A game of edge removal on graphs

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Abstract

Two players are presented with a finite, simple graph $G = (V, E)$ that has no isolated vertices. They take turns deleting an edge from the graph in such a way that no isolated vertex is created. The winner is the last player able to remove an edge. We analyze this game when the graph G is a path of arbitrary length. In addition, some observations are made in the situation that the graph has an automorphism of a special type.

1 Introduction

Two players, $\mathcal{A}$ and $\mathcal{B}$, are presented with a finite, simple graph $G = (V, E)$ that has no isolated vertices. They play the "edge-delete game" on G by alternately removing an edge from the graph in such a way that no isolated vertex is created. Player $\mathcal{A}$ begins and the winner is the last player able to remove an edge. We say that the graph G is a *Player $\mathcal{A}$ graph* if there is a strategy that $\mathcal{A}$ can follow that will guarantee that she wins the game. Otherwise, G is said to be a *Player $\mathcal{B}$ graph*.

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Of course, after $\mathcal{A}$ has removed the first edge from G a new graph H has been created. If there is an edge that can legally be removed by Player $\mathcal{B}$, then this can be considered as a new game in which the first player is now $\mathcal{B}$ and the game is being played on H.

To be able to analyze the game played on a particular graph we now formally use the setting of directed graphs. For any undefined terms see Chapter 15 of [6]. A directed graph, or digraph for short, D, consists of a finite set of vertices $V(D)$ and some collection $A(D)$ of ordered pairs of distinct vertices. For $u, v \in V(D)$ such that $(u, v) \in A(D)$ we say that v is a *successor* of u and that u is a *predecessor* of v. In addition, we say the arc (u, v) is directed from u to v. The *outset* of a vertex x is the set $O(x) = \{y \mid (x, y) \in A(D)\}$. The *inset* of x is denoted by $I(x)$ and is defined as $\{y \mid (y, x) \in A(D)\}$. A *sink* of D is a vertex x having an empty outset. A set I of $V(D)$ is *independent* if for all $u, v \in I$, $(u, v) \notin A(D)$. The set I is *absorbant* if for each $u \notin I$ there is an arc (u, v) such that $v \in I$. A set K that is both independent and absorbant is called a *kernel*. A sequence $W : v_1, v_2, \ldots, v_k$ of vertices in D is called a *walk* if $v_i \in I(v_{i+1})$ for each $1 \leq i \leq k-1$. The walk is *closed* if $v_1 = v_k$.

For a given undirected graph $G = (V, E)$ having no isolates, let $\mathcal{D}(G)$ be the directed graph whose vertex set consists of all spanning subgraphs of G that can result from some sequence of legal moves in the edge-delete game played on G. In particular, the vertices correspond to the spanning subgraphs of G having no isolated vertices. For two such subgraphs G_1 and G_2 there is an arc directed from G_1 to G_2 in $\mathcal{D}(G)$ if $G_2 = G_1 - f$ for some edge f of G_1. That is, we have the arc (G_1, G_2) in $\mathcal{D}(G)$ if and only if some legal move in the edge-delete game played on G_1 yields G_2.

Since the number of edges is decreased by one with each move of the game, it is clear that there are no closed walks in $\mathcal{D}(G)$, and each sequence of legal moves is finite. A complete game corresponds to a directed path in $\mathcal{D}(G)$, starting at G and ending at a sink. A player thus wins the game by moving to a sink of $\mathcal{D}(G)$.

A vertex H of $\mathcal{D}(G)$ is considered to be a *position* or a *state* of the game played on G, but $\mathcal{D}(H)$ is also an induced subdigraph of $\mathcal{D}(G)$. Hence we may consider H as a position in the original game or as the starting position in the game played on H. In fact, H could have k connected components $C_1, C_2, \ldots, C_k$ in which case it is consistent to consider H as a vertex in the directed graph arising from the game sum of k edge-delete games. (For the formal definition of the sum of games see the chapter by A. S. Fraenkel in [2].) In either case H is called an *N-position* (i.e., next) if the player starting the edge-delete game on H has a winning strategy. Otherwise, H

is called a *P-position* (i.e., previous). More precisely, H is a P-position if for every edge $e \in E(H)$ such that $H - e$ has no isolated vertices, the graph $H - e$ is an N-position. If it is the case that $H - e$ has an isolated vertex for every edge e of H, then the game has been won by the player who removed an edge from some graph to leave H, and so H is a P-position. Note that in this latter case the vertex H is a sink of $\mathcal{D}(G)$ and of $\mathcal{D}(H)$.

For a given undirected graph G we partition the vertex set of the directed graph $\mathcal{D}(G)$ as $\mathcal{P} \cup \mathcal{N}$ where

- $\mathcal{P}$ is the set of all P-positions, and
- $\mathcal{N}$ is the set of all N-positions.

Here $\mathcal{P}$ and $\mathcal{N}$ suggestively stand for "previous" and "next". Note that every sink of $\mathcal{D}(G)$ belongs to $\mathcal{P}$. Suppose now that a player must make a move on a graph H. If $H \in \mathcal{N}$, then by definition there exists an edge e of H such that $H - e \in \mathcal{P}$. On the other hand, if $H \in \mathcal{P}$, then for every arc (H, F) in $\mathcal{D}(G)$, $F \in \mathcal{N}$. Thus, if $G \in \mathcal{N}$, then there is a strategy that $\mathcal{A}$ can follow to guarantee herself a win. But if $G \in \mathcal{P}$, then $\mathcal{B}$ can follow a strategy that guarantees him a win.

It follows from the definitions that $\mathcal{P}$ is a kernel of the digraph $\mathcal{D}(G)$, and since $\mathcal{D}(G)$ has no circuits the following result of Von Neumann [7] implies the partition is unique.

Theorem 1 *A digraph without circuits possesses a unique kernel.*

Of course, the difficulty is in finding the partition $\mathcal{P} \cup \mathcal{N}$ efficiently. In Section 2 we determine those values of k for which the path of order k belongs to $\mathcal{N}$. In Section 3 we consider the more general case and find a number of classes of graphs on which $\mathcal{B}$ has a winning strategy.

2 Paths

Throughout this section we let P_n denote the path of order n having vertices labelled $1, 2, \ldots, n$ in the natural order. Each vertex of $\mathcal{D}(P_n)$ is a collection of paths each of order at least two. In addition, the orders of these subpaths of P_n add to n. For simplicity we denote by $(p_1, p_2, \ldots, p_k)$ the vertex of $\mathcal{D}(P_n)$ that has k components having orders $p_1, p_2, \ldots, p_k$. Note that we are allowing orders to be repeated. For example, $(2, 3, 3, 6)$ is a vertex of $\mathcal{D}(P_{14})$

and this vertex has two successors, namely $(2,3,3,2,4)$ and $(2,3,3,3,3)$, which has no successors. The only successor of $(2,3,3,2,4)$ is $(2,3,3,2,2,2)$, which is a sink. Thus, $(2,3,3,2,4) \in \mathcal{N}$. There are several sequences of moves that give rise to the position $(2,3,3,6)$. How does one determine whether the original P_{14} is in $\mathcal{P}$ or in $\mathcal{N}$?

We define a function g, the so-called Grundy function, that assigns a non-negative integer to each vertex of $\mathcal{D}(P_n)$ in such a way that $\mathcal{P} = \{x \mid g(x) = 0\}$. This method was introduced in 1939 by P. M. Grundy [1]. First we partition the vertex set of $\mathcal{D}(P_n)$ into levels as follows. Let $\mathcal{L}_0$ be the set of sinks and let $\mathcal{L}_1$ be the set of vertices of $\mathcal{D}(P_n)$ all of whose successors belong to $\mathcal{L}_0$. For $k > 1$, let $\mathcal{L}_k$ denote the set of vertices $x \notin \cup_{i<k}\mathcal{L}_i$ such that $O(x) \cap \mathcal{L}_{k-1} \neq \emptyset$ and $O(x) \subseteq \cup_{i<k}\mathcal{L}_i$. For $x \in \mathcal{L}_0$ let $g(x) = 0$. If $x \in \mathcal{L}_k$ for $k \geq 1$ let $g(x)$ be the smallest non-negative integer that does not belong to the set $\{g(y) \mid y \in O(x)\}$.

This implies that $\mathcal{P} = \{x \in V(\mathcal{D}(P_n)) \mid g(x) = 0\}$ and $\mathcal{N} = \{x \in V(\mathcal{D}(P_n)) \mid g(x) > 0\}$. To see this we show that $K = \{x \in V(\mathcal{D}(P_n)) \mid g(x) = 0\}$ is a kernel in $\mathcal{D}(P_n)$. If $u, v \in K$ then $g(u) = 0 = g(v)$ and so by the definition of g, u is not a successor of v nor is v a successor of u. Also, if $w \notin K$, then $g(w) > 0$ and so by definition w has a successor x such that $g(x) = 0$. Hence, K is independent and absorbant. That is, K is a kernel of $\mathcal{D}(P_n)$. But by Theorem 1 the kernel of $\mathcal{D}(P_n)$ is unique and so $\mathcal{P} = \{x \in V(\mathcal{D}(P_n)) \mid g(x) = 0\}$.

We now appeal to the method presented by A. Fraenkel in [2]. See especially pages 117–120. For two non-negative integers r and s we denote by $r \oplus s$ the *Nim-sum* of r and s, which is computed as follows. Suppose $r = \sum_{i=0}^{k} r_i 2^i$ where each r_i is non-negative; similarly, $s = \sum_{i=0}^{k} s_i 2^i$. Then $r \oplus s$ is $\sum_{i=0}^{k} t_i 2^i$, where for each i, $t_i = r_i + s_i \in \{0,1\}$ computed in Z_2.

If F_1 and F_2 are games that both belong to $\mathcal{P}$ or if one belongs to $\mathcal{P}$ and the other is in $\mathcal{N}$, we can determine where their game sum, denoted $F_1 \cup F_2$, lies by using the Grundy function defined on each.

Lemma 2 *Let F_1 and F_2 be games. Then*

(i) If $F_1, F_2 \in \mathcal{P}$, then the game sum $F_1 \cup F_2$ is also in $\mathcal{P}$.

(ii) If $F_1 \in \mathcal{P}$ and $F_2 \in \mathcal{N}$, then $F_1 \cup F_2 \in \mathcal{N}$.

Proof To prove (i) let $F_1, F_2 \in \mathcal{P}$. Then, $g(F_1) = 0 = g(F_2)$, and every successor of F_1 and every successor of F_2 has a positive Grundy value. For any successor F_1' of F_1 and any successor F_2' of F_2 it follows that

$g(F_1') \oplus g(F_2) > 0$ and $g(F_1) \oplus g(F_2') > 0$. Therefore, $g(F_1 \cup F_2)$, being the smallest non-negative value not assumed by g for all possible successors of $F_1 \cup F_2$, is 0. This implies that $F_1 \cup F_2 \in \mathcal{P}$.

For (ii), since $F_2 \in \mathcal{N}$ let $F_2' \in \mathcal{P}$ be a successor of F_1. Then $F_1 \cup F_2'$ is a successor of $F_1 \cup F_2$ and belongs to $\mathcal{P}$ by (i). Therefore, in this case $F_1 \cup F_2$ has a successor that is in $\mathcal{P}$, and so $F_1 \cup F_2 \in \mathcal{N}$. □

To compute the Grundy value, $g(n)$, for the path P_n we first note that the set of successors in $\mathcal{D}(P_n)$ is $\{(2, n-2), (3, n-3), \ldots, (n-2, 2)\}$. As in [2] the Grundy value $g(k, n-k)$ is $g(k) \oplus g(n-k)$ and hence $g(n)$ is the smallest non-negative integer not in the set

$$\{g(2) \oplus g(n-2), g(3) \oplus g(n-3), \ldots, g(n-2) \oplus g(2)\}.$$

To be able to find the partition of $\mathcal{D}(P_n)$ as $\mathcal{P} \cup \mathcal{N}$ for a given n we will need to know all of the above Grundy values. One step in that process is the following lemma. Note that $(m, m+34)$ is a position in the game played on P_{2m+34}, but it could also be two of some number of components that arise in the course of a game played on a longer path. The meaning should be clear from context.

Lemma 3 *For every positive integer $m \geq 56$, the position $(m, m+34)$ is in $\mathcal{P}$.*

Proof The calculation of Nim-sums is tedious but straightforward. A computer program has verified that $(m, m+34) \in \mathcal{P}$ for $56 \leq m \leq 145$. Proceeding by induction we let $M \geq 146$ and assume that for all m, such that $56 \leq m < M$ the position $(m, m+34)$ belongs to $\mathcal{P}$. Let $x = (M, M+34)$ and let y be any successor of x.

First suppose that $y = (a, b, M+34)$ where $M = a + b$ and $b \geq a$. Let $z = (a, b, a, b+34) \in O(y)$. By Lemma 2 $(a, a) \in \mathcal{P}$. Since $b \geq a$, it follows that $b \geq \frac{M}{2} \geq 56$. Therefore, by the induction hypothesis $(b, b+34)$ also belongs to $\mathcal{P}$. Thus, z is the sum of two games in $\mathcal{P}$and hence $z \in \mathcal{P}$. As z, a successor of y, is in $\mathcal{P}$, we have $y \in \mathcal{N}$.

Next, suppose that $y = (M, c, d)$ where $c + d = M + 34$ and $c \leq d$. Since $c \leq M - 2$ the position $z = (c, M - c, c, d)$ is a successor of y. But $M - c = d - 34$ and hence $z = (c, c, d-34, (d-34)+34)$. Now, $d \geq \frac{M+34}{2} \geq 90$ and so $d - 34 \geq 56$. By induction $(d-34, d) \in \mathcal{P}$ and hence z, being the sum of two games in $\mathcal{P}$, is also in $\mathcal{P}$. As z, a successor of y, is in $\mathcal{P}$, we have $y \in \mathcal{N}$.

Since we have shown that every successor y of x is in $\mathcal{N}$, we have that $x = (M, M + 34) \in \mathcal{P}$. □

Now we can complete the analysis of the edge delete game on paths. Using a computer program that computes Grundy numbers we have shown that for paths of order n less than 90, the edge delete game on P_n can be won by $\mathcal{A}$ unless $n \in \{2, 3, 7, 11, 17, 23, 27, 31, 37, 41, 45, 57, 61, 65, 75, 79\}$. The proof of the following theorem now follows immediately by using Lemmas 2 and 3.

Theorem 4 *The edge delete game on the path of order n can be won by $\mathcal{B}$ if and only if n belongs to one of the following sets:*

- $\{2, 3, 7, 11, 17, 23, 27, 31, 37\}$
- $\bigcup_{k \geq 0} \{41 + 34k, 45 + 34k, 57 + 34k, 61 + 34k, 65 + 34k\}$.

3 Graphs with symmetry

Many other graphs for which it is easy to see a winning strategy fit into the following general framework.

Theorem 5 *Suppose $G = (V, E)$ has an automorphism $f : V \to V$ with the properties: (1) f has no fixed points; (2) $f(f(x)) = x$ for every vertex x in V; and (3) x is not adjacent to $f(x)$ for any x in V. Then $\mathcal{B}$ has a winning strategy on G.*

Proof Given the structure of the graph, we observe that whenever $\mathcal{A}$ deletes an edge xy, $\mathcal{B}$ can respond by deleting the edge $f(x)f(y)$. After each pair of moves (in other words, after $\mathcal{B}$ has played each time), f is still an automorphism of the resulting graph. So, if $\mathcal{A}$ removing an edge is a legal move (i.e., no vertex is isolated), then $\mathcal{B}$'s move is also legal. □

Many situations fit into this "mirror type" situation. For example,

(i) Even cycles.

(ii) Any graph which is the disjoint union of 2 copies of the same graph (or any even number of copies of the same graph for that matter).

(iii) Any complete bipartite graph of the form $K_{2t,2t}$ (the automorphism interchanges pairs within the same color class).

(iv) A complete graph of even order which has had a perfect matching removed (the automorphism interchanges the vertices at the ends of removed edges in the perfect matching).

(v) Hypercubes (the automorphism is the function that interchanges 0's and 1's in the representation of the hypercube of dimension n as a bit string of length n in which two bit strings are adjacent if and only if they differ in exactly one coordinate).

(vi) Any even by even grid graph (for instance, for the $2n$ by $2m$ grid, vertex (i, j) corresponds to vertex $(2n + 1 - i, 2m + 1 - j)$).

Note that any even by odd grid, say $2n$ by $2m + 1$, is an $\mathcal{A}$ graph, since $\mathcal{A}$ could remove the edge joining vertex $(n, m + 1)$ to vertex $(n + 1, m + 1)$ creating a graph as described in Theorem 5 where vertex (i, j) corresponds to vertex $(2n + 1 - i, 2m + 2 - j)$) and thus a $\mathcal{B}$ winning situation. We also observe that $\mathcal{A}$ graphs can be built by taking two vertex-disjoint $\mathcal{B}$ graphs, say G and H, and joining a vertex u in G with a vertex v in H by an edge.

4 Related Problems

The edge delete game actually first arose as a special case of the following two person game [3] played on a graph G, where E is the number of edges. The set $\{1, 2, \ldots, E\}$ will be the set of labels, each of which can be used at most once. Each edge can be assigned at most one label. The players alternate assigning an unused label from $\{1, 2, \ldots, E\}$ to an unlabeled edge. For a given vertex x, let $S(x)$ be the set consisting of all the edges meeting x. If not all elements of $S(x)$ have been assigned a label yet, we call the *partial weight* of x the sum of the labels of those elements of $S(x)$ that have a label. Once all of $S(x)$ has a label we call the *weight* of the vertex x the sum of the labels assigned to all of its incident edges. The first time some vertex has all incident edges labeled, the sum of those labels is called the *magic constant, k*. If the first time this occurs it involves two vertices (by labeling the edge joining them), both weights must be the same. Once the magic constant has been set, all other weights must be equal to this magic constant. If this cannot be accomplished at some vertex, then one of the incident edges must remain unlabeled. Note that as long as an incident edge remains unlabeled, the partial weight may actually exceed k.

The last player able to make a legal move wins. Thus, in general, the game will be over before all the edges have been labeled. Now observe that if the first player uses the label 1 on an edge to a leaf on his/her first move, the constant k is set to 1 and this cannot be achieved at any other vertex. Hence the players can assign any label to edges but must always leave some edge unlabeled at every vertex. In essence, this is simply the edge delete game. Thus the first player would make such a move on her first play if she could win the edge delete game on the rest of the graph.

The reader is also referred to [4] for a collection of graphs with the very special property that regardless of how the players move the outcome is determined. For example, consider an arbitrary graph which has an odd number of edges. If one attaches a leaf to every vertex of this graph to form a new graph H, then no matter how the players move on H, $\mathcal{A}$ will always win.

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