contradiction.

If $bz \notin E(G)$, then $bc \in E(G)$, because of $d(b) \geq 3$. Since $bz, bd \notin E(G)$, we obtain $d(b) = 3$, but $G[N(b)] \neq \emptyset$, a contradiction.

4.2.2.3.2: Let $zd \notin E(G)$.

In this case, we observe that $zb \in E(G)$.

4.2.2.3.2.1: Let $ad \in E(G)$.

If $d(a) = 4$, then $a \rightarrow R$, and thus we are done. Now let $d(a) = 3$ and thus we have $ab \in E(G)$ or $ac \in E(G)$, say $ab \in E(G)$ and $ac \notin E(G)$. By our choice of u, we observe $xb, xd, bd \notin E(G)$. Since $d(d) \geq 3$, we have $dy, dc \in E(G)$ and $d(d) = 3$, a contradiction to the choice of u.

4.2.2.3.2.2: Let $ad \notin E(G)$.

Again, we assume that $d(a) = 3$, because otherwise $\gamma(G) \leq 2$. Then $ab, ac \in E(G)$, $xb, xc, bc \notin E(G)$ and since $d(x) \geq 3$, we deduce $xd \in E(G)$. Now we have to consider the cases $dy \in E(G)$ or $dy \notin E(G)$. If $dy \in E(G)$, then $db, dc \in E(G)$, since $d(d) \geq 3$ and the choice of u. Hence $G \in \mathcal{B}$, a contradiction.

If $dy \notin E(G)$, then $db, dc \in E(G)$, and thus $G \in \mathcal{B}$, a contradiction.

4.3: Let $n(G) = 8$ and $\delta(G) \geq 4$.

If $\delta(G) \geq 4$, then we are done by Theorem 1.8.

Since we have discussed all possible cases, the proof is complete. □

Corollary 3.8 *Let G be a graph of diameter 2 . If $n(G) \geq 9$, then $\gamma(G) \leq \lfloor n(G)/3 \rfloor$.*

Corollary 3.9 *The two 3-regular graphs in $\mathcal{B}$ are the only 3-regular graphs with 8 vertices and diameter 2.*

Proof: We assume that there exists a 3-regular graph G, such that $G \notin \mathcal{B}$, $n(G) = 8$ and $\mathrm{dm}(G) = 2$. Since $G \notin \mathcal{B}$, we obtain $\gamma(G) = 2$. Let $\{a, b\}$ be a γ-set. By using the fact $d(a) = d(b) = 3$ and $\gamma(G) = 2$, we observe that a and b are not adjacent and do not have a common neighbor, a contradiction to $\mathrm{dm}(G) = 2$. □

3.2 Domination in graphs of diameter 3

Theorem 3.8 *Let G be a graph of diameter 3.*

i) If $\lambda(G) < \delta(G)$, then $\gamma(G) \leq 2\lambda(G) < 2\delta(G)$.

ii) If $\lambda(G) = \delta(G)$ and there exists an induced complete subgraph H of order $\delta(G)$ in G such that $d(x) = \delta(G)$ for all $x \in V(H)$, then $\gamma(G) \leq \delta(G) + \Delta(G) - 1 \leq 2\Delta(G) - 1$.

iii) If $\lambda(G) = \delta(G)$ and G is not super-λ and there does not exist induced complete subgraph H of order $\delta(G)$ in G such that $d(x) = \delta(G)$ for all $x \in V(H)$, then $\gamma(G) \leq 2\delta(G)$.

Proof: i) Let G be a graph satisfying $\lambda(G) < \delta(G)$ and let S be a minimum edge-cut. Furthermore, let X be a minimum vertex sets of the components from $G - S$. Since $\lambda(G) < \delta(G)$ we observe $|X| \geq 2$. By degree-considerations for the vertices in X, we obtain $|X| \geq \delta(G) + 1$. We denote by $X_1 \subseteq X$ and $\bar{X}_1 \subseteq \bar{X}$ the sets of vertices which are incident to at least one edge in $[X, \bar{X}]$. Furthermore, let $X_0 = X \setminus X_1$ and $\bar{X}_0 = \bar{X} \setminus \bar{X}_1$. It follows that $|X_1|, |\bar{X}_1| \leq \lambda(G)$ and $|X_0|, |\bar{X}_0| \geq 1$. Since $\mathrm{dm}(G) = 3$, each vertex in X_0 has at least one neighbor in X_1 and each vertex in $\bar{X}_0$ has at least one neighbor in $\bar{X}_1$. Hence, $D = X_1 \cup \bar{X}_1$ is a dominating set with $|D| \leq 2\lambda(G)$ and thus $\gamma(G) \leq 2\lambda(G)$.

ii) Now let G be a graph satisfying the condition in ii) and let $X = V(H)$ and $\bar{X} = V(G) \setminus X$. Then, $|[X, \bar{X}]|$ is a minimum edge-cut such that $|X| = |X_1| = \delta(G)$, where $X_1, \bar{X}_1$ is defined as above, and thus the graph G is not super-λ. Let u be an arbitrary vertex in $\bar{X}_1$ and let $u' \in N(u) \cap X$. Now we will show that $D = N(u) \cup (\bar{X}_1 \setminus \{u\})$ is a dominating set. Since $G[X]$ is isomorphic to $K_{\delta(G)}$, we observe that $v \in N[u']$ for each $v \in X$. If we assume that there exists a vertex $v \in \bar{X}$ such that $v \notin N[D]$, then $d(u', v) \geq 4$, a contradiction. Because of $|\bar{X}_1 \setminus \{u\}| \leq \delta(G) - 1$ and $|N(u) \setminus \bar{X}_1| \leq \Delta(G)$, it follows the desired result.

iii) Let G be a graph satisfying the conditions in iii), and let S be a minimum edge-cut. Furthermore, let $X, \bar{X}, X_0, X_1, \bar{X}_1, \bar{X}_0$ be defined as in the proof of i). Clearly, $|X_1| \leq \delta(G)$. The hypotheses implies that $|X| \geq \delta(G) + 1$, which leads to $|\bar{X}| \geq \delta(G) + 1$, and thus $X_0, \bar{X}_0 \neq \emptyset$. As in the proof of i), we can show that $X_1 \cup \bar{X}_1$ is a dominating set and thus $\gamma(G) \leq |X_1 \cup \bar{X}_1| \leq 2\delta(G)$. □

Corollary 3.10 *Let G be a graph of diameter 3.*
If $\Delta(G) > \delta(G)$ and G is not super-λ, then $\gamma(G) \leq 2\Delta - 1$.
If $\Delta(G) = \delta(G)$ and G is not super-λ, then $\gamma(G) \leq 2\delta$.

Lemma 3.2 *Let G be a graph of diameter 3. For each vertex v, the vertex sets $N_2(v) \cup \{v\}$ and $N_1(v) \cup N_3(v)$ are dominating sets.*

Proof: Firstly, we prove that $N[N_2(v) \cup \{v\}] = V(G)$. Since the vertex set $N_1(v) = N(v)$ is dominated by v, and the vertex set $N_3(v)$ is dominated by $N_2(v)$, the vertex set $N_2(v) \cup \{v\}$ is a dominating set. Now we prove that $N_1(v) \cup N_3(v)$ is a dominating set. Since the vertex v and the vertex set $N_2(v)$ are dominated by $N_1(v)$, the vertex set $N_1(v) \cup N_3(v)$ is a dominating set. □

By using Lemma 3.2, the two following corollaries are obviously.

Corollary 3.11 *If G is a connected graph of diameter 3, then*

$$\gamma(G) \leq \min\left\{\min_{v\in V(G)}\{|N_3(v)| + d(v)\}, \min_{v\in V(G)}\{n(G) - |N_3(v)| - d(v)\}\right\}.$$

Corollary 3.12 *Let G be a connected graph of diameter 3. If there exists a vertex v in $V(G)$ such that $N_3(v) = \emptyset$, then $\gamma(G) \leq d(v)$.*

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