Minimum coverings of the complete graph with 5-cycles

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Abstract

Let K_v be the complete graph on v vertices, and C_5 be a cycle of length five. A simple minimum $(v, C_5, 1)$-covering, is a pair (V, C) where $V = V(K_v)$ and C is a family of edge-disjoint 5-cycles of minimum cardinality which partition $E(K_v) \cup E$, for some $E \subset E(K_v)$. The collection of edges E is called the excess. In this paper we determine the necessary and sufficient conditions for the existence of a simple minimum $(v, C_5, 1)$-covering. More precisely, for each $v \geq 6$, we prove that there is a simple minimum $(v, C_5, 1)$-covering having all possible excesses.

1 Introduction

A *G-design* of order v or a *G-decomposition* of K_v is a pair (V, B), where V is a v-set and B an edge-disjoint decomposition of K_v into copies of a simple graph G. The existence of a G-design, with $V(G) \leq 5$, has been studied in the literature [3, 4, 5].

A *simple covering* of K_v with copies of G, denoted by $(v, G, 1)$-covering, is an ordered triple (V, C, E), where $V = V(K_v)$, $E \subset E(K_v)$ is called the *padding* or *excess* and C is a collection of edge-disjoint copies of G which partition $E(K_v) \cup E$. The number v is called the *order* of the covering. If E is as small as possible, then (V, C, E) is called a *simple minimum covering*. In the case that a G-design exists, the excess is empty.

A *packing* of K_v with copies of G, denoted by $(v, G, 1)$-packing, is an ordered triple (V, B, L), where $V = V(K_v)$, B is a collection of edge-disjoint copies of G, and L is the set of edges not belonging to a block of B. The number v is called the *order* of the packing and the set of unused edges L is called the *leave*. If B is as large as possible, then (V, B, L) is called a *maximum packing* .

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Maximum packings of K_v with copies of G have been considered in [25] for $G = K_3$; in [8] for $G = K_4$; in [22, 27, 28] for $G = K_5$; in [13] for $G = K_4 - e$. Various results on maximum packings of K_v with cycles can be found in [9, 10, 12, 14, 15, 24, 26] and with other graphs in [1, 6, 18, 23, 29].

Results concerning minimum coverings of K_v with copies of G are given in [11] for $G = K_3$; in [20, 21] for $G = K_4$; in [17] for $G = K_4 - e$; in [12, 16] for $G = C_4$ and C_6; in [3, 4, 5] for the trees of order seven; in [23] for all graphs of four vertices or less and in [19] for graphs having at least one vertex of degree one and such that $V(G) = E(G) = 5$. Recently the existence of a minimum covering of K_v with special 5-cycles (Steiner Pentagon Systems) has been investigated by Abel et al. in [2]. In [2] the authors established the existence of covering designs having only one possible excess and with some exceptions.

In this paper we prove that, for all $v \geq 5$, there is a simple minimum $(v, C_5, 1)$-covering having all possible excesses.

2 Preliminaries and necessary conditions

In this section we determine the necessary conditions for the existence of a simple minimum $(v, C_5, 1)$-covering with excess E.

Theorem 2.1. *The only possible excesses E of a simple minimum $(v, C_5, 1)$-covering, for all $v \geq 5$, are given in the following table:*

Table 1	
v	*excess*
$\equiv 0 \pmod{10}$	*a 1-factor*
$\equiv 1, 5 \pmod{10}$	*empty set*
$\equiv 7, 9 \pmod{10}$	C_4
$\equiv 4, 6 \pmod{10}$	$X_i, i = 1, 2, 3$
$\equiv 2, 8 \pmod{10}$	$Y_i, i = 1, 2, ...7$
$\equiv 3 \pmod{10}$	$Z_i, i = 1, 2, 3, 4$
6	X_1
8	$Y_i, i = 1, 2, 3, 4$

Here C_4 is a cycle of length 4 and X_i, Y_i, Z_i, are given in Appendix I.

Proof. Let (V, B) be a simple minimum $(v, C_5, 1)$-covering with excess E. Since K_v has degree $v - 1$ and every vertex in a 5-cycle has degree two, every vertex in E has even degree when v is odd, and odd degree when v is even. Then, for v even, E must be a spanning subgraph of K_v. Consider six cases.

Case 1 : $v \equiv 0 \pmod{10}$. In this case, since $\frac{v(v-1)}{2} \equiv 0 \pmod 5$, the smallest possible excess is a 1-factor.

Case 2 : $v \equiv 1, 5 \pmod{10}$. In this case, since there exists a $(v, C_5, 1)$-design, E is the empty set.

Case 3 : $v \equiv 7, 9 \pmod{10}$. In this case we have $\frac{v(v-1)}{2} \equiv 1 \pmod 5$, hence the smallest possible excess would have four edges. Then, since each vertex of E has even degree, the only such simple graph is a 4-cycle.

Case 4 : $v \equiv 4, 6 \pmod{10}$. In this case we have $\frac{v(v-1)}{2} \equiv 1, 0 \pmod 5$, hence the smallest possible excess would have $\frac{v}{2} + 2$ edges. Since the sum of the degrees of the vertices of E is $v + 4$, we obtain the following degree sequences for E: (5, 1, 1, ...1), (3, 3, 1, 1,...1). In the first case we have $E{=}X_2$ and in the second case $E{=}X_1$ or X_3. For $v = 6$, it is easy to see that the only possible excess is X_1.

Case 5 : $v \equiv 2, 8 \pmod{10}$. In this case, we have $\frac{v(v-1)}{2} \equiv$ 1,3 (mod 5), hence the smallest possible excess would have $\frac{v}{2} + 3$ edges. Since the sum of the degrees of its vertices is $v + 6$, we obtain the following degree sequences for E: (7, 1, 1,...1), (5, 3, 1, 1,...1), (3, 3, 3, 1, 1,...1).
In the first case we have $E = Y_1$ and in the second case $E = Y_2$ or Y_5. In the final case we have $E = Y_i$, $i = 3, 4, 6, 7$. For $v = 8$, it is easy to see that the only possible excesses are Y_i, $i = 1, 2, 3, 4$.

Case 6 : $v \equiv 3 \pmod{10}$. In this case we have $\frac{v(v-1)}{2} \equiv 3 \pmod 5$, hence the smallest possible excess would have seven edges, with each vertex having even degree. Since the sum of the degrees of the vertices of E is 14, we obtain the following degree sequences for E:
(1) (2, 2, 2, 2, 2, 2, 2), (2) (2, 2, 2, 2, 2, 4),
(3) (2, 2, 2, 4, 4), (4) (2, 4, 4, 4).

In the first case we obtain $E{=}C_7{=}Z_1$ or $E = Z_2$. In the cases (2) and (3) $E{=}Z_3$ or Z_4 respectively. The final case is impossible.□

We complete this section by collecting some definitions and results which will be useful later on.

A *5-cycle system* of order v with a *hole* of size h is a triple (V, H, B), where $V = V(K_v)$, $H = V(K_h)$ with $H \subseteq V$ and B is an edge-disjoint decomposition of $K_v - K_h$ into 5-cycles.
We will also need the following auxiliary results

Lemma 2.2. *([14]). If $k \geq 3$ is odd and $t \geq 3$, the complete multipartite graph $K_{2k,2k,\ldots,2k}$ with t parts of sizes $2k$ can be decomposed into k-cycles.*

Lemma 2.3. *([24]). If $v \equiv 3 \pmod{10}$, there exists a $(v, C_5, 1)$-packing of K_v which leave a triangle. If $v \equiv 7, 9 \pmod{10}$, there exists a $(v, C_5, 1)$-packing of K_v which leave two vertex disjoint triangles.*

Lemma 2.4. *([7]). There exists a $(v, C_5, 1)$-design of order v with a hole of size u when :* $(v, u) \in \{(13,3),(17,7),(19,9),(23,13),(27,9),(29,17)\}$.

Now we need to mention one auxiliary device.

Let $K_{2n} + I$ be the multigraph obtained by adding the edges of a 1-factor I to K_{2n}. Let $\bar{K_n}$ denote the complement of K_n. If G and H are two graphs, the *join* $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Lemma 2.5. *Let I be a 1-factor of K_{10}. If v is a positive even integer, $2 \leq v \leq 20$, then there is a decomposition of the graph $(K_{10} + I) \vee \bar{K_v}$ into 5-cycles.*

Proof. Let (V, B) be a simple minimum $(10, C_5, 1)$-covering (see Example 3.3), with $V = \{a_0, a_1, ...a_9\}$ and excess I. For $q \in \{1, 2, ..., 10\}$, let $c_s = (a_0, a_1, a_2, a_3, a_4)$, $0 \leq s \leq q-1$, be a 5-cycle of B and x_{c_s} and y_{c_s} two new points. On $V \cup \{x_{c_s}, y_{c_s}\}$ define the following set of 5-cycles:
$B(c_s, x_{c_s}, y_{c_s}) = \{(x_{c_s}, a_0, a_1, y_{c_s}, a_5), (x_{c_s}, a_1, a_2, y_{c_s}, a_6), (x_{c_s}, a_2, a_3, y_{c_s}, a_7), (x_{c_s}, a_3, a_4, y_{c_s}, a_8), (x_{c_s}, a_4, a_0, y_{c_s}, a_9)\}$.
Let $\bar{B} = (\cup_{s=0}^{q-1} B(c_s, x_{c_s}, y_{c_s})$. Then it is easy to se that $\bar{B} \cup (B - \{c_0, c_1, ..., c_{q-1}\})$ is a decomposition of $(K_{10} + I) \vee \bar{K_{2q}}$ into 5-cycles on $V \cup (\cup_{s=0}^{q-1}\{x_{c_s}, y_{c_s}\}$ and the proof is complete. □

3 Direct constructions

In this section,we present constructions for simple minimum $(v, C_5, 1)$-coverings for certain small values of v.

Example 3.1. *A simple minimum $(6, C_5, 1)$-covering (V, B) with excess X_1.*

Elements: $V = Z_6$.
Blocks: $B = \{(0,1,2,5,3),(0,2,4,3,5),(0,4,1,3,2),(0,4,5,1,3)\}$.
The excess is $\{20, 04, 03, 31, 35\}$.

Example 3.2. *Four simple minimum $(8, C_5, 1)$-coverings (V, B_i) with excesses Y_i, $i = 1,2,3,4$.*

1. A simple minimum $(8, C_5, 1)$-covering (V, B_1) with excess Y_1.
   Elements: $V = Z_8$.
   Blocks: $B_1 = \{(0,1,2,6,7),(0,1,3,2,5),(0,5,1,4,2),(0,6,1,7,2),$
   $(0,4,6,5,3),(0,6,3,7,4),(0,3,4,5,7)\}$.
   The excess is $\{01, 02, 03, 04, 05, 06, 07\}$.

2. A simple minimum $(8, C_5, 1)$-covering (V, B_2) with excess Y_2.
   Elements: $V = Z_8$.
   Blocks: $B_2 = \{B_1 - \{(0,3,4,5,7)\}\} \cup \{(2,3,4,5,7)\}$.
   The excess is $\{01, 02, 04, 05, 06, 27, 23\}$.

3. A simple minimum $(8, C_5, 1)$-covering (V, B_3) with excess Y_3.
   Elements: $V = Z_8$.
   Blocks: $B_3 = \{B_1 - \{(0,3,4,5,7), (0,6,3,7,4)\}\} \cup \{(2,3,4,5,7), (5,6,3,7,4)\}$.
   The excess is $\{01, 02, 05, 54, 56, 27, 23\}$.

4. A simple minimum $(8, C_5, 1)$-covering (V, B_4) with excess Y_4 .
   Elements: $V = Z_8$.
   Blocks: $B_4 = \{(0,1,5,7,2), (0,1,6,2,3), (0,2,1,6,4), (0,3,4,2,5), (0,6,3,5,7), (1,2,4,7,3), (1,4,5,6,7)\}$.
   The excess is $\{16, 12, 10, 24, 20, 03, 57\}$.

Example 3.3 ([2]). *A simple minimum* $(10, C_5, 1)$*-covering* (V, B) *with excess* E *as in Table 1.*

Elements: $V = Z_{10}$.
Blocks: $B = \{(1,9,4,3,5), (2,0,5,4,6), (4,1,7,6,8), (5,2,8,7,9), (6,3,9,8,0), (0,7,2,1,3), (1,0,7,3,8), (0,4,7,5,9), (5,6,9,2,8), (2,4,6,1,3)\}$.
The excess is {07,13,28,59,46}.

Example 3.4. *Seven simple minimum* $(12, C_5, 1)$*-coverings* (V, D_i) *with excesses* Y_i*,* $i = 1, 2, ..7$*, as in Table 1.*

Let (Z_8, B_i), $i = 1, 2, 3, 4$, be the $(8, C_5, 1)$-covering, with excess E_i, given in Example 3.2.

1. Four simple minimum $(12, C_5, 1)$-coverings (V, D_i) with excesses Y_i, $i = 1, 2, 3, 4$.
   Elements: $V = Z_8 \cup \{a, b, c, d\}$.
   Blocks: $D_i = B_i \cup \{0, a, b, 1, c), (0, d, 1, a, b), (4, a, 5, c, d), (4, b, 5, d, c), (2, a, c, 3, d), (3, a, d, 7, b), (6, b, c, 7, a), (2, b, d, 6, c)\}, i = 1, 2, 3, 4$.
   The excesses are $E_i \cup \{ab, cd\}, i = 1, 2, 3, 4$.

2. A simple minimum $(12, C_5, 1)$-coverings (V, D_5) with excess Y_5.
   Elements: $V = Z_8 \cup \{a, b, c, d\}$.
   Blocks: $D_5 = \{D_1 - \{(0,3,4,5,7)\} \cup \{(a,3,4,5,7)\}$.
   The excess is $\{01, 02, 04, 05, 06, ab, a3, a7, cd\}$.

3. A simple minimum $(12, C_5, 1)$-covering (V, D_6) with excess Y_6.
   Elements: $V = Z_8 \cup \{a, b, c, d\}$.
   Blocks: $D_6 = \{D_5 - \{(0,1,3,2,5)\} \cup \{(b,1,3,2,5)\}$.
   The excess is $\{02, 04, 06, ab, a3, a7, cd, b1, b5\}$.

4. A simple minimum $(12, C_5, 1)$-covering (V, D_7) with excesses Y_7.
   Elements: $V = Z_8 \cup \{a, b, c, d\}$.
   Blocks: $D_7 = \{D_5 - \{(0,1,3,2,5)\} \cup \{(c,1,3,2,5)\}$.
   The excess is $\{02, 04, 06, ab, a3, a7, cd, c1, c5\}$.

Example 3.5. *Three simple minimum* $(14, C_5, 1)$*-coverings* (V, M_i) *with excesses* X_i, $i = 1, 2, 3$.

Let (Z_{10}, B) be the simple minimum $(10, C_5, 1)$-covering with excess E given in Example 3.3. Let $M = B - \{2, 4, 6, 1, 3\}$.

1. A simple minimum $(14, C_5, 1)$-covering (V, M_1) with excess X_1. .
   Elements: $V = Z_{10} \cup \{a, b, c, d\}$.
   Blocks: $M_1 = M \cup \{(1, a, 2, 4, b), (a, 4, 6, c, 3), (a, 6, 1, d, 5), (1, 3, b, 5, c), (2, b, 6, d, 3), (0, c, 2, d, a), (0, b, c, 4, d), (b, c, a, 7, d), (0, b, d, 8, a), (a, b, 8, c, 9), (b, 7, c, d, 9)\}$.
   The excess is $\{07, 0a, 0b, bc, bd, 13, 28, 59, 46\}$.

2. A simple minimum $(14, C_5, 1)$-covering (V, M_2) with excess X_2.
   Elements: $V = Z_{10} \cup \{a, b, c, d\}$.
   Blocks: $M_2 = \{M_1 - \{(b, c, a, 7, d)\} \cup \{(0, c, a, 7, d)\}$.
   The excess is $\{07, 0a, 0b, 0c, 0d, 13, 28, 59, 46\}$.

3 A simple minimum $(14, C_5, 1)$-covering (V, M_3) with excess X_3.
   Elements: $V = Z_{10} \cup \{a, b, c, d\}$.
   Blocks: $M_3 = \{M_2 - \{(0, c, a, 7, d)\} \cup \{(2, c, a, 7, d)\}$.
   The excess is $\{07, 0a, 0b, 13, 28, 2c, 2d, 59, 46\}$.

Example 3.6. *Three simple minimum* $(16, C_5, 1)$*-coverings* (V, N_i) *with excesses* X_i, $i = 1, 2, 3$.

Let N be the block set of the decomposition of $(K_{10}+I) \vee \bar{K}_6$ into 5-cycles (see Lemma 2.5). Let $V_1 = \{a_i, i \in Z_{10}\}$ and $I = \{a_0a_7, a_1a_3, a_2a_8, a_5a_9, a_4a_6\}$. Let (Z_6, B) be the simple minimum $(6, C_5, 1)$-covering, with excess $E = \{20, 04, 03, 31, 35\}$, given in Example 3.1. Let (a_0, a_7) be an edge of I. Define the following sets of blocks:
$N_2 = \{B - \{(0, 4, 1, 3, 2), (0, 1, 2, 5, 3)\}\} \cup \{(a_0, 4, 1, 3, 2), (0, 1, 2, 5, a_0)\} \cup N$, $N_3 = \{B - \{(0, 4, 1, 3, 2)\}\} \cup \{(a_0, 4, 1, 3, 2)\} \cup N$.

1. A simple minimum $(16, C_5, 1)$-covering (V, N_1) with excess X_1.
   Elements: $V = Z_6 \cup V_1$.
   Blocks: $N_1 = B \cup N$
   The excess is $E \cup I$.

2. A simple minimum $(16, C_5, 1)$-covering (V, N_2) with excess X_2.
   Elements: $V = Z_6 \cup V_1$.
   Blocks: N_2.
   The excess is $I \cup \{a_04, a_02, a_00, a_05, 13\}$.

3. A simple minimum $(16, C_5, 1)$-covering (V, N_3) with excess X_3.
   Elements: $V = Z_6 \cup V_1$.

Blocks: N_3.
The excess is $I \cup \{a_0 4, a_0 2, 30, 13, 35\}$.

Example 3.7. *Seven simple minimum* $(18, C_5, 1)$*-coverings* (V, S_i) *with excesses* Y_i, $i = 1, 2, .., 7$, *as in Table 1.*

Let S be the block set of the decomposition of $(K_{10}+I) \vee \bar{K}_8$ into 5-cycles (see Lemma 2.5). Let $V_1 = \{a_i, i \in Z_{10}\}$ and $I = \{a_0a_7, a_1a_3, a_2a_8, a_5a_9, a_4a_6\}$. Let (Z_8, B_i) be the simple minimum $(8, C_5, 1)$-coverings with excess E_i, i=1,2,3,4, given in Example 3.2. Let (a_0, a_7) and (a_1, a_3) be two edges of I. Define the following sets of blocks:
$S_5 = \{B_1 - \{(0, 1, 2, 6, 7)\}\} \cup \{(a_0, 1, 2, 6, 7)\} \cup S$,
$S_6 = \{S_5 - \{(0, 4, 6, 5, 3)\}\} \cup \{(a_7, 4, 6, 5, 3)\}$,
$S_7 = \{S_5 - \{(0, 4, 6, 5, 3)\}\} \cup \{(a_1, 4, 6, 5, 3)\}$.

1. Four simple minimum $(18, C_5, 1)$-coverings (V, S_i) with excesses Y_i, $i = 1, 2, 3, 4$.
   Elements: $V = Z_8 \cup V_1$.
   Blocks: $S_i = B_i \cup S$, $i = 1, 2, 3, 4$.
   The excesses are $I \cup E_i$, $i = 1, 2, 3, 4$.

2. A simple minimum $(18, C_5, 1)$-covering (V, S_5) with excess Y_5.
   Elements: $V = Z_8 \cup V_1$.
   Blocks: S_5.
   The excess is $I \cup \{a_0 1, a_0 7, 02, 03, 04, 05, 06\}$.

3. A simple minimum $(18, C_5, 1)$-covering (V, S_6) with excess Y_6.
   Elements: $V = Z_8 \cup V_1$.
   Blocks: S_6.
   The excess is $I \cup \{a_0 1, a_0 7, 02, a_7 3, a_7 4, 05, 06\}$.

4. A simple minimum $(18, C_5, 1)$-covering (V, S_7) with excess Y_7.
   Elements: $V = Z_8 \cup V_1$.
   Blocks: S_7.
   The excess is $I \cup \{a_0 1, a_0 7, 02, a_1 3, a_1 4, 05, 06\}$.

Lemma 3.8. *There exists a simple minimum* $(v, C_5, 1)$*-covering,* $v = 22, 24, 26, 28$, *with excess as in Table 1.*

Proof. Let $v = 10+h$, where $h = 12, 14, 16, 18$. Let I be a 1-factor of K_{10}. From Lemma 2.5, there exists a decomposition of $(K_{10} + I) \vee \bar{K}_h$, $h = 12, 14, 16, 18$, into 5-cycles. Now, replacing the hole of size h by a simple minimum $(h, C_5, 1)$-covering with excess E as in Table 1 (such design has been given in Examples 3.4, 3.5, 3.6 and 3.7), we obtain the required designs with excess $E \cup I$. □

4 Main result

In this section we prove that the conditions of the Theorem 2.1 are sufficient for the existence of a simple minimum $(v, C_5, 1)$-covering, $v \geq 6$, with excess E as in Table 1.

Lemma 4.1. *If* $v \equiv 3 \pmod{10}$, $v \geq 13$, *then there exists a simple minimum* $(v, C_5, 1)$*-covering with excess as in Table 1.*

Proof. Let $v \equiv 3 \pmod{10}$. By Lemma 2.3, there exists a $(v, C_5, 1)$-packing (V, B, L) of order v with leave a triangle L. Let $V = \{a, b, c, d, e, f, g, h\} \cup \{a_i, i = 1, 2, ..., v-8\}$, $L = \{ab, bc, ac\}$. Let (a, d, e, f, g) be a 5-cycle of B. Define four sets of 5-cycles as follows:
$B_1 = B \cup \{(a, b, c, d, e), (a, c, f, g, h)\}$,
$B_2 = \{B - \{(a, d, e, f, g)\}\} \cup \{(a, b, d, g, f), (a, c, f, e, d), (a, g, b, c, e)\}$,
$B_3 = B \cup \{(a, b, c, d, e), (a, c, e, f, g)\}$,
$B_4 = B \cup \{(a, b, c, d, e), (a, c, e, b, d)\}$.
Then (V, B_i), $i = 1, 2, 3, 4$, is a simple $(v, C_5, 1)$-covering with excess Z_i as in Table 1. This completes the proof. □

Lemma 4.2. *If* $v \equiv 7, 9 \pmod{10}$, $v \geq 7$, *then there exists a simple minimum* $(v, C_5, 1)$*-covering with excess as in Table 1.*

Proof. Let $v \equiv 7, 9 \pmod{10}$. By Lemma 2.3, there exists a $(v, C_5, 1)$-packing (V, B, L) of order v with leave two vertex disjoint triangles L. Let $V = \{a, b, c, d, e, f\} \cup \{a_i, i = 1, 2, ..., v-6\}$, $L = \{ab, bc, ac, de, df, ef\}$. Let $B_1 = B \cup \{(a, b, c, d, e), (a, c, e, f, d)\}$.
Then (V, B_1) is a simple $(v, C_5, 1)$-covering with excess C_4. This completes the proof. □

Lemma 4.3. *If* $v \equiv 2, 4, 6, 8 \pmod{10}$, $v \geq 12$, *then there exists a simple minimum* $(v, C_5, 1)$*-covering with excess as in Table 1.*

Proof. Write $v = h + 10n$, where $h \in \{2, 4, 6, 8\}$. The designs of order $v = 12, 14, 16, 18$ are all given in Examples 3.4, 3.5, 3.6 and 3.7. The case $n = 2$ was solved in Lemma 3.8. Now let $n \geq 3$, X be a set of size $10n$, H a h-set, $h \in \{2, 4, 6, 8\}$ with $X \cap H = \emptyset$, $H_r = Z_{10} \times \{r\}$, $r = 0, 1, \ldots, n-1$, and $V(K_v) = (\cup_{i=0}^{n-1} H_i) \cup H$. Let I_r be a 1-factor on $V(H_r)$, where $r = 0, 1, .., n-1$.

Now we obtain the required design on V as follows.
On the set $H \cup H_0$ place a simple minimum $(10+h, C_5, 1)$-covering of order $10+h$, $h \in \{2, 4, 6, 8\}$, and excess E. On the set $H \cup H_i$, $i = 1, 2, \ldots, n-1$ place a decomposition of the graph $(K_{10}+I) \vee \bar{K}_h$ into 5-cycles (see Lemma 2.4). On the $10n$-set $X = H_0 \cup H_1 \cup \cdots \cup H_{n-1}$ place a decomposition of the complete n-partite graph $K_{10,10,...10}$ into 5-cycles (see Lemma 2.2). Now, making use of a simple minimum $(u, C_5, 1)$-covering of order $u \in \{12, 14, 16, 18\}$ with excess

E as in Table 1, we obtain a simple minimum $(v, C_5, 1)$-covering with excess $E \cup (\cup_{i=1}^{n-1} H_i)$ as in Table 1. This completes the proof. □

Theorem 4.4. *If* $v \equiv 0,2,3,4,6,7,8,9 \pmod{10}$, $v \geq 6$, *then there exists a simple minimum* $(v, C_5, 1)$*-covering with excess as in Table 1.*

Proof. The case $v \equiv 0 \pmod{10}$ can be found in [2]. The designs of order $v = 6, 8$ are given in Examples 3.1 and 3.2. The other cases follow immediately from Lemmas 4.1, 4.2 and 4.3. □

References

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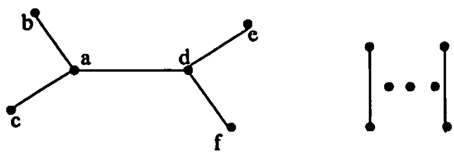
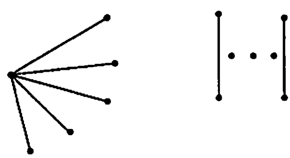
Appendix I

I.1 The excesses X_i, $i = 1, 2, 3$, for $v \equiv 4, 6 \pmod{10}$, $n \geq 14$. For $v = 6$, the excess is X_1.

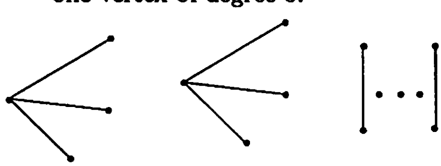
X_1 A one factor on $v - 6$ vertices and a graph $\{ab, ac, ad, de, df\}$.



X_2 A one factor on $v - 6$ vertices and a tree on 6 vertices with one vertex of degree 5.

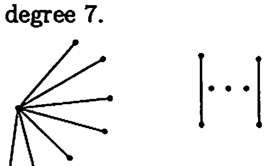


X_3 A one factor on $v - 8$ vertices and two trees each on 4 vertices with one vertex of degree 3.

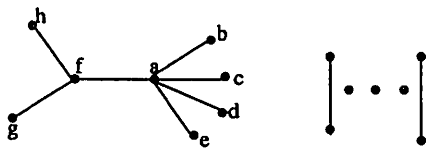


I.2 The excesses Y_i, $i = 1, 2, ...7$, for $v \equiv 2, 8 \pmod{10}$, $n \geq 12$. For $v = 8$, the excesses are Y_i, $i = 1, 2, 3, 4$.

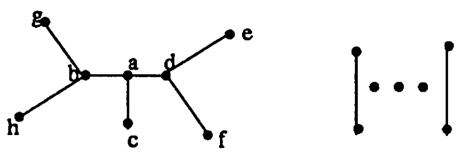
Y_1 A one factor on $v - 8$ vertices and a tree on 8 vertices with one vertex of degree 7.



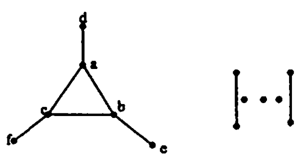
Y_2 A one factor on $v-8$ vertices and a graph $\{ab, ac, ad, ae, af, fg, fh\}$.



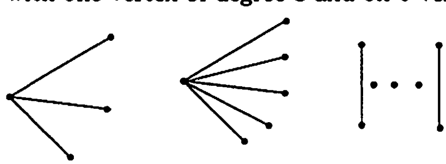
Y_3 A one factor on $v-8$ vertices and a graph $\{ab, ac, ad, de, df, bg, bh\}$.



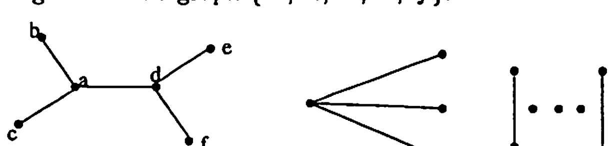
Y_4 A one factor on $v-6$ vertices and a graph $\{ab, ac, bc, ad, be, cf\}$.



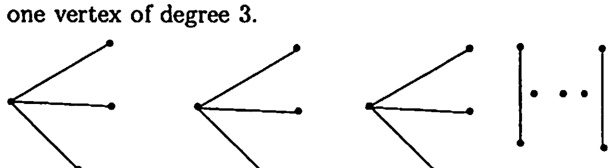
Y_5 A one factor on $v-10$ vertices and two trees respectively on 4 vertices with one vertex of degree 3 and on 6 vertices with one vertex of degree 5.



Y_6 A one factor on $v-10$ vertices, a tree on 4 vertices with one vertex of degree 3 and a graph $\{ab, ac, ad, de, df\}$.



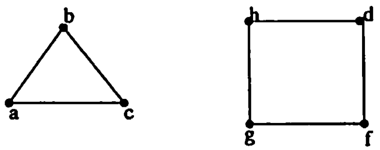
Y_7 A one factor on $v-12$ vertices and three trees each on 4 vertices with one vertex of degree 3.



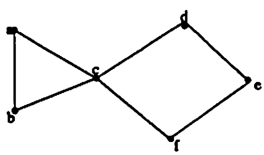
I.3 The excesses Z_i, $i = 1, 2, 3, 4$, for $v \equiv 3 \pmod{10}$.

Z_1 A cycle of length 7.

Z_2 A graph on seven vertices $\{ab, ac, bc, hd, df, fg, gh\}$.



Z_3 A graph on six vertices $\{ab, ac, bc, cd, de, ef, fc\}$.



Z_4 A graph on five vertices $\{ab, ac, bc, ad, db, af, bf\}$.