On Strong Chromatic Index of Halin Graph

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Abstract

A strong k-edge-coloring of a graph G is an assignment of k colors to the edges of G in such a way that any two edges meeting at a common vertex, or being adjacent to the same edge of G, are assigned different colors. The strong chromatic index of G is the smallest number k for which G has a strong k-edge-coloring. A Halin graph is a planar graph consisting of a tree with no vertex of degree two and a cycle connecting the leaves of the tree. A caterpillar is a tree such that the removal of the leaves becomes a path. In this paper, we show that the strong chromatic index of cubic Halin graph is at most 9. That is, every cubic Halin graph is edge-decomposable into at most 9 induced matchings. Also we study the strong chromatic index of a cubic Halin graph whose characteristic tree is a caterpillar.

Keywords : Strong chromatic index, necklace, Halin graph, caterpillar
AMS 2000 MSC : 05C15

1 Introduction and notations

All graphs in this paper are finite and simple. All undefined symbols and concepts may be looked up from [2]

For k being a positive integer, let $[\![k]\!] = \{1, 2, \ldots, k\}$. A *strong k-edge-coloring* of a graph $G = (V, E)$ is a mapping $c : E \to [\![k]\!]$ in such a way that any two edges meeting at a common vertex, or being adjacent to the same edge of G, are assigned different values (colors). The *strong chromatic index* of G, denoted by $s\chi'(G)$, is the smallest number k for which G has a strong k-edge-coloring. A matching in a graph G is *induced* if no two edges in the matching are joined by an edge in G. So $s\chi'(G) \leq k$ if and only if G is edge-decomposable into k induced matchings.

A *Halin graph* $G = T \cup C$ is a plane graph that consists of a plane embedding of a tree T and a cycle C connecting the leaves (vertices of degree 1) of the tree such that C is the boundary of the exterior face and the degree of each interior vertex (also called node) of T is at least three. The tree T and the cycle C are called the *characteristic tree* and the *adjoint cycle* of G, respectively.

A tree is called a $(3,1)$-*tree* if the degree of each node is 3. A $(3,1)$-*caterpillar* T is a $(3,1)$-tree if the removal of the leaves (together with their incident edges) becomes a path which is called the *spine* of T. In this paper, "caterpillar" means $(3,1)$-caterpillar.

Suppose G is a Halin graph of order $2h+2$ with a caterpillar T as its characteristic tree, $h \geq 1$. We name the vertices along the spine P_h by 1, 2, ..., h. The vertices adjacent with 1 are named by 0 and $1'$. The vertices adjacent with h are named by $h+1$ and h'. Other leaf adjacent with i is named by i', $2 \leq i \leq h-1$. Note that $0, 1', \ldots, h', h+1$ are vertices lying on the adjoint cycle C_{h+2}. We shall use this vertex labeling through this paper. Let $\mathcal{G}_h$ be the set of all cubic Halin graphs whose characteristic trees are caterpillars of order $2h+2$. Figure 1.1 shows all graphs in $\mathcal{G}_4$.

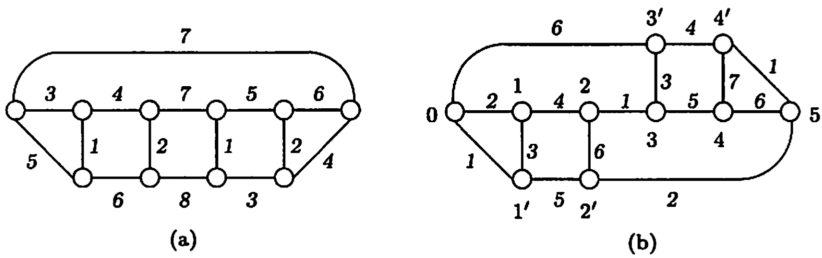

Figure 1.1. All two graphs in $\mathcal{G}_4$: (a) Ne_4, (b) another graph in $\mathcal{G}_4$.

Let $G \in \mathcal{G}_h$. If $\{0, 1'\}$, $\{1', 2'\}$, ..., $\{(h-1)', h'\}$, $\{h', h+1\}$, and $\{h+1, 0\}$ are edges of the adjoint cycle of G (i.e., vertices $0, 1' \cdots, h', h+1$ in C_{h+2} are in order), then G is called a *necklace*. It is denoted by Ne_h (see Figure 1.2). It is easy to see that, $\mathcal{G}_h = \{Ne_h\}$ for $h = 1, 2, 3$.

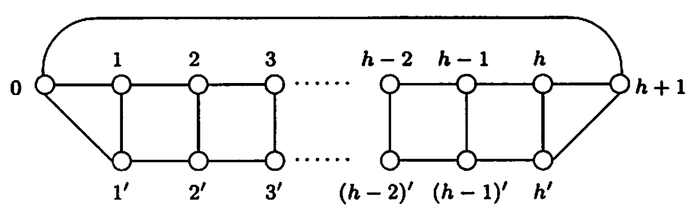

Figure 1.2. Necklace Ne_h.

In this paper, we investigate the strong chromatic index of a cubic Halin graph whose characteristic tree is a caterpillar.

2 Conjectures and Known Results

We use Δ to denote the maximum degree of a graph G. J. L. Fouquet and J. L. Jolivet [7, 8] first studied the strong edge-coloring of cubic planar graphs. In 1988, P. Erdős and J. Nešetřil [4, 5] posed the following conjecture.

Conjecture 1 [4, 5]: For any simple graph G,

$$s\chi'(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even;} \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

Faudree, Gyárfás, Schelp and Tuza [6] asked in 1990 whether $s\chi'(G) \leq 9$ if G is cubic and planar. The upper bound is attained by the complement of C_6. So if the upper bound is valid, it would be the best possible. The problem is still open. The following theorem can be found in [6, 8].

Theorem 2.1 [6, 8] $s\chi'(G) \leq 2\Delta(\Delta - 1)$.

On the other hand, a trivial upper bound for the strong chromatic index of G is given by $s\chi'(G) \leq 2\Delta^2 - 2\Delta + 1$ (see [9]). This inequality only shows that the conjecture of Erdős and Nešetřil is true for $\Delta \leq 2$.

If $\Delta = 3$, then $s\chi'(G) \leq 10$ by Conjecture 1. This result was proved by L. Andersen [1], and independently, Horák, Qing and Trotter [9]. For $\Delta = 4$, by Conjecture 1 that $s\chi'(G) \leq 20$. Recently, Cranston [3] obtained that $s\chi'(G) \leq 21$ for $\Delta = 4$.

In [6], an obvious lower bound for the strong chromatic index of G is given by the inequality $s\chi'(G) \geq \max_{uv \in E} \{\deg(u) + \deg(v) - 1\}$. The equality holds for trees.

Theorem 2.2 [6] *If G is a tree, then* $s\chi'(G) = \max_{uv \in E} \{\deg(u) + \deg(v) - 1\}$.

3 Main Results

In the following, we consider the strong chromatic index of a cubic Halin graph whose characteristic tree is a caterpillar. Also we find sharp bounds for the strong chromatic index of cubic Halin graphs.

It is easy to check that $s\chi'(Ne_1) = 6$ (see Figure 3.2). It is straightforward to see that all edges of Ne_2 must be assigned distinct colors. Hence, $s\chi'(Ne_2) = 9$ (see Figure 3.3). Also we shall show in Theorem 3.3 that

$s\chi'(Ne_3) = 6$ (see Figure 3.1). It suffices to consider the strong chromatic index graphs in $\mathcal{G}_h$ for $h \geq 4$.

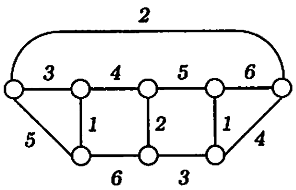

Figure 3.1. Ne_3.

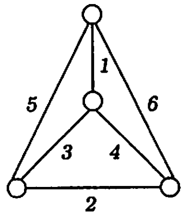

Figure 3.2. Ne_1.

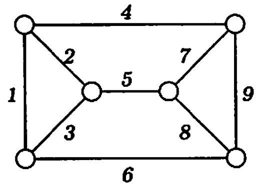

Figure 3.3. Ne_2.

Theorem 3.1 *For $h \geq 4$ and $G \in \mathcal{G}_h$, we have* $6 \leq s\chi'(G) \leq 8$.

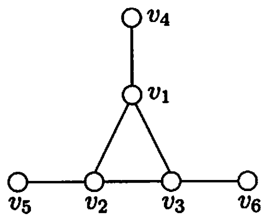

Figure 3.4. The graph W.

Proof: Suppose $G \in \mathcal{G}_h$. It is easy to see that there are at least two triangles contained in G when $h > 1$ (readers may also find the proof from Theorem 2.2 of [10]). So G contains a subgraph isomorphic to the graph W (see Figure 3.4). In fact, G contains only two such subgraphs. It is easy to see that $s\chi'(W) = 6$. Hence $s\chi'(G) \geq 6$.

Now we are going to give a strong 8-edge-coloring of G. For convenience, we let $0' = 0$ and $(h+1)' = h+1$. Consider the subgraph W that contains the triangle $011'$ first. Let the third vertex adjacent with 0 be r' $(r \neq 1)$, and the third vertex adjacent with $h+1$ be m' $(m \neq h)$. If $v_1 = 1$, $v_2 = 0$ and $v_3 = 1'$, then $v_4 = 2$ and is adjacent with either v_5 or v_6. If $v_6 = 2'$, then $v_5 = r'$ $(r \neq 2)$. If $v_5 = 2'$, then rename v_3 as 0 and v_2 as $1'$ such that $v_6 = r'$ $(r \neq 2)$. By a similar argument, the other subgraph which is isomorphic to W can be named by $v_1 = h$, $v_2 = h+1$, $v_3 = h'$, $v_4 = h-1$, $v_5 = m'$ $(m \neq h-1)$ and $v_6 = (h-1)'$. So G is described as either Figure 3.5 or Figure 3.6. If G is the graph in Figure 3.5, then either $r = m = 0$ (or $h+1$) or $3 \leq r \leq m \leq h-2$. For the latter case, it implies that $h \geq 5$. If G is the graph in Figure 3.6, then $3 \leq r \leq h-1$ and $2 \leq m \leq h-2$. First we are going to color the edges of

$$\tilde{G} = G - \Big\{\{h-1,h\},\{h,h+1\},\{h-1,(h-1)'\},\{h,h'\},\{(h-1)',h'\},\{h',h+1\}\Big\}$$

by using the color set $[\![7]\!]$. First we use 6 colors to color the edges $\{0,1\}$, $\{1,1'\}$, $\{0,1'\}$, $\{1,2\}$, $\{1',2'\}$, $\{0,r'\}$ arbitrarily. By the construction of G, G contains a subgraph H (Figure 3.7) for $0 \leq j < i < k \leq h+1$.

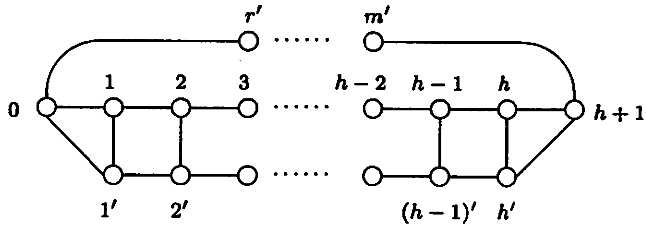

Figure 3.5.

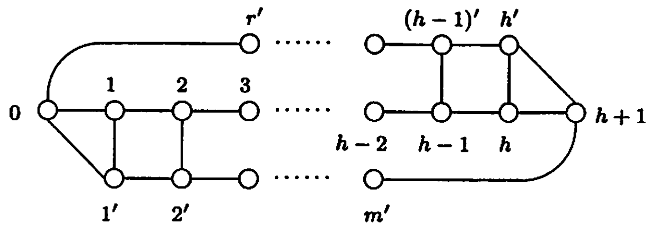

Figure 3.6.

For $2 \leq i \leq h-2$, suppose $\{i-1, i\}$ and $\{j', i'\}$ of H have been colored. We color the remaining edges of H in such a way that $\{i, i'\}$ is the first edge to be colored, follow by the edges $\{i, i+1\}$ and then $\{i', k'\}$. Then at most six colors are forbidden for each of the edges $\{i, i'\}$, $\{i, i+1\}$ and $\{i', k'\}$. So we have a strong 7-edge-coloring for $\widetilde{G}$.

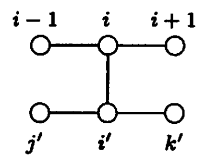

Figure 3.7. The graph H.

To color the remaining edges of G, we look at three cases. We consider G as in Figure 3.5 first. Let c be the strong edge-coloring of $\widetilde{G}$ defined above.

Case 1: If $m = h - 2$ (Figure 3.8), then we color the edges in the following order: $\{h-1, (h-1)'\}$, $\{h-1, h\}$, $\{(h-1)', h'\}$, $\{h, h+1\}$, $\{h, h'\}$, $\{h', h+1\}$. Then at most six colors are forbidden for both $\{h-1, (h-1)'\}$ and $\{h-1, h\}$; at most seven colors are forbidden for $\{(h-1)', h'\}$, $\{h, h+1\}$ and $\{h, h'\}$. Finally, assign $c(\{h-2, h-1\})$ to $\{h', h+1\}$. Hence, all edges of G require at most eight colors.

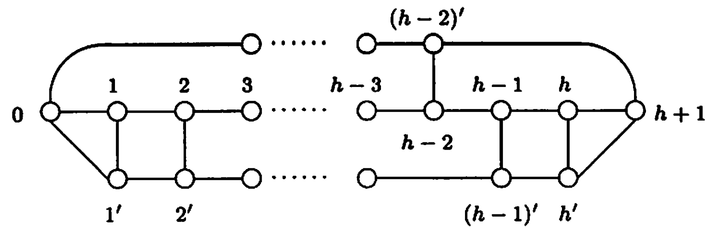

Figure 3.8.

Case 2: Suppose $0 < m \leq h-3$ (Figure 3.9). By the above assignment we have used $1, 2, \ldots, 7$ to color the edges of $\widetilde{G}$. Let l' be the vertex adjacent with m', $0 \leq l \leq m-1$. Without loss of generality, we may assume $c(\{h-2, h-1\}) = 1$, $c(\{h-2, (h-2)'\}) = 2$, $c(\{(h-2)', (h-1)'\}) = 3$, $c(\{p', (h-2)'\}) = 4$, where $2 \leq p \leq h-3$, $c(\{h-3, h-2\}) = 5$. Let $c(\{m', h+1\}) = x$, $c(\{l', m'\}) = y$ and $c(\{m, m'\}) = z$, where $x, y, z \in [\![7]\!]$.

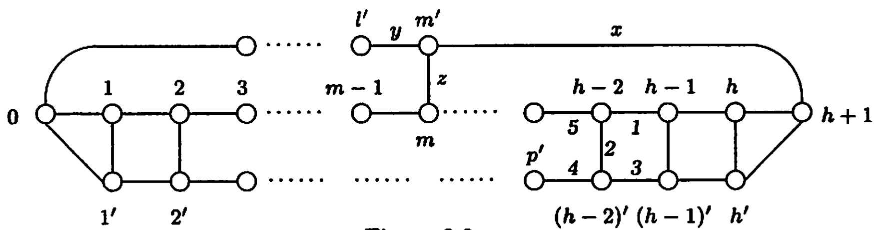

Figure 3.9.

Case 2-1: Suppose $x = 7$. Then $y, z \in [\![6]\!]$. Define $c(\{h-1, (h-1)'\}) = 7$, $c(\{(h-1)', h'\}) = 5$, $c(\{h, h'\}) = 2$ and $c(\{h-1, h\}) = 4$.

(a) If $1 \notin \{y, z\}$, then assign 2 to $\{h', h+1\}$ and 8 to $\{h, h+1\}$.

(b) If $1 \in \{y, z\}$, then assign 8 to $\{h', h+1\}$ and either 3 or 6 to $\{h, h+1\}$ depending on the values of y and z.

Case 2-2: Suppose $x \neq 7$.

(a) If $1 \notin \{y, z\}$, then define $c(\{h-1, (h-1)'\}) = 6$, $c(\{(h-1)', h'\}) = 7$, $c(\{h', h+1\}) = 1$ and $c(\{h-1, h\}) = 8$. At most 7 colors are forbidden for $\{h, h+1\}$. After coloring $\{h, h+1\}$, at most 6 colors are forbidden for $\{h, h'\}$. So we can color G by 8 colors.

(b) If $1 \in \{y, z\}$, then recolor $\{h-1, h-2\}$ by 8. Define $c(\{h-1,(h-1)'\}) = 6$, $c(\{(h-1)', h'\}) = 1$, $c(\{h', h+1\}) = 8$ and $c(\{h-1, h\}) = 7$. Now the edge $\{h, h'\}$ may be colored by 2, 4 or 5 depending on the values of x, y, z (note that $1 \in \{y, z\}$). Similarly the edge $\{h, h+1\}$ may be colored by at least one color from $\{2, 3, 4, 5\}$.

So, at most eight colors are used to color all the edges of G in this case.

Case 3: If $m = 0$, then $G \cong Ne_h$. To finish the coloring, follow Case 2 and replace $\{l', m'\}$, $\{m, m'\}$ and p' by $\{0, 1'\}$, $\{0, 1\}$ and 0 respectively in Case 2. Actually, we can find the strong chromatic index of Ne_h. Please see Theorem 3.3.

Similarly, we have a strong 8-edge-coloring in both Cases 1 and 2 for G in Figure 3.6.

Consequently, we find a strong 8-edge-coloring for G. Therefore, $6 \leq s\chi'(G) \leq 8$. □

In fact, Theorem 3.1 gives a strong edge-coloring of necklace. In the following, we will provide another coloring for necklace and determine the strong chromatic index of it.

Lemma 3.2 *Suppose G is a graph with $s\chi'(G) \geq 6$. Let two adjacent vertices A and B of G be of degree 3. Let X, Y and Z, W be the other two neighbors of A and B, respectively (X, Y, Z, W are not necessarily distinct). Let $\widetilde{G}$ be a graph obtained from G by replacing the edge-induced subgraph $G[\{AB\}]$ by a ladder graph of length 4 (see Figure 3.10). Then $s\chi'(\widetilde{G}) \leq s\chi'(G)$.*

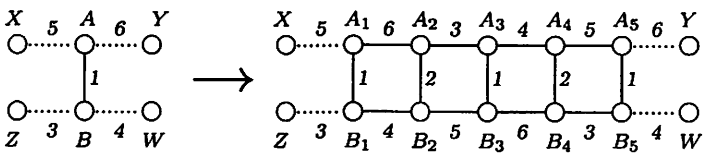

Figure 3.10.

Proof: Figure 3.10 shows that $s\chi'(\widetilde{G})$ does not exceed $s\chi'(G)$. □

Theorem 3.3 *Suppose $h \geq 1$.*

$$s\chi'(Ne_h) = \begin{cases} 6 & \textit{if } h \textit{ is odd,} \\ 7 & \textit{if } h \geq 6 \textit{ and is even,} \\ 8 & \textit{if } h = 4, \\ 9 & \textit{if } h = 2. \end{cases}$$

Proof: For h being odd, it suffices to give a strong 6-edge-coloring for Ne_h. It is shown in Figures 3.2 and 3.1 that $s\chi'(Ne_h) = 6$ for $h = 1$ and 3, respectively. Applying Lemma 3.2 repeatedly we get $s\chi'(Ne_h) = 6$ for all odd positive integers h.

For h being even, we have seen that $s\chi'(Ne_2) = 9$. So we may assume $h \geq 4$. The edges $\{1, 1'\}$, $\{h+1, 0\}$, $\{0, 1\}$, $\{1, 2\}$, $\{0, 1'\}$ and $\{1', 2'\}$ must be colored in different colors. Without loss of generality, we may assume they are colored by *1*, *2*, *3*, *4*, *5* and *6*, respectively. Since the edges of a subgraph H' (see Figure 3.11) of Ne_h require all six colors, the edges $\{2, 3\}$, $\{2', 3'\}$ and $\{2, 2'\}$ must receive colors *5*, *3* and *2* respectively. Continuing in this fashion, we see that the edge $\{j, j'\}$, where $1 \leq j \leq h-1$ is *1* or *2*, according to whether j is odd or even, respectively. In particular $\{h-1, (h-1)'\}$ is colored by *1*. We can also see that $\{h-1, h\}$ and $\{(h-1)', h'\}$ are either colored by *4* and *6* respectively, or the other way round. So the remaining three edges $\{h, h+1\}$, $\{h', h+1\}$ and $\{h, h'\}$ cannot be properly colored by six colors. Thus, $s\chi'(Ne_h) \geq 7$ for $h \geq 4$.

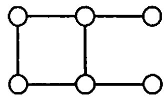

Figure 3.11. The graph H'.

To color Ne_4, we first note that it has 15 edges. It is also straightforward to verify that no color may be used for three times. Therefore $s\chi'(Ne_4) \geq 8$. A strong 8-edge-coloring of Ne_4 is given in Figure 1.1.

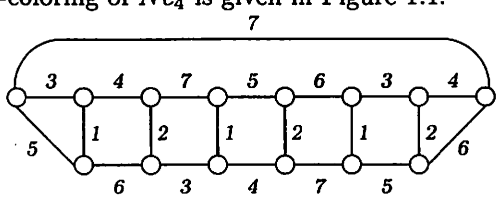

Figure 3.12. A strong 7-edge-coloring for Ne_6.

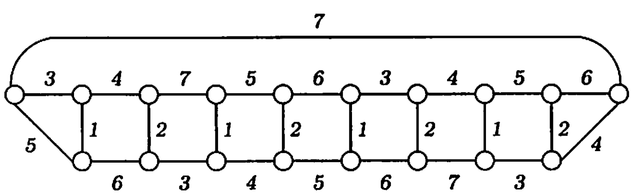

Figure 3.13. A strong 7-edge-coloring for Ne_8.

To prove $s\chi'(Ne_h) = 7$ for even $h \geq 6$, it suffices to find a 7-edge-coloring for Ne_h. It is shown in Figures 3.12 and 3.13 that $s\chi'(Ne_h) = 7$ for $h = 6$ and 8, respectively. Applying Lemma 3.2 repeatedly we get $s\chi'(Ne_h) = 7$ for all even positive integers h which are greater than 4. □

We illustrate a strong 6-edge-coloring of Ne_5 in Figure 3.14.

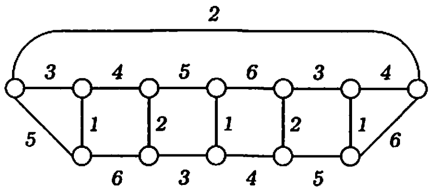

Figure 3.14. A strong 6-edge-coloring for Ne_5.

We are now going to find general bounds for the strong chromatic index of cubic Halin graphs. We have mentioned that any cubic Halin graphs G contains at least two triangles. It is easy to see that $G \in \mathcal{G}_h$ for some $h \geq 2$ if and only if G contains only two triangles.

Theorem 3.4 *If G is a cubic Halin graph, then $6 \leq s\chi'(G) \leq 9$ and the bounds are sharp.*

Proof: As mentioned before, every cubic Halin graph G contains a subgraph isomorphic to the subgraph W (Figure 3.4). Since the edges of W must be assigned distinct colors, we have $s\chi'(G) \geq s\chi'(W) = 6$

Let $G = T \cup C$, where T is a $(3,1)$-tree and $C = C_n$. Let $v_1, v_2, \ldots, v_n$ be vertices lying in C_n clockwisely. Let e_i be edge in T which is incident with v_i, $1 \leq i \leq n$.

If G contains two triangles sharing an edge, then $G \cong K_4$. Hence $s\chi'(G) = 6$. If G contains only two triangles, $G \in \mathcal{G}_h$ for some $h \geq 2$. By Theorem 3.1 $s\chi'(G) \leq 9$. Thus from now on, we assume that G contains at least three triangles. Then $n \geq 6$. When $n = 6$, the characteristic tree of G is a complete cubic tree of height 2 (Figure 3.15). Then $s\chi'(G) \leq 7$ (actually $s\chi'(G) = 7$, see [11]). So we assume $n \geq 7$. The number of leaves is two more than the number of nodes in a $(3,1)$-tree T. So $|V(T)| \geq 12$ and even. By Theorem 2.2 $s\chi'(T) = 5$. Let c_0 be a strong 5-edge-coloring for T. We shall extend c_0 to be a strong 9-edge-coloring of G.

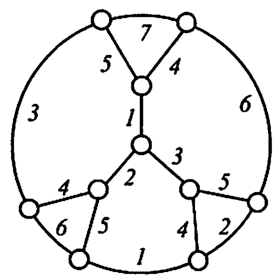

Figure 3.15.

Since $n \geq 7$, either there is an edge e_t, for some t, that is not an edge of any triangle or G contains at least four triangles. For the first case, after renumbering the vertices in C_n we may assume $t = n$ and $\{v_1, v_2\}$ is an edge of a triangle. We shall use notation Δ_s to denote the triangle containing the edge $\{v_s, v_{s+1}\}$, $1 \leq s \leq n-1$.

Suppose Δ_s is a triangle in G. If $1 \leq s \leq n-2$, then by exchanging the colors of e_s and e_{s+1} if necessary, we may assume $c_0(e_{s-1}) \neq c_0(e_s)$ (where $e_0 = e_n$). Note that, $c_0(e_{n-2})$ may equal to $c_0(e_{n-1})$ if e_{n-1} and e_n are edges of the triangle Δ_{n-1}.

First we perform the change colors procedure below (we shall call this procedure CCP):

Starting from $j = 2$ to $j = n-1$,
if $c_0(e_j) = c_0(e_{j+1})$, then we redefine $c_0(e_{j+1})$ by *6*.

Note that after performing CCP, no two consecutive edges are recolored. Let the new coloring be denoted by c. Then c is still a proper coloring of T. We can see that the edges of triangles in T are not colored by *6* except e_{n-1} may be. Without loss of generality, we may assume $c(e_1) = 1$, $c(e_2) = 2$. Also we may assume Δ_1, Δ_i and Δ_j are three consecutive triangles along the adjoint cycle C_n, $3 \leq i$, $i+2 \leq j \leq n-2$.

Case 1: Suppose $n \equiv 0 \pmod 3$. We color the edges of C_n starting at $\{v_1, v_2\}$ clockwisely by the colors *7, 8, 9* cyclically.

Case 2: Suppose $n \equiv 1 \pmod 3$. We have the following cases:

Case 2-1: Suppose $c(e_3) \neq 6$ and $c(e_n) \neq 6$. Then define $c(\{v_1, v_2\}) = 6$ and color the remaining edges of C_n starting at $\{v_2, v_3\}$ clockwisely by the colors *7, 8, 9* cyclically.

Case 2-2: Suppose $c(e_3) \neq 6$ and $c(e_n) = 6$. It means that $c(e_3) \neq 2$. We redefine $c(e_2) = 6$ and define $c(\{v_1, v_2\}) = 2$. And then color the remaining edges of C_n starting at $\{v_2, v_3\}$ clockwisely by the colors *7, 8, 9* cyclically.

Case 2-3: Suppose $c(e_3) = 6$ and $c(e_n) \neq 6$. We redefine $c(e_1) = 6$ and define $c(\{v_1, v_2\}) = 1$. The rest is same as Case 2-2.

Case 2-4: Suppose $c(e_3) = 6$ and $c(e_n) = 6$. Consider Δ_i. If $c(e_{i-1}) \neq 6$, then the case can be referred to Case 2-1 or 2-3. If $c(e_{i-1}) = 6$, then change back the original color assigned to e_k for $3 \leq k \leq i-1$ first. And then exchange the colors of e_1 and e_2. Perform CCP for edges from e_3 to e_i. Then the case can be referred to Case 2-2.

Case 3: Suppose $n \equiv 2 \pmod 3$. From Case 2 we can see that the recoloring procedure only influences the edge $\{v_1, v_2\}$ and edges e_k for $1 \le k \le i$. Thus we apply the recoloring procedure described in Case 2 from Δ_1 to Δ_i and from Δ_j to Δ_i (anti-clockwisely). The new coloring is still denoted by c.

After that we have colored the $\{v_1, v_2\}$, $\{v_j, v_{j+1}\}$ and edges e_k for $1 \le k \le j+1$ by colors in $[\![6]\!]$. But the new colors assigned to e_i and e_{i+1} may be the same and equal to *6*. If it happens, then it means that the previous colors assigned to e_{i-1} and e_i are the same, also the previous colors assigned to e_{i+1} and e_{i+2} are the same. Since e_i and e_{i+1} are adjacent, the previous colors assigned to e_i and e_{i+1} are different. Recolor e_i by $c(e_{i+2})$ and e_{i+1} by $c(e_{i-1})$.

Up to now, there are $n-2$ edges in C_n that have not been colored. Color those edges starting at any edge clockwisely by the colors *7, 8, 9* cyclically.

So we get a strong 9-edge-coloring for G. The proof is complete. □

Remark: From Theorem 3.4 we get that every cubic Halin graph is edge-decomposable into at most 9 induced matchings.

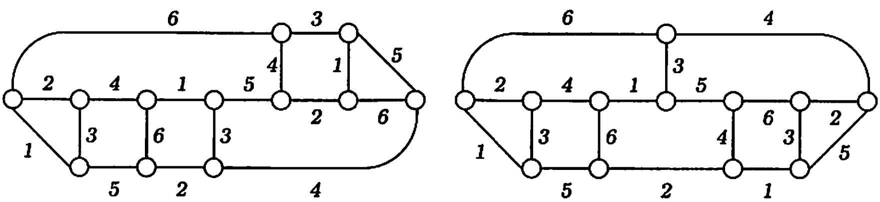

Figure 3.16. Strong 6-edge-colorings of the other two graphs in $\mathcal{G}_5$

There are only two (non-isomorphic) graphs contained in $\mathcal{G}_4$. We showed in Figure 1.1 that $s\chi'(G) = 7$, where G is described in Figure 1.1(b). Also there are only three graphs contained in $\mathcal{G}_5$. We show in Figures 3.14 and 3.16 that $s\chi'(G) = 6$ for $G \in \mathcal{G}_5$. It can be checked that $\mathcal{G}_6$ has 6 members. The strong chromatic indices of 5 graphs in $\mathcal{G}_6$ are 7 and the strong chromatic index of the remaining one is 6. We wonder whether the graphs in $\mathcal{G}_h$ are strong 7-edge-colorable for $h \ge 5$. So we conclude by presenting the following conjectures:

Conjecture 2: *For $h \ge 5$, $s\chi'(G) \le 7$ for any $G \in \mathcal{G}_h$.*

Conjecture 3: *For $h \ge 5$ and h odd, $s\chi'(G) = 6$ for any $G \in \mathcal{G}_h$.*

Conjecture 4: *Suppose* $G = T \cup C$ *is a Halin graph. Then* $s\chi'(G) \leq s\chi'(T) + 4$.

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