Proof of a Hypergeometric Identity

Peter J. Larcombe

Derbyshire Business School
University of Derby, Kedleston Road, Derby DE22 1GB, U.K.
{P.J.Larcombe@derby.ac.uk}

Abstract

A (previously reported) surprising and attractive hypergeometric identity is established from first principles using three hypergeometric transformations.

Introduction

In [1], a hitherto unseen hypergeometric identity arose naturally as a consequence of producing a new form for the general term of the so called Fennessey-Larcombe-French sequence (see (24) of Remark 2 therein, p.90). Adopting standard notation, it is thus:

<u>**Theorem**</u> For integer $n \geq 0$,

$$(2n+1)\binom{2n}{n}^2 {}_3F_2\left(\begin{matrix} -n, \frac{1}{2}, -\frac{1}{2} \\ \frac{1}{2}-n, -\frac{1}{2}-n \end{matrix} \middle| -1\right) = 8^n \, {}_4F_3\left(\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}-\frac{1}{2}n, -\frac{1}{2}n \\ \frac{1}{4}, 1, 1 \end{matrix} \middle| 1\right).$$

The identity, as far as anyone is currently aware, is not, it seems, a special case of an existing hypergeometric identity. On the contrary, as will be shown, its formulation requires no little effort and ingenuity, on which point the author is indebted to Professor Christian Krattenthaler for kindly providing a proof outline (computer-assisted via his symbolic package "HYP"[1]). Because it is non-trivial, the setting down of the full proof will no doubt

[1]See **http://www.mat.univie.ac.at/~kratt/hyp_ hypq/hyp.html#HYP**.

JCMCC **57** (2006), pp. 65-73

interest people working in this specialised area of mathematics.

In this short paper a first principles proof of the result is detailed, based on certain hypergeometric transformations. For ease of reference we list them at the outset. Result I appears in, for example, Slater's well known 1966 text [2], and Result II—generally (but wrongly) accepted to be due to J. Thomae—is given in Gasper and Rahman [3] in a slightly different guise. Result III is a contiguous relation which is easily verified (by showing that the coefficient of a general term z^s, say, is the same on both l.h.s. and r.h.s.) and used here for $p = 2, 3$ as needed.

<u>Result I</u> For integer d or f (or both) ≤ 0 [2, (2.4.2.1), p.65],

$$ {}_3F_2\left(\begin{matrix} f, 1+f-a, d \\ a, g \end{matrix} \middle| -1\right) = $$
$$ \frac{\Gamma(g)\Gamma(g-f-d)}{\Gamma(g-f)\Gamma(g-d)} \, {}_4F_3\left(\begin{matrix} d, 1+f-g, \frac{1}{2}f, \frac{1}{2}+\frac{1}{2}f \\ a, \frac{1}{2}+\frac{1}{2}f+\frac{1}{2}d-\frac{1}{2}g, 1+\frac{1}{2}f+\frac{1}{2}d-\frac{1}{2}g \end{matrix} \middle| 1\right). $$

<u>Result II</u> For integer b or c (or both) ≤ 0 (see the Appendix),

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| 1\right) = \frac{\Gamma(e-b-c)\Gamma(e)}{\Gamma(e-b)\Gamma(e-c)} \, {}_3F_2\left(\begin{matrix} d-a, b, c \\ d, 1+b+c-e \end{matrix} \middle| 1\right). $$

<u>Result III</u> For $p \geq 1$,

$$ {}_{p+1}F_p\left(\begin{matrix} a, A_1, \ldots, A_p \\ B_1, \ldots, B_p \end{matrix} \middle| z\right) = $$
$$ {}_{p+1}F_p\left(\begin{matrix} a-1, A_1, \ldots, A_p \\ B_1, \ldots, B_p \end{matrix} \middle| z\right) $$
$$ + \left(\frac{A_1 \cdots A_p}{B_1 \cdots B_p}\right) z \, {}_{p+1}F_p\left(\begin{matrix} a, A_1+1, \ldots, A_p+1 \\ B_1+1, \ldots, B_p+1 \end{matrix} \middle| z\right). $$

Whilst Result II is used fairly commonly, the Result I transformation between a nearly poised ${}_3F_2(-1)$ series and a ${}_4F_3(1)$ series is quite an exotic one which is rarely applied. For this reason the route through our proof cannot be described as obvious, the limiting arguments required in places perhaps adding elements of appeal to those interested in this type of proof construction. The identity in question seems quite attractive, possessing a rather pleasing compactness (viewing the proof in reverse, this is seen to arise essentially from the somewhat fortunate combining of terms generated by Result III), though time alone will tell whether or not the result becomes one of any significance within the study of binomial coefficient or hypergeometric series identities.

Denoting by $(u)_m$, in usual fashion, the rising factorial function

$$(u)_m = u(u+1)(u+2)(u+3)\cdots(u+m-1) \tag{1}$$

defined for integer $m \geq 0$ (where $(u)_0 = 1$), our Theorem can be re-cast slightly:

<u>**Theorem**</u> For integer $n \geq 0$,

$$ {}_3F_2\left(\begin{matrix} -n, \frac{1}{2}, -\frac{1}{2} \\ \frac{1}{2}-n, -\frac{1}{2}-n \end{matrix} \,\middle|\, -1\right) = \frac{n!^2}{2^{n+1}(\frac{1}{2})_n(\frac{1}{2})_{n+1}}\, {}_4F_3\left(\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}-\frac{1}{2}n, -\frac{1}{2}n \\ \frac{1}{4}, 1, 1 \end{matrix} \,\middle|\, 1\right). $$

It is this version which will be proved. It is trivial to see that it holds for $n = 0$, and the case $n = 1$ may be checked by hand without difficulty. For a minor technical reason explained later, we assume $n \geq 2$. Three other sub-results are employed in places, in addition to Results I-III already given. The first,

$$x\Gamma(x) = \Gamma(x+1), \tag{2}$$

is a familiar one. Replacing x with $x-1$ in (2) and iterating it $m-1$ times leads, using (1), to

$$\frac{\Gamma(x)}{\Gamma(x-m)} = (x-m)_m, \tag{3}$$

our second result. Finally, it is easy to deduce from (1) that

$$(x)_m = (-1)^m(-x-m+1)_m. \tag{4}$$

The Proof

Our starting point is to set $f = -n$, $d = -\frac{1}{2}$, $a = \frac{1}{2} - n$ and $g = -\frac{1}{2} - n + \varepsilon$ in Result I, which reads

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} -n, \frac{1}{2}, -\frac{1}{2} \\ \frac{1}{2}-n, -\frac{1}{2}-n+\varepsilon \end{matrix} \,\middle|\, -1\right) \\ &= \frac{\Gamma(-\frac{1}{2}-n+\varepsilon)\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)\Gamma(-n+\varepsilon)}\, {}_4F_3\left(\begin{matrix} -\frac{1}{2}, \frac{3}{2}-\varepsilon, -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n \\ \frac{1}{2}-n, \frac{1}{2}-\frac{1}{2}\varepsilon, 1-\frac{1}{2}\varepsilon \end{matrix} \,\middle|\, 1\right) \\ &= \frac{(1-\varepsilon)_n}{(\frac{3}{2}-\varepsilon)_n}\, {}_4F_3\left(\begin{matrix} -\frac{1}{2}, \frac{3}{2}-\varepsilon, -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n \\ \frac{1}{2}-n, \frac{1}{2}-\frac{1}{2}\varepsilon, 1-\frac{1}{2}\varepsilon \end{matrix} \,\middle|\, 1\right), \end{aligned} \tag{P1}$$

since, applying (3),(4) consecutively,

$$\begin{aligned}\frac{\Gamma(-\frac{1}{2}-n+\varepsilon)\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)\Gamma(-n+\varepsilon)} &= \frac{\Gamma(-\frac{1}{2}+\varepsilon-n)}{\Gamma(-\frac{1}{2}+\varepsilon)}\frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon-n)} \\ &= \frac{1}{(-\frac{1}{2}+\varepsilon-n)_n}(\varepsilon-n)_n \\ &= \frac{1}{(\frac{3}{2}-\varepsilon)_n}(1-\varepsilon)_n. \end{aligned} \tag{P2}$$

Thus, in the limit $\varepsilon \to 0$ the l.h.s. ${}_3F_2(-1)$ series of the Theorem can, noting that $(1)_n = n!$, be written as

$$\begin{aligned}&{}_3F_2\left(\begin{matrix}-n,\frac{1}{2},-\frac{1}{2}\\ \frac{1}{2}-n,-\frac{1}{2}-n\end{matrix}\,\middle|\,-1\right) \\ &\quad = \frac{n!}{(\frac{3}{2})_n}\,{}_4F_3\left(\begin{matrix}\frac{3}{2},-\frac{1}{2},-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n\\ \frac{1}{2},1,\frac{1}{2}-n\end{matrix}\,\middle|\,1\right)\end{aligned} \tag{P3}$$

from (P1), having imposed a convenient re-ordering of upper/lower parameters in the r.h.s. ${}_4F_3(1)$ series. This is in readiness for the next step—which is to transform the latter via a $p = 3$ instance of Result III (with $a = \frac{3}{2}$, $A_1 = -\frac{1}{2}, \ldots, B_3 = \frac{1}{2} - n$; $z = 1$)—whereupon, after some simplification, it takes the form of a combination of ${}_3F_2(1)$ series:

$$\begin{aligned}&{}_4F_3\left(\begin{matrix}\frac{3}{2},-\frac{1}{2},-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n\\ \frac{1}{2},1,\frac{1}{2}-n\end{matrix}\,\middle|\,1\right) = \\ &\quad {}_3F_2\left(\begin{matrix}-\frac{1}{2},-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n\\ 1,\frac{1}{2}-n\end{matrix}\,\middle|\,1\right) \\ &\quad + \frac{n(n-1)}{2(2n-1)}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},1-\frac{1}{2}n,\frac{3}{2}-\frac{1}{2}n\\ 2,\frac{3}{2}-n\end{matrix}\,\middle|\,1\right).\end{aligned} \tag{P4}$$

We now apply Result II to the two ${}_3F_2(1)$ series of (P4), both of which require careful treatment. In the first instance, putting $a = -\frac{1}{2}$, $b = -\frac{1}{2}n$, $c = \frac{1}{2} - \frac{1}{2}n$, $d = 1$ and $e = \frac{1}{2} - n + \varepsilon$, it gives

$${}_3F_2\left(\begin{matrix}-\frac{1}{2},-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n\\ 1,\frac{1}{2}-n+\varepsilon\end{matrix}\,\middle|\,1\right) = f(n,\varepsilon)\,{}_3F_2\left(\begin{matrix}\frac{3}{2},-\frac{1}{2}n,\frac{1}{2}-\frac{1}{2}n\\ 1,1-\varepsilon\end{matrix}\,\middle|\,1\right), \tag{P5}$$

where

$$f(n,\varepsilon) = \frac{\Gamma(\varepsilon)\Gamma(\frac{1}{2}-n+\varepsilon)}{\Gamma(\frac{1}{2}-\frac{1}{2}n+\varepsilon)\Gamma(-\frac{1}{2}n+\varepsilon)}. \tag{P6}$$

There are clearly separate cases to consider in employing Result II, depending on whether n is odd or even (which results in either one of b or c being a (strictly) negative integer).

Case A: Suppose n (even, ≥ 2) $= 2m$, where $m = 1, 2, 3, 4, \ldots$, so that $b = -\frac{1}{2}n = -1, -2, -3, -4, \ldots$ (with c rational). Then

$$\begin{aligned} f(n,\varepsilon) &= f(m(n),\varepsilon) \\ &= \frac{\Gamma(\varepsilon)\Gamma(\frac{1}{2}-2m+\varepsilon)}{\Gamma(\frac{1}{2}-m+\varepsilon)\Gamma(-m+\varepsilon)} \\ &= \frac{(1-\varepsilon)_m}{(\frac{1}{2}+m-\varepsilon)_m}, \end{aligned} \tag{P7}$$

again making use of (3),(4). Letting $\varepsilon \to 0$,

$$\begin{aligned} f(n) &= f(m(n)) \\ &= \frac{(1)_m}{(m+\frac{1}{2})_m} \\ &= \frac{(\frac{1}{2}n)!}{(\frac{1}{2}n+\frac{1}{2})_{\frac{1}{2}n}} \\ &= \frac{1\cdot 2\cdot 3\cdot \quad\cdots\quad \cdot\frac{1}{2}n}{(\frac{1}{2}n+\frac{1}{2})(\frac{1}{2}n+\frac{3}{2})(\frac{1}{2}n+\frac{5}{2})\cdots(n-\frac{1}{2})} \\ &= \frac{2\cdot 4\cdot 6\cdot \quad\cdots\quad \cdot n}{(n+1)(n+3)(n+5)\cdots(2n-1)} \\ &= 2^n\binom{2n}{n}^{-1}. \end{aligned} \tag{P8}$$

Case B: Suppose instead n (odd, ≥ 3) $= 2m+1$, where $m = 1, 2, 3, 4, \ldots$, so that $c = \frac{1}{2} - \frac{1}{2}n = -1, -2, -3, -4, \ldots$ (with b rational). Then

$$\begin{aligned} f(n,\varepsilon) &= f(m(n),\varepsilon) \\ &= \frac{\Gamma(\varepsilon)\Gamma(-\frac{1}{2}-2m+\varepsilon)}{\Gamma(-m+\varepsilon)\Gamma(-\frac{1}{2}-m+\varepsilon)} \\ &= \frac{(1-\varepsilon)_m}{(\frac{3}{2}+m-\varepsilon)_m} \end{aligned} \tag{P9}$$

from (3),(4) yet once more. This time in the limit,

$$f(n) = \frac{[\frac{1}{2}(n-1)]!}{(\frac{1}{2}n+1)_{\frac{1}{2}(n-1)}}$$

$$
\begin{aligned}
&= \frac{1\cdot 2\cdot 3\cdot \quad \cdots \quad \cdot\frac{1}{2}(n-1)}{(\frac{1}{2}n+1)(\frac{1}{2}n+2)(\frac{1}{2}n+3)\cdots(n-\frac{1}{2})} \\
&= \frac{2\cdot 4\cdot 6\cdot \quad \cdots \quad \cdot(n-1)}{(n+2)(n+4)(n+6)\cdots(2n-1)} \\
&= 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1}. \qquad \text{(P10)}
\end{aligned}
$$

In both Cases A,B $f(n)$ is, as anticipated, found to be the same expression, and (P5) duly yields

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} -\frac{1}{2}, -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n \\ 1, \frac{1}{2}-n \end{matrix} \,\middle|\, 1\right) \\
&\quad = 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} {}_3F_2\left(\begin{matrix} \frac{3}{2}, -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n \\ 1, 1 \end{matrix} \,\middle|\, 1\right). \qquad \text{(P11)}
\end{aligned}
$$

As stated, Result II is applied also to the second r.h.s. ${}_3F_2(1)$ series of (P4), this time choosing $a = \frac{1}{2}$, $b = 1 - \frac{1}{2}n$, $c = \frac{3}{2} - \frac{1}{2}n$, $d = 2$ and $e = \frac{3}{2} - n + \varepsilon$, whereby we obtain

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}, 1-\frac{1}{2}n, \frac{3}{2}-\frac{1}{2}n \\ 2, \frac{3}{2}-n+\varepsilon \end{matrix} \,\middle|\, 1\right) = g(n,\varepsilon)\, {}_3F_2\left(\begin{matrix} \frac{3}{2}, 1-\frac{1}{2}n, \frac{3}{2}-\frac{1}{2}n \\ 2, 2-\varepsilon \end{matrix} \,\middle|\, 1\right), \qquad \text{(P12)}
$$

with

$$
g(n,\varepsilon) = \frac{\Gamma(-1+\varepsilon)\Gamma(\frac{3}{2}-n+\varepsilon)}{\Gamma(\frac{1}{2}-\frac{1}{2}n+\varepsilon)\Gamma(-\frac{1}{2}n+\varepsilon)}. \qquad \text{(P13)}
$$

Again the parity of n needs consideration, and at this point the assumption that $n \geq 2$ comes into play since $n = 2$ is the smallest value which guarantees one of b or c is, as required, a non-positive integer.

Case A: Here n (even) $= 2m$, where $m = 1, 2, 3, 4, \ldots$ Then $b = 1 - \frac{1}{2}n = 0, -1, -2, -3, \ldots$, and

$$
\begin{aligned}
g(n,\varepsilon) &= g(m(n),\varepsilon) \\
&= \frac{\Gamma(-1+\varepsilon)\Gamma(\frac{3}{2}-2m+\varepsilon)}{\Gamma(\frac{1}{2}-m+\varepsilon)\Gamma(-m+\varepsilon)} \\
&= \frac{\frac{1}{2}-2m+\varepsilon}{-1+\varepsilon}\,\frac{\Gamma(\varepsilon)\Gamma(\frac{1}{2}-2m+\varepsilon)}{\Gamma(\frac{1}{2}-m+\varepsilon)\Gamma(-m+\varepsilon)} \qquad \text{(P14)}
\end{aligned}
$$

by (2). Comparison of $g(n,\varepsilon)$ with $f(n,\varepsilon)$ in the penultimate line of (P7) (see the previous Case A for n even) gives by inspection that, in the limit

$\varepsilon \to 0,$

$$g(n) = \frac{\frac{1}{2} - n}{-1} 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} = \left(n - \frac{1}{2}\right) 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} \tag{P15}$$

directly from (P8).

Case B: Here n (odd) $= 2m + 1$, where $m = 1, 2, 3, 4, \ldots$, so that $c = \frac{3}{2} - \frac{1}{2}n = 0, -1, -2, -3, \ldots$ By a similar process to that in Case A just discussed, a consistent form of $g(n)$ is obtained (this is left as a straightforward reader exercise).

Having established that $g(n)$ is as in (P15) for both n odd/even, (P12) gives

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} \frac{1}{2}, 1 - \frac{1}{2}n, \frac{3}{2} - \frac{1}{2}n \\ 2, \frac{3}{2} - n \end{matrix} \middle| 1\right) \\ &\quad = \left(n - \frac{1}{2}\right) 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} {}_3F_2\left(\begin{matrix} \frac{3}{2}, 1 - \frac{1}{2}n, \frac{3}{2} - \frac{1}{2}n \\ 2, 2 \end{matrix} \middle| 1\right). \end{aligned} \tag{P16}$$

By (P11) and (P16), equation (P4) now reads

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n \\ \frac{1}{2}, 1, \frac{1}{2} - n \end{matrix} \middle| 1\right) \\ &\quad = 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} \left[{}_3F_2\left(\begin{matrix} \frac{3}{2}, -\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n \\ 1, 1 \end{matrix} \middle| 1\right) \right. \\ &\qquad \left. + \frac{n(n-1)}{4} {}_3F_2\left(\begin{matrix} \frac{3}{2}, 1 - \frac{1}{2}n, \frac{3}{2} - \frac{1}{2}n \\ 2, 2 \end{matrix} \middle| 1\right) \right], \end{aligned} \tag{P17}$$

and in turn

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n \\ \frac{1}{2}, 1, \frac{1}{2} - n \end{matrix} \middle| 1\right) \\ &\quad = 2^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} \left[{}_3F_2\left(\begin{matrix} \frac{1}{2}, -\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n \\ 1, 1 \end{matrix} \middle| 1\right) \right. \\ &\qquad \left. + \frac{n(n-1)}{2} {}_3F_2\left(\begin{matrix} \frac{3}{2}, 1 - \frac{1}{2}n, \frac{3}{2} - \frac{1}{2}n \\ 2, 2 \end{matrix} \middle| 1\right) \right] \end{aligned} \tag{P18}$$

on employing Result III (for $p = 2$) to the first ${}_3F_2(1)$ series in the r.h.s. of (P17) as appropriate. Our proof ends with the substitution of the ${}_4F_3(1)$

series above in (P18) back into the r.h.s. of (P3), which becomes, bearing in mind the easily established relations

$$\left(\frac{3}{2}\right)_n = (2n+1)\left(\frac{1}{2}\right)_n, \qquad 2^n \binom{2n}{n}^{-1} = \frac{(2n+1)n!}{2^{n+1}(\frac{1}{2})_{n+1}}, \tag{P19}$$

$$\begin{aligned}
&{}_3F_2\left(\begin{matrix}-n, \frac{1}{2}, -\frac{1}{2}\\ \frac{1}{2}-n, -\frac{1}{2}-n\end{matrix}\,\middle|\, -1\right)\\
&= \frac{n!^2}{2^{n+1}(\frac{1}{2})_n(\frac{1}{2})_{n+1}}\left[{}_3F_2\left(\begin{matrix}\frac{1}{2}, -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n\\ 1,1\end{matrix}\,\middle|\, 1\right)\right.\\
&\qquad\left. + \frac{n(n-1)}{2}\,{}_3F_2\left(\begin{matrix}\frac{3}{2}, 1-\frac{1}{2}n, \frac{3}{2}-\frac{1}{2}n\\ 2,2\end{matrix}\,\middle|\, 1\right)\right]\\
&= \frac{n!^2}{2^{n+1}(\frac{1}{2})_n(\frac{1}{2})_{n+1}}\,{}_4F_3\left(\begin{matrix}\frac{5}{4}, \frac{1}{2}, \frac{1}{2}-\frac{1}{2}n, -\frac{1}{2}n\\ \frac{1}{4}, 1, 1\end{matrix}\,\middle|\, 1\right),
\end{aligned} \tag{P20}$$

contraction of the r.h.s. here being the immediate consequence of a particular instance of Result III (for $p = 3$) with parameters/argument assigned accordingly (we omit the details, which are elementary, as we did in its last application to arrive at (P18)). Equation (P20) is the Theorem.□

Summary

In this paper an unusual hypergeometric identity has been established from first principles. Whilst computer-assisted, it has the advantage over a fully automated proof using the WZ method of Wilf and Zeilberger[2] (which, being but a pure verification, would not be particularly illuminating) in that it puts the result into some kind of perspective relative to others and, as with some classical formulations such as this, there is always the possibility that it might lead to a generalised version of the original identity or even a q-analogue expressed in terms of basic hypergeometric series. In any case, as stated in the Introduction, it is felt that the details of the proof are sufficiently interesting from an analytical point of view to warrant dissemination.

Appendix

Here, for completeness, we derive Result II. It is, as alluded to in the Introduction, but a simple re-write of (3.1.1) in [3, p.59], which with a

[2]See, for instance, Petkovšek, M., Wilf, H.S. and Zeilberger, D. (1996). A=B, A.K. Peters, Wellesley, U.S.A., for more information.

corrected l.h.s. ${}_3F_2$ series argument is, for $n = 0, 1, 2, 3, \ldots$,

$$
{}_3F_2\left(\begin{matrix}-n, a, b\\ c, d\end{matrix}\,\middle|\,1\right) = \frac{(d-b)_n}{(d)_n}\,{}_3F_2\left(\begin{matrix}-n, c-a, b\\ c, 1+b-d-n\end{matrix}\,\middle|\,1\right). \tag{A1}
$$

We write

$$
{}_3F_2\left(\begin{matrix}-n, a, b\\ c, d\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(d-b+n)\Gamma(d)}{\Gamma(d-b)\Gamma(d+n)}\,{}_3F_2\left(\begin{matrix}-n, c-a, b\\ c, 1+b-d-n\end{matrix}\,\middle|\,1\right), \tag{A2}
$$

since

$$
\begin{aligned}
\frac{(d-b)_n}{(d)_n} &= \frac{((d-b+n)-n)_n}{((d+n)-n)_n}\\
&= \frac{\Gamma(d-b+n)\Gamma(d)}{\Gamma(d-b)\Gamma(d+n)}
\end{aligned} \tag{A3}
$$

using (3). Switching variables according to $d \to e$, $c \to d$, $n \to -c$, (A2) becomes

$$
{}_3F_2\left(\begin{matrix}c, a, b\\ d, e\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(e-b-c)\Gamma(e)}{\Gamma(e-b)\Gamma(e-c)}\,{}_3F_2\left(\begin{matrix}c, d-a, b\\ d, 1+b-e+c\end{matrix}\,\middle|\,1\right), \tag{A4}
$$

which now holds for $c = 0, -1, -2, -3, \ldots$ Clearly b and c are interchangeable in (A4), and we have Result II.

References

[1] Jarvis, A.F., Larcombe, P.J. and French, D.R. (2005). Power series identities generated by two recent integer sequences, *Bull. I.C.A.*, **43**, pp.85-95.

[2] Slater, L.J. (1966). Generalized hypergeometric functions, Cambridge University Press, London, U.K.

[3] Gasper, G. and Rahman, M. (1990). Basic hypergeometric series (Encyclopedia of mathematics and its applications, No. 35), Cambridge University Press, Cambridge, U.K.