Constructing Edge-labellings of K_n with Constant-length Hamilton Cycles

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Abstract

We present an optimal algorithm to label the edges of a complete graph with integer lengths so that every Hamilton cycle has the same length. The algorithm is complete in the sense that every edge-labelling with this property is the output labelling of some run of this algorithm. Such edge-labellings are induced by half-integer vertex-labellings by adding the vertex labels on an edge's ends to determine its label. The Fibonacci sequence arises in this connection.

1 Introduction

In [4], we introduced the notion of a *trivial-TSP* edge-labelling of a complete graph K_n. This is a function $\lambda\colon E(K_n) \to \mathbb{Z}$ for which the sum $\sum_{A \in E(H)} \lambda(A)$ is the same for each Hamilton cycle H of K_n. The nomenclature derives from the observation that such a labelling corresponds to a trivial instance of the Travelling Salesman Problem, in the sense that every TSP-tour has the same length (see [8] for background on the TSP). The stature of the TSP as a centrally important problem in combinatorial optimization and theoretical computer science (see, e.g., [1]) suggests that

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new insights even into its special cases become welcome additions to the mathematical literature.

A key result of [4] illuminates a fundamental connection between trivial-TSP edge-labellings and certain vertex-labellings:

Theorem 1 *For $n \geq 3$, an edge-labelling $\lambda: E(K_n) \to \mathbb{Z}$ is trivial-TSP if and only if there is a vertex-labelling $\nu: V(K_n) \to \frac{1}{2}\mathbb{Z}$ such that*

$$\lambda(ij) = \nu_i + \nu_j \text{ for each edge } ij \text{ of } K_n. \tag{1}$$

The sequence $(\nu_i)_{i=1}^{n}$ is uniquely determined by λ.

An essential ingredient in our proof [4] of Theorem 1 is a characterization of the trivial-TSP edge-labellings in terms of a local condition on the edge-labels, called the C_4-matching property; see Theorem 2 below. One of our purposes here is to illustrate further the strength of this characterization by demonstrating how far we can proceed using the C_4-matching property as our primary tool and staying strictly within the realm of edge-labelling. In Section 2, we present an optimal algorithm for producing a trivial-TSP edge-labelling of K_n without resorting to Theorem 1. The algorithm is complete in the sense that every trivial-TSP edge-labelling is a possible output labelling of some run of this algorithm. We shall see in Section 3 that the Fibonacci sequence arises naturally through the connection (1). Now we pause to discuss the necessary background material and related definitions.

Background

Since our graph-theoretic notation is fairly standard, we refer the reader to any basic text—e.g. [13]—for omitted definitions. We use $[n] := \{1, \ldots, n\}$ for the vertex set of K_n. If an edge A has ends i, j, then we write $A = ij$ or $A = \{i, j\}$. A cycle visiting the vertices $v_1, v_2, \ldots, v_r$ in this order and then returning to v_1 is denoted by $(v_1, v_2, \ldots, v_r)$. If two graphs G, H are isomorphic, then we write $G \cong H$.

An *edge-labelling* (resp. *vertex-labelling*) of K_n is a function $\lambda: E(K_n) \to S$ (resp. $\nu: [n] \to S$) into some set S of *labels*. For edges, we use the label sets $S = \mathbb{Z}$ and $\mathbb{Z}^+$ (of integers and positive integers); for vertices, influenced by Theorem 1, we use $S = \frac{1}{2}\mathbb{Z}$ and $\frac{1}{2}\mathbb{N}$ (resp. half-integers and half-nonnegative integers). If λ is an edge-labelling and $A \in E(K_n)$, then $\lambda(A)$ is called the *label* or *length* of A. We use analogous terminology for vertex-labellings ν, but shall denote the label of a vertex i by ν_i. If λ and ν satisfy (1), then we say that λ is *induced* from ν (via (1)). When a pair of edge- and vertex-labellings are linked by a relation such as (1), we enter the domain of graph labelling, a subject that enjoys an extensive literature; see [3] for a still-evolving survey.

An edge-labelling $\lambda: E(K_n) \to \mathbb{Z}^+$ is *metric* if it satisfies the triangle-inequality: $\lambda(ik) \leq \lambda(ij) + \lambda(jk)$ for every triple $i, j, k \in [n]$. We call λ *trivial-MTSP* if it is both metric and trivial-TSP. Notice that the sequence $(\nu_i)_{i=1}^n$ in Theorem 1 is nonnegative if and only if the induced edge-labelling λ is metric, since, for any three vertices i, j, k, we have

$$\lambda(ik) \leq \lambda(ij) + \lambda(jk) \Leftrightarrow \nu_j \geq 0.$$

A labelling $\lambda: E(K_n) \to \mathbb{Z}$ has the *C_4-matching property*—abbreviated by C_4-MP—if, for each 4-cycle in K_n, say with consecutive edges A, B, C, D, we have $\lambda(A)+\lambda(C) = \lambda(B)+\lambda(D)$. It was surprising—at least to these authors—that the trivial-TSP edge-labellings of K_n can be recognized by verifying this local condition only:

Theorem 2 ([4]) *An edge-labelling of K_n is trivial-TSP if and only if it satisfies the C_4-matching property.*

In [4], we established a more extensive set of equivalent conditions for an edge-labelling of K_n to be trivial-TSP, but these are not essential for the present paper.

Any constant function on $E(K_n)$ provides a simple (indeed, trivial!) example of a trivial-TSP edge-labelling. One way to avoid this triviality is to consider only those λ which are injective, as in Fig. 1. The sequence $(\nu_i)_{i=1}^n$ of vertex labels inducing such an edge-labelling λ has the property that the sums $\nu_i + \nu_j$, for $i \neq j$, are all different. Following Kotzig [7] (see also [10]), we call such a sequence *well-spread*, though the term *weakly Sidon* has also been used; see [12] for a related survey. In this language, if λ is induced by ν, then $(\nu_i)_{i=1}^n$ is well-spread if and only if λ is injective.

Of course, Theorem 1 suggests a simple algorithm to construct a trivial-TSP edge-labelling of K_n: just begin with the vertex labels $(\nu_i)_{i=1}^n$ and apply (1). If an injective edge-labelling λ is desired, then one needs to ensure that $(\nu_i)_{i=1}^n$ is well-spread. To achieve this property, even a recursive approach requires ensuring that a newly added vertex label ν_{n+1} does not introduce any violations of the well-spread property. Besides showcasing the C_4-MP, another advantage of our approach in Section 2 is that the condition to ensure an injective λ (see Step 2 in Algorithm 4) is simpler than verifying well-spreadedness. We revisit well-spread sequences of nonnegative integers in Section 3, after focusing exclusively on edge-labellings in the following section.

2 The construction

At the heart of our algorithm is the following lemma, which asserts a property of trivial-TSP edge-labellings of interest in its own right.

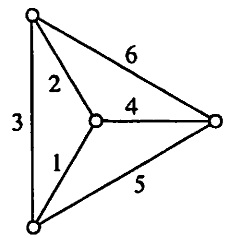

Figure 1: An injective trivial-MTSP edge-labelling of K_4

Lemma 3 *Suppose* $\lambda: E(K_n) \to \mathbb{Z}$ *satisfies the* C_4*-matching property. Then there is a vertex* v_0 *such that, for every* $v \in [n]$*, some minimum-length edge* A_v *incident with* v *is also incident with* v_0*; that is, if* $v \neq v_0$*, then we may choose* $A_v = \{v, v_0\}$*. If* λ *is injective, then* v_0 *and each* A_v *are uniquely determined.*

Proof. As all the conclusions are trivial when $1 \leq n \leq 3$, we may assume that $n \geq 4$.

First we argue that there are two shortest-length edges that are incident. Consider two such edges A, B. If they have different lengths, let us suppose that $\lambda(A) < \lambda(B) \leq \lambda(C)$ for edges $C \neq A, B$. In this case, we may conclude immediately that A and B are incident (otherwise, let C, D be edges completing a 4-cycle with A, B; then $\lambda(A)+\lambda(B) < \lambda(C)+\lambda(D)$, but this violates the C_4-MP). Now suppose that $\lambda(A) = \lambda(B) \leq \lambda(C)$ for all edges $C \neq A, B$. If A and B fail to be incident, then again let C, D be edges completing a 4-cycle with A, B. Then $\lambda(A) + \lambda(B) \leq \lambda(C) + \lambda(D)$, and the C_4-MP forces equality; this implies that $\lambda(A) = \lambda(B) = \lambda(C) = \lambda(D)$. Now A and C are incident minimum-length edges.

To fix notation, now we write A, B for two incident shortest-length edges with $\lambda(A) \leq \lambda(B) \leq \lambda(C)$ for edges $C \neq A, B$. Let v_0 be the common endpoint of A, B, so that $A = \{v_0, v_1\}$ and $B = \{v_0, v_2\}$ for some distinct vertices $v_1, v_2 \neq v_0$. Notice that v_0 is uniquely determined in case $\lambda(A) < \lambda(B)$, in particular, in case λ is injective. We claim that v_0 has the desired property.

Since $A_{v_0} = A_{v_1} = A$ and $A_{v_2} = B$ are suitable choices, and these edges are both incident with v_0, it remains to establish that

$$A_v = \{v, v_0\} \text{ for all } v \in [n] \setminus \{v_0, v_1, v_2\} \tag{2}$$

defines a set of legal choices. This we verify by arguing that in each of the other three possibilities for a legal choice—$A_v = \{v, v_1\}$, $A_v = \{v, v_2\}$ and $A_v = \{v, x\}$ for a vertex $x \neq v_0, v_1, v_2, v$—the assignment (2) also specifies an appropriate choice.

Case 1: $A_v = \{v, v_1\}$.
Consider the 4-cycle (v_0, v_2, v_1, v). Since the C_4-MP gives $\lambda(A_v) + \lambda(B) = \lambda(vv_0) + \lambda(v_1v_2)$, and we have $\lambda(A_v) \leq \lambda(vv_0)$ and $\lambda(B) \leq \lambda(v_1v_2)$, if either of these inequalities were strict, we would obtain a contradiction. Thus, $\lambda(A_v) = \lambda(vv_0)$, and we see that $A_v = \{v, v_0\}$ is also a legal choice.

Case 2: $A_v = \{v, v_2\}$.
Now consider the 4-cycle (v_0, v_1, v, v_2). Here, the C_4-MP yields $\lambda(A) + \lambda(A_v) = \lambda(B) + \lambda(vv_1)$. Since $\lambda(A) \leq \lambda(B)$ and $\lambda(A_v) \leq \lambda(vv_1)$, strictness of either of these inequalities again leads to a contradiction. Now $\lambda(A_v) = \lambda(vv_1)$ means that $\{v, v_1\}$ is a minimum-length edge incident with v, so by Case 1, another legal choice is $A_v = \{v, v_0\}$.

Case 3: $A_v = \{v, x\}$ for some $x \in [n] \setminus \{v_0, v_1, v_2, v\}$.
Finally consider the 4-cycle (v_0, v_1, x, v). Now the C_4-MP implies that $\lambda(A) + \lambda(A_v) = \lambda(v_1x) + \lambda(vv_0)$, and since $\lambda(A) \leq \lambda(v_1x)$ and $\lambda(A_v) \leq \lambda(vv_0)$, in fact we have $\lambda(A_v) = \lambda(vv_0)$. Thus we could legally redefine $A_v = \{v, v_0\}$.

In each case, we see that $\{v, v_0\}$ is a minimum-length edge incident with v, as claimed. Of course, this edge is uniquely determined when all the edge labels are distinct. ∎

Remark As suggested in the introduction, the spirit of the present section is to constrain ourselves to edge-labelling without reference to vertex labels. There is a sense in which this is akin to fighting with one hand (viz. Theorem 1) tied behind our backs. With this hand available, Lemma 3 becomes almost transparent: the vertex v_0 is simply one minimizing its ν-label. If λ is injective, then so is ν, whence v_0 is unique.

Algorithm 4 `triv_TSP_label`(n): *to compute an edge-labelling* λ *of* K_n, $n \geq 3$, *with the* C_4*-matching property and, if desired, distinct edge labels.*

Basis $(n = 3)$

if $n = 3$ let $\lambda: E(K_3) \to \mathbb{Z}$ be any edge-labelling, an injection if distinct labels are desired.
[Any such λ vacuously satisfies the C_4-MP.]

`return`(λ)

Recursive step $(n \geq 4)$

else

1. set $\lambda :=$ `triv_TSP_label`$(n-1)$.
[Then $\lambda: E(K_{n-1}) \to \mathbb{Z}$ is an (injective if desired) edge-labelling satisfying the C_4-MP; Steps 2,3 extend λ to $E(K_n)$ while preserving the C_4-MP and label distinctness, if desired.]

2. let $\Lambda \in \mathbb{Z}$; choosing Λ to exceed $\max_{A \in E(K_{n-1})} \lambda(A)$ is sufficient to ensure distinct edge labels.

3. let the new vertex be n and the vertex v_0 of Lemma 3 be 1. For each new edge $\{i, n\}$, $1 \leq i \leq n-1$, choose a vertex $j \neq 1, i, n$; assign length $\lambda(in) := \Lambda + (\lambda(ij) - \lambda(1j))$.
[Now $\lambda: E(K_n) \to \mathbb{Z}$ is extended.]

`return`(λ)

Fig. 2 depicts two early steps of Algorithm 4; notice that the output labelling agrees with the one in Fig. 1.

Theorem 5 *Algorithm 4 is correct; i.e., for* $n \geq 3$, `triv_TSP_label`(n) *returns an (injective if desired) edge-labelling* $\lambda: E(K_n) \to \mathbb{Z}$ *satisfying the* C_4*-MP.*

Proof. By induction on n. The basis being trivial, let us fix $n > 3$ and suppose the algorithm produces (in Step 1) an edge-labelling $\lambda: E(K_{n-1}) \to \mathbb{Z}$ satisfying the C_4-MP. Suppose further that when the distinctness instructions are followed, an injective λ results. We need to show that the recursive step (in Steps 2,3) preserves the C_4-MP and, if instructed, maintains distinct labels.

Claim 1: the C_4-MP is preserved.

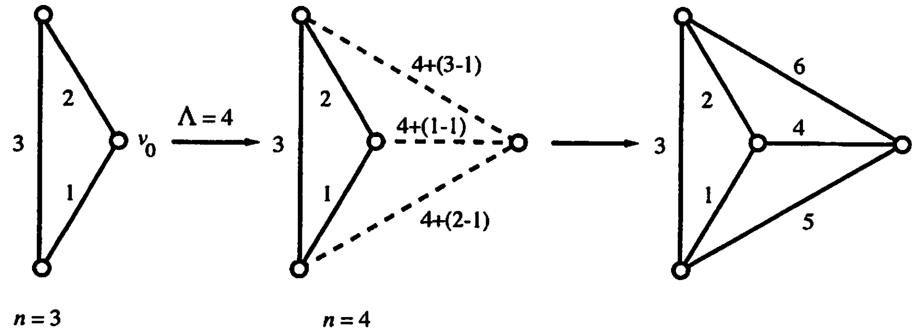

Figure 2: The basis and first recursive step of Algorithm 4

Proof of claim. The inductive hypothesis guarantees that no 4-cycle with vertices in $[n-1]$ can violate the C_4-MP. So consider a 4-cycle $C = (n, i_1, k, i_2)$, which traverses two edges through vertex n: $\{i_1, n\}$ and $\{i_2, n\}$. If j_1, j_2 respectively denote the vertices j chosen in Step 3 for the new length assignments $\lambda(i_1n)$, $\lambda(i_2n)$ to these edges, then

$$j_1 \notin \{1, i_1, n\} \text{ and } j_2 \notin \{1, i_2, n\}. \tag{3}$$

To show that C does not violate the C_4-MP, we need to establish that

$$\lambda(i_1n) + \lambda(i_2k) = \lambda(i_2n) + \lambda(i_1k),$$

or, using Step 3, that

$$\lambda(i_1j_1) + \lambda(1j_2) + \lambda(i_2k) = \lambda(i_2j_2) + \lambda(1j_1) + \lambda(i_1k). \tag{4}$$

That we have (3) and

$$|\{n, i_1, k, i_2\}| = 4 \tag{5}$$

(C is a 4-cycle) leaves open six cases depending on which, if any, of the seven vertex labels 1, n, k, i_1, i_2, j_1, j_2 name the same vertex.

First consider the case when these are distinct vertices, with the possible exception that k might be 1. The C_4-MP applied to the 4-cycles $(1, j_1, i_1, j_2)$ and (i_1, j_2, i_2, k)—both in K_{n-1}—implies in turn that the left side of (4) is

$$\begin{aligned} \lambda(i_1j_1) + \lambda(1j_2) + \lambda(i_2k) &= \lambda(1j_1) + \lambda(i_1j_2) + \lambda(i_2k) \\ &= \lambda(1j_1) + \lambda(i_1k) + \lambda(i_2j_2), \end{aligned}$$

the right side of (4). Thus in this case, the C_4-MP is not violated by C.

The common pattern in verifying the remaining cases is that there is always at least one 4-cycle in K_{n-1} that forces (4) for C by the inductive truth of the C_4-MP. Since these cases all yield to similar reasoning, we omit the details in favour of Table 1 describing the cases along with the 4-cycles in K_{n-1} essential for deduction of (4).

Case	Subcase	4-cycle(s) in K_{n-1} implying (4) for C (or other justification)
$j_1 = j_2$ (includes when one of i_1, i_2 is 1; say $i_2 = 1$ in this event)	$j_1 = j_2 = k$ $j_1 = j_2 \neq k$	(4) holds trivially (i_1, j_1, i_2, k)
$j_1 \neq j_2$; $i_1 \neq 1 \neq i_2$; one of j_1, j_2 is k; wlog $j_2 = k$, $j_1 \neq k$		$(1, j_1, i_1, j_2)$
$j_1 \neq j_2$; $i_1 \neq 1 \neq i_2$; $i_2 = j_1 \neq k \neq j_2 = i_1$	$k = 1$ $k \neq 1$	(4) holds trivially $(1, i_1, k, i_2)$
$j_1 \neq j_2$; $i_1 \neq 1 \neq i_2$; $j_1 \neq k \neq j_2$; at least one j is not the other i; wlog $j_2 \neq i_1$		$(1, j_1, i_1, j_2)$, (i_1, j_2, i_2, k) (Note this includes the case covered in detail.)

Table 1: Exhausting the cases $4 \leq |\{1, n, k, i_1, i_2, j_1, j_2\}| \leq 7$, subject to (3), (5)

Since in each case we find that C satisfies (4), the claim is proved. ∎

Claim 2: the recursive step maintains distinct labels provided Step 2 chooses Λ to exceed $\max_{A \in E(K_{n-1})} \lambda(A)$.

Proof of claim. We need to check that the labels $\lambda(in)$, $1 \leq i \leq n-1$, are different from each other and from the edge labels on K_{n-1}.

If $\lambda(i_1 n) = \lambda(i_2 n)$ for some indices $1 \leq i_1 \neq i_2 \leq n-1$, pick $k \in [n-1] \setminus \{i_1, i_2\}$ and consider the 4-cycle (n, i_1, k, i_2). The C_4-MP—already verified for λ on all of K_n—ensures that

$$\lambda(i_1 n) + \lambda(i_2 k) = \lambda(i_2 n) + \lambda(i_1 k),$$

so we obtain $\lambda(i_2 k) = \lambda(i_1 k)$. But these are edge-labels of K_{n-1} and hence, by the inductive hypothesis, are distinct. The contradiction shows that the new labels are distinct from one another.

It remains to see why no new label $\lambda(in) = \Lambda + (\lambda(ij) - \lambda(1j))$ coincides with an edge-label of K_{n-1}. Since $v_0 = 1$ (Step 3), Lemma 3 yields $A_j = \{j, 1\}$, whence $\lambda(1j) \leq \lambda(ij)$. By Step 2, now $\lambda(in) \geq \Lambda > \max_{A \in E(K_{n-1})} \lambda(A)$, which shows that $\lambda(in)$ is distinct from any edge-label of K_{n-1}. ∎

With Claims 1 and 2 established, the proof is complete. ∎

It is sometimes desirable—e.g. when we consider metric labellings—to restrict the set of edge labels to the positive integers. After a simple modification, Algorithm 4 produces a trivial-TSP edge-labelling $\lambda\colon E(K_n) \to \mathbb{Z}^+$:

Theorem 6 *If each occurrence of* $\mathbb{Z}$ *in* `triv_TSP_label(n)` *is replaced by* $\mathbb{Z}^+$*, then the modified algorithm produces an (injective if desired) trivial-TSP edge-labelling with positive labels.*

Proof. Since $\mathbb{Z}^+ \subseteq \mathbb{Z}$, Theorem 2 and the proof of Theorem 5 show that the new algorithm returns an (injective if desired) trivial-TSP edge-labelling. So it remains only to verify the positivity of the labels. Again we use induction and leave the basis for the reader. For $n > 3$, let us suppose that the new algorithm produces (in Step 1) a positive integer edge-labelling of K_{n-1}. To see that the recursive step maintains positive labels, recall from the preceding proof that $\lambda(in) \geq \Lambda$ for each $i \in [n-1]$. Since the modified algorithm ensures (in Step 2) that $\Lambda > 0$, we have $\lambda(in) > 0$. ∎

A moment's reflection shows that in proving Theorem 5, we actually proved a little more, namely that Algorithm 4 is *complete.* That is, *every* trivial-TSP labelling of $E(K_n)$ can arise via a call to `triv_TSP_label(n)`. Theorem 2 asserts that if we want to generate such a labelling, we must arrange for the C_4-MP to hold. This implies that relative to the label $\Lambda = \lambda(1n)$, the labels $\lambda(in)$, for $2 \leq i \leq n-1$, must agree with the assignment in Step 3 of the algorithm. Aside from ensuring this necessary condition, the algorithm allows total freedom in the choices of Λ (in Step 2) and $\lambda|_{E(K_3)}$ (in the Basis).

We close this section with an assertion on the optimality of the algorithm `triv_TSP_label(n)`.

Theorem 7 *Algorithm 4 is optimal, up to a constant factor; i.e., the running time of* `triv_TSP_label(n)` *is in* $O(n^2)$.

Proof. We shall analyze the version of Algorithm 4 that does not seek distinct labels, but with an appropriate implementation, the overhead needed to ensure an injective λ does not change the order of the running time.

Let $T(n)$ denote the running time of `triv_TSP_label(n)`. From the statement of Algorithm 4, we see that

$$T(n) = \begin{cases} c & \text{if } n = 3 \\ T(n-1) + (n-1) + d & \text{if } n \geq 4 \end{cases}$$

for some constants c, d. Since the general solution of this recurrence relation is $T(n) = (c-3)+(n-3)d+n(n-1)/2$, for $n \geq 3$, we see that $T(n) \in O(n^2)$.

Clearly this is optimal (up to a constant factor), since any algorithm to label the edges of K_n (in any manner whatsoever) must at least spend constant time on each of the $\binom{n}{2} \in \Omega(n^2)$ edges. ∎

3 Reduced Fibonacci numbers

As discussed in the introduction, the relation (1) determines several characteristics of the sequence of vertex labels inducing a trivial-TSP edge-labelling. For example, an injective, trivial-MTSP edge-labelling corresponds to a well-spread, nonnegative, half-integer sequence of vertex labels. With its first term deleted, the Fibonacci sequence furnishes one example of such a sequence. In this section, we examine how this example dovetails with Algorithm 4.

We define the *reduced Fibonacci sequence* $(f_n)_{n\geq 1}$ by $f_1 := 0$, $f_2 := 1$ and

$$f_n := f_{n-1} + f_{n-2} + 1 \text{ for } n \geq 3. \qquad (6)$$

It is easy to see that $(f_n)_{n\geq 1}$ is obtained from the Fibonacci sequence $(F_n)_{n\geq 1}$ by dropping the first term and decrementing each successive term—i.e. $f_n = F_{n+1} - 1$ for $n \geq 1$—so our use of "reduced" seems doubly appropriate. Though it has received its share of attention—an early reference is [9], and it is entry A000071 in [11]—this sequence as yet does not appear to have been named, so we felt free to assign our own.

In seeking an integral vertex-labelling ν inducing an injective trivial-TSP edge-labelling via (1), it is actually the reduced Fibonacci sequence that arises naturally. For suppose $\nu_1 < \nu_2 < \cdots < \nu_{n-1} < \nu_n$ and the first $(n-1)$ of these vertex labels induce such a labelling of $E(K_{n-1})$. Then the maximum edge label on K_{n-1} is $\nu_{n-1} + \nu_{n-2}$, and the minimum label on an edge $\{i, n\}$, $1 \leq i \leq n-1$, is $\nu_n + \nu_1$. One (greedy) way to ensure that these final labels are different from any of those on $E(K_{n-1})$ is to require that $\nu_n + \nu_1 > \nu_{n-1} + \nu_{n-2}$. If we also *begin* greedily and set $\nu_1 = 0$, then the least integral choice for ν_n is

$$\nu_n = \nu_{n-1} + \nu_{n-2} + 1,$$

the recurrence (6) defining $(f_n)_{n \geq 1}$. The next result formalizes these observations and connects them with Algorithm 4.

Theorem 8 *If in Algorithm 4 the basis returns labels* $\lambda(12) = 1$, $\lambda(13) = 2$, $\lambda(23) = 3$, *and the recursive step always selects* $\Lambda = \max_{A \in E(K_{n-1})} \lambda(A) + 1$, *then the vertex-labelling inducing* λ *via* (1) *consists of labels* $\nu_i = f_i$, *for* $1 \leq i \leq n$.

Proof. By induction on n. For the basis, notice that the vertex labels $(\nu_1, \nu_2, \nu_3) = (f_1, f_2, f_3) = (0, 1, 2)$ induce edge labels $\{1, 2, 3\}$, as desired.

Now fix $n > 3$, and assume the result is true for complete graphs on $n-1$ vertices. Thus, if G denotes the subgraph of K_n induced on the vertices $[n-1]$, then the recursive call `triv_TSP_label(n - 1)` (Step 1) returns an edge-labelling $\lambda|_{E(G)}$ induced by the vertex-labelling ν given by $\nu_i = f_i$ for $1 \leq i \leq n-1$. We may now determine ν_n by considering (1) for the new edge $\{1, n\}$. Steps 2,3 show that

$$\nu_n = \lambda(1n) - \nu_1 = \Lambda - f_1 = \max_{A \in E(G)} \lambda(A) + 1 = (f_{n-1} + f_{n-2}) + 1 = f_n. \blacksquare$$

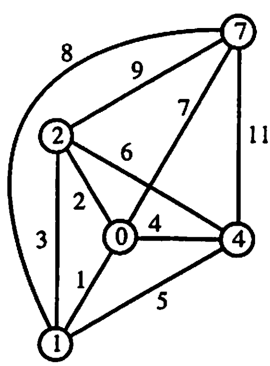

Figure 3: Reduced Fibonacci vertex labels induce an edge-labelling of K_5 produced by Algorithm 4.

The $n = 5$ case of Theorem 8 is illustrated in Fig. 3, where the vertex *labels*, not their names, appear on the vertices. With $\Lambda = 7$, this labelling also results from the next recursive call in Algorithm 4 following the labelling of K_4 in Fig. 2.

Remark In the proof of Theorem 8, it may be unsettling that in computing ν_n, we only considered (1) for $ij = 1n$. One might question the veracity of the remaining equations in (1) involving ν_n, namely of

$$\lambda(in) = \nu_i + \nu_n \text{ for } 1 < i < n-1. \tag{7}$$

To illustrate that our choice $\nu_n := \lambda(1n) - \nu_1$ also satisfies these relations, we fix i, $1 < i < n$, and derive the ith equation in (7). Since $n > 3$, there is an index $j \in [n] \setminus \{1, i, n\}$, so that $(1, j, i, n)$ is a 4-cycle. Since λ satisfies the C_4-MP, we have

$$\lambda(1j) + \lambda(in) = \lambda(ij) + \lambda(1n),$$

which, because $\lambda(ij) = \nu_i + \nu_j$ for each edge ij of $G \cong K_{n-1}$, yields

$$(\nu_1 + \nu_j) + \lambda(in) = (\nu_i + \nu_j) + \lambda(1n),$$

or

$$\lambda(in) = \nu_i + (\lambda(1n) - \nu_1) = \nu_i + \nu_n.$$

Therefore, our choice of ν_n indeed satisfies the ith equation in (7).

Optimal edge-label growth rate

It is natural to ask how quickly the labels in a "most efficient" edge-labelling scheme grow, and to compare the answer with the growth-rate of the labels induced by the reduced Fibonacci numbers. For the latter, if φ is the golden ratio and λ is induced by $(f_i)_{i=1}^n$, then $M(n) := \max_{A \in E(K_n)} \lambda(A) = f_n + f_{n-1} \in \Theta(\varphi^n)$ (see, e.g., [2]), so these labels grow exponentially. For the former, we shall measure efficiency by the length of an initial segment of the nonnegative integers into which we can squeeze all the edge labels. Thus, we should compare $M(n)$ with

$$m(n) := \min_{\lambda} \max_{A \in E(K_n)} \lambda(A),$$

where the minimum is taken over all injective trivial-MTSP edge-labellings λ. Notice that the labellings over which λ ranges in the definition of $m(n)$ include the labelling in the definition of $M(n)$. In [4], we conjectured, and in [5] the second author proved, that $m(n)/2n^2 \to 1$ as $n \to \infty$ (see also [6] for related developments). Thus the edge labels induced by the reduced Fibonacci numbers are a far cry from those realizing $m(n)$, and, here at least, greed does not pay.

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References

[1] D. Applegate, R. Bixby, V. Chvátal and W. Cook, *The Traveling Salesman Problem Homepage*, `http://www.math.princeton.edu/tsp/`, 2004.

[2] P.J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, Cambridge, 1994.

[3] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **5** (1998), #DS6, updated 19 Oct. 2003.
`http://www.combinatorics.org/Surveys/index.html`

[4] S.O. Jones, P.M. Kayll, B. Mohar and W.D. Wallis, On constant-weight TSP-tours, *Discuss. Math. Graph Theory* **23** (2003), 287–307.

[5] P.M. Kayll, Well-spread sequences and edge-labellings with constant Hamilton-weight, *Discrete Math. Theor. Comput. Sci.* **6** 2004, 401–408 (electronic).

[6] P.M. Kayll, A note on weak Sidon sequences, *Discrete Math.*, to appear.

[7] A. Kotzig, On well spread sets of integers, *Centre Res. Math.* (Université de Montréal) *CRM*-**161** (1972), 83pp.

[8] E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., *The Traveling Salesman Problem*, Wiley, New York, 1985.

[9] D.A. Lind, On a class of nonlinear binomial sums, *Fibonacci Quart.* **3** (1965), 292–298.

[10] N.C.K. Phillips and W.D. Wallis, Well-spread sequences (Papers in honour of Stephen T. Hedetniemi), *J. Combin. Math. Combin. Comput.* **31** (1999), 91–96.

[11] N.J.A. Sloane, ed., *The On-Line Encyclopedia of Integer Sequences* (2004).
`http://www.research.att.com/~njas/sequences/`

[12] V.T. Sós, An additive problem in different structures, in: Y. Alavi, F.R.K. Chung, R.L. Graham and D.F. Hsu, eds., *Graph theory, combinatorics, algorithms, and applications (San Francisco State University, San Francisco, CA, 1989)*, SIAM, Philadelphia, 1991, 486–510.

[13] D.B. West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, Upper Saddle River, NJ, 2001.