A geometric construction of large vertex transitive graphs of diameter two

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Abstract

The Moore upper bound for the order $n(\Delta, 2)$ of graphs with maximum degree Δ and diameter two is $n(\Delta, 2) \leq \Delta^2 + 1$. The only general lower bound for vertex symmetric graphs is $n_{vt}(\Delta, 2) \geq \lfloor\frac{\Delta+2}{2}\rfloor\lceil\frac{\Delta+2}{2}\rceil$. Recently a construction of vertex transitive graphs of diameter two, based on voltage graphs, with order $\frac{8}{9}(\Delta + \frac{1}{2})^2$ has been given in [5] for $\Delta = (3q-1)/2$ and q a prime power congruent with 1 mod 4. We give an alternative geometric construction which provides vertex transitive graphs with the same parameters and, when q is a prime power not congruent to 1 modulo 4, it gives vertex transitive graphs of diameter two and order $\frac{1}{2}(\Delta+1)^2$, where $\Delta = 2q - 1$. For $q = 4$, we obtain a vertex transitive graph of degree 6 and order 32.

1 Introduction

The well-known (Δ, D)-problem asks for the largest possible number $n(\Delta, D)$ of vertices in a graph with given maximum degree Δ and diameter D. The *Moore bound* $n(\Delta, d) \leq \frac{\Delta(\Delta-1)^D-2}{\Delta-2}$ can only be reached when either $\Delta = 2$ (the cycle C_{2D+1}), or $D = 1$ (the complete graph $K_{\Delta+1}$) or when $D = 2$ and $\Delta = 1, 2, 3, 7$ and perhaps 57 [4]. A survey of the current best know constructions of graphs with large order for given maximum degree and diameter can be found in [6]. For $D = 2$, incidence graphs of projective planes folded by a polarity give $n(\Delta, 2) \geq \Delta^2 - \Delta + 1$ whenever $\Delta - 1$ is a primer power [1, 2] and they can be used to provide families of graphs of diameter two and order $O(\Delta^2)$ for each value of Δ, see [5].

When the graphs are required to be vertex transitive, the only general lower bound available seems to be

$$n_{vt}(\Delta, 2) \geq \lfloor \frac{\Delta+2}{2} \rfloor \lceil \frac{\Delta+2}{2} \rceil,$$

attained by the Cayley graphs $Cay(\mathbb{Z}_a \times \mathbb{Z}_b, S)$, where $a = \lfloor \frac{\Delta+2}{2} \rfloor$, $b = \lceil \frac{\Delta+2}{2} \rceil$ and $S = (\mathbb{Z}_a \times \{0\}) \cup (\{0\} \times \mathbb{Z}_b) \setminus \{(0,0)\}$. Larger vertex symmetric graphs were obtained by Hafner [3]. McKay, Miller and Širáň [5] gave an infinite family of vertex transitive graphs of diameter two and order $\frac{8}{9}(\Delta + \frac{1}{2})^2$ when $q = (2\Delta + 1)/3$ is a prime power congruent to 1 mod 4. In view of the (unattainable) Moore bound $n(\Delta, 2) \leq \Delta^2 + 1$ this is a remarkable result. These graphs, which we call MMS in what follows, still provide some of the best constructions for large vertex transitive graphs of diameter two. Their construction is based on the covering graph technique and use lifts of complete bipartite graphs. A simplified description of the MMS graphs was given by Šiagiová [7] by using abelian lifts of graphs with two vertices; she also proved [8] that this kind of technique can not produce graphs of order larger than about $0.933\Delta^2$.

Here we present a geometric construction of large graphs of diameter two. This geometric construction is based on the incidence graphs of finite affine planes. When the chosen affine plane comes from a Desarguesian projective plane, the construction allows one to give an alternative way of obtaining the MMS graphs. For non Desarguesian planes the construction gives graphs with the same parameters of order, degree and diameter but, for the known examples of non Desarguesian planes, the corresponding graphs fail to be vertex transitive. For simplicity we give here the proofs by using the Desarguesian affine planes.

The proposed technique also provides the following lower bound for vertex transitive graphs of diameter two:

$$n_{vt}(\Delta, 2) \geq \frac{1}{2}(\Delta^2 + 1),$$

when $q = (\Delta + 1)/2$ is a prime power.

2 The construction

Let $q \neq 2$ be a prime power and denote by $\mathbb{F}_q$ the Galois field of order q. Let $L_0, L_1, \ldots, L_q$ be the parallel classes of lines in the affine plane $A(2, q)$. We denote by $L = L_0 \cup \cdots \cup L_q$ the set of lines of $A(2, q)$. Assume that the plane is coordinatised in such a way that L_0 consists of the lines with equation $x = c$ for each $c \in \mathbb{F}_q$. We denote the line with equation $y = mx + b$ in $L \setminus L_0$ by $[m, b]$.

Consider the incidence graph B_q of the points in $A(2, q)$ and the set $L \setminus L_0$ of all lines in $A(2, q)$ except the ones in the parallel class L_0. The graph B_q is bipartite, q-regular, has $2q^2$ vertices and is vertex transitive. Indeed, the group of translations of the affine plane $A(2, q)$ acts regularly on the set of points and leaves L_0 invariant, and the map ϕ which exchanges the point (a, b) with the line $[a, -b]$ is an automorphism of B_q which exchanges stable sets of the graph. Every two points not in a line of L_0 determine a unique line in $L \setminus L_0$ and thus they are at distance two in B_q. Similarly, two lines not in the same parallel class of $L \setminus L_0$ intersect in a unique point and so they are at distance two in the graph. The construction is completed by inserting appropriate copies of graphs of diameter two in the set of points of each line of L_0, and in the set of lines in each parallel class of lines except L_0. Since we require our graphs to be vertex transitive, additional care must be taken about the way these graphs are inserted.

Let S_1, S_2 be subsets of $\mathbb{F}_q$ satisfying the following three conditions:

(i) There is $\alpha \in \mathbb{F}_q$ such that $\alpha S_1 = S_2$ and $\alpha S_2 = S_1$.

(ii) $S_1 \cup S_2$ covers all non zero elements of $\mathbb{F}_q$.

(iii) $S_i = -S_i$, $i = 1, 2$.

Let G_i be the Cayley graph on the additive group of $\mathbb{F}_q$ with generating set S_i, $i = 1, 2$. Then, by condition (i), the two graphs are isomorphic. Actually, if we denote still by α the map $\alpha(x) = \alpha x$, then $\alpha(G_1) = G_2$ and $\alpha G_2 = G_1$. Moreover, by (ii), we have $|S_1| = |S_2| \geq (q - 1)/2$. Then both G_1 and G_2 have diameter at most 2.

Consider the graph $B_q(G_1, G_2)$ constructed from B_q by adding copies of G_1 and G_2 as follows.

1. For each line in L_0, we embed a copy of G_1 in its set of points. More precisely, if S_1 is the generating set of the Cayley graph G_1, for each line in L_0 with equation $x = c$ the neighborhood of vertex (c, y) in the inserted copy of G_1 is the set of points $(c, y) + (\{0\} \times S_1)$.

2. Similarly, to each parallel class L_i, $i \neq 0$, we embed a copy of G_2 in such a way that the neighborhood of the line $[a, b]$ is the set of lines $[a, b] + (\{0\} \times S_2)$ in the same parallel class.

An example of the construction for $q = 3$ and $S_1 = S_2 = \mathbb{F}_3^*$ is shown in Figure 1.

Proposition 1 *The graph $B_q(G_1, G_2)$ is vertex transitive and has diameter two.*

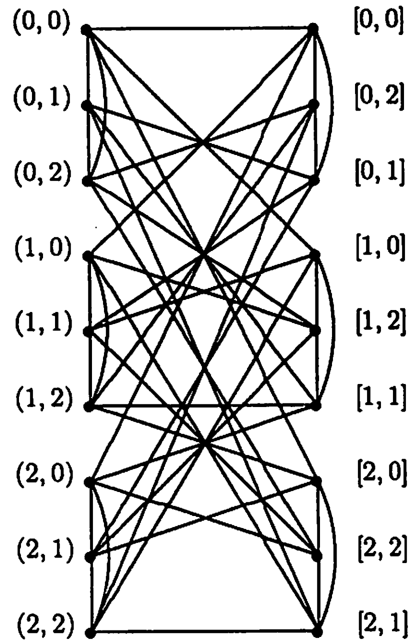

Figure 1: The graph $B_3(K_3, K_3)$.

Proof. Let us first show that $B_q(G_1, G_2)$ is vertex transitive. Note that the translations in the affine plane $A(2, q)$ still act as automorphisms of $B_q(G_1, G_2)$. Indeed, a translation sends each line $[m, b]$ to a parallel one of the form $[m, b + x_m]$ for some $x_m \in \mathbb{F}_q$, so that it acts as a translation of the induced graph G_2 in the parallel class, i.e. as an automorphism of G_2. Similarly, a translation sends the set of points of a line in L_0 to the points of another line in L_0 and thus an induced copy of G_1 to another one.

Let us denote by α' the bijection on the vertex set of $B_q(G_1, G_2)$ defined as $\alpha'(x, y) = (x, \alpha y)$ in the set of points and $\alpha'[x, y] = [\alpha x, \alpha y]$ on the set of lines, where $\alpha \in \mathbb{F}_q$ satisfies $\alpha S_1 = S_2$ (and $\alpha S_2 = S_1$). Clearly α' preserves the incidence relations in $A(2, q)$ and it is therefore an automorphism of the graph B_q. Moreover, α' exchanges the copies of G_1 with the copies of G_2. Hence, $\alpha'\phi$ is an automorphism of $B_q(G_1, G_2)$ which exchanges points with lines. Therefore, the group of automorphims generated by the translations and $\alpha'\phi$ acts transitively on the set of vertices of the graph.

Finally, let us show that $(0, 0)$ has eccentricity two. All points not in the line of L_0 incident to $(0, 0)$ are at distance two from $(0, 0)$ in the subgraph

B_q, and the points in this line are also at distance at most 2 in the copy of G_1 embedded in it. On the other hand, $(0,0)$ has the lines $[m,0]$, $m \in \mathbb{F}_q$ at distance one in B_q. The lines $[m,u], u \in S_2$ are at distance one in a copy of G_2 from $[m,0]$ and the lines $[m,v]$, $v \in S_1$ at distance one from $(0,v)$ which in turn is at distance one from $(0,0)$ in a copy of G_1. Since $S_1 \cup S_2 = \mathbb{F}_q^*$, all lines are at distance at most 2 from $(0,0)$. Therefore, $B_q(G_1, G_2)$ has diameter two. □

The above construction provides instances of large graphs with diameter two by appropriate choices of sets S_1 and S_2 satisfying conditions (i)-(iii) above. Part (2) of the following theorem is proved in [5] using a different construction.

Theorem 1 *Let q be a prime power, $q \neq 2$. Then*

1. $n_{vt}(\Delta, 2) \geq \frac{1}{2}(\Delta+1)^2$ *for* $\Delta = 2q-1$.
2. $n_{vt}(\Delta, 2) \geq \frac{8}{9}(\Delta+\frac{1}{2})^2$ *for* $\Delta = (3q-1)/2$ *and* $q \equiv 1 \pmod 4$.
3. $n_{vt}(6, 2) \geq 32$.

Proof. A trivial choice for the sets S_1, S_2 involved in the construction of $B_q(G_1, G_2)$ is $S_1 = S_2 = \mathbb{F}_q^*$.

Then, both G_1 and G_2 are complete graphs and $B_q(G_1, G_2)$ has degree $\Delta = 2q-1$ and order $n = \frac{1}{2}(\Delta+1)^2$. This proves (1).

When q is a prime power congruent to 1 mod 4 then a better choice for S_1 and S_2 with cardinality $(q-1)/2$ (the minimum possible) can be found. Let α be a primitive root of $\mathbb{F}_q$,

$$S_1 = \{\alpha^{2k+1},\ k = 0, 1, \ldots, (q-3)/2\} \text{ and } S_2 = \alpha S_1.$$

Since $-\alpha^i = \alpha^{i+(q-1)/2}$ and $(q-1)/2$ is even, we have $S_1 = -S_1$. Moreover, $\alpha S_2 = S_1$ and $S_1 \cup S_2 = \mathbb{F}_q^*$. The resulting graph $B_q(G_1, G_2)$ has degree $\Delta = (3q-1)/2$ and order $n = \frac{8}{9}(\Delta + \frac{1}{2})^2$. This proves (2).

Finally, for $q = 4$, we may choose $S_1 = \{1, \alpha\}$ and $S_2 = \{1, \alpha^2\} = \alpha^2 S_1$ which results in a graph of degree 6 and order 32. This proves (3). □

When $q \not\equiv 1 \pmod 4$ then the only choice for S_1 and S_2 satisfying conditions (i)–(iii) is $S_i = \mathbb{F}_q^*$. However, there are other choices for such sets satisfying only conditions (ii) and (iii). This leads to graphs $B_q(G_1, G_2)$ which are no longer vertex transitive but still have diameter two and degree $q + \max\{|S_1|, |S_2|\}$. The following two examples give the families described in [5, Theorem 1].

For $q \equiv 0 \pmod 4$ we may choose $S_1 = \{1, \alpha, \ldots, \alpha^{q/2-1}\}$ and $S_2 = \{1, \alpha^{q/2}, \ldots, \alpha^{q-1}\}$. We have $S_i = -S_i$, $i = 1, 2$, and $S_1 \cup S_2 = \mathbb{F}_q^*$. The resulting graph $B_q(G_1, G_2)$ has diameter two and order $n = \frac{8}{9}\Delta^2$. When $q \equiv -1 \pmod 4$, a possible choice is

$$S_1 = \pm\{1, \alpha, \ldots, \alpha^{(q-3)/4}\} \text{ and } S_2 = \pm\{1, \alpha^{(q+1)/4}, \ldots, \alpha^{(q-3)/2}\},$$

which gives rise to a graph with diameter two and degree $n = \frac{8}{9}(\Delta^2 - 1/2)$.

References

[1] W.G. Brown, On graphs that do not contain a Tompsen graph, *Canad. Math. Bull.* **9** (1966) 281–285.

[2] Ch. Delorme, Grands graphes the degré et diamètre donnés,*Europ. J. Combin.* **6** (1985) 291–302.

[3] P.R. Hafner *Large Cayley graphs and digraphs with small degree and diameter* Computational Algebra and Number Theory (W. Bosma and van der Poorten, Eds) Kluwer, Amsterdam. pp.291-302. 1995.

[4] A. J. Hoffman, R. R. Singelton *On Moore graphs with diameter 2 and 3* IBM J. Res Develop. vol.4. pp.497-504. 1960.

[5] B. D. McKay, M. Miller, J. Širáň, *A note on large graphs of diameter two and given maximum degree* J. Combin. Theory (B) vol.74. pp.110-118. 1998.

[6] *The (Degree, Diameter) Problem for Graphs* A World Combinatorics Exchange resource at
http://www-mat.upc.es/grup/_de/_grafs/grafs/taula/_delta/_d.html

[7] J. Šiagiová, A note on the McKay-Miller-Širáň graphs, J. Combinat. Theory B **81** (2001), 205–208.

[8] J. Šiagiová, A Moore-like bound for graphs of diameter 2 and given degree obtained from abelian lifts of dipoles, acta Math. Univ. Comenianae LXXI (2002) No. 2, 157–161.