On a class of quasi-symmetric designs

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Abstract

We show that for each positive integer t, for which there is a skew-type Hadamard matrix of order $4t$, there is a quasi-symmetric $((4t-1)^2, (4t-1)(2t-1), t(4t-3))$ design.

1 Introduction

A ± 1-matrix with mutually perpendicular rows (and columns) is called a *Hadamard matrix*. A Hadamard matrix H is said to be of *skew-type* if $H = I + S$, where I denotes the identity matrix and S is a skew $(0, \pm 1)$-matrix. Throughout the paper J denotes a matrix of all entries 1. A block design **D** with parameters (v, k, λ), where v is the number of points, k is the constant block size and λ is the number of blocks containing any pair of distinct points, is called a *quasi-symmetric design* if there are constants x and y such that every pair of blocks intersects in x or y points. Such a quasi-symmetric design is called *proper* if $x \neq y$, that is, if **D** is not a symmetric design.

Very few families of proper quasi-symmetric designs seem to be known. For a good reference on quasi-symmetric designs refer to [2].

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2 Main Results

Lemma 1. *Let* $4t$ *be the order of a skew-type Hadamard matrix. There there is a symmetric Hadamard* $(4t-1, 2t-1, t-1)$ *design with* $(0,1)$*-incidence matrix* Q *such that:*

$$Q + Q^t + I = J.$$

Proof. Let H be a skew-type Hadamard matrix of order $4t$ and write

$$H = \begin{pmatrix} 1 & e \\ -e^t & I+P \end{pmatrix},$$

where e is the row matrix of order $1 \times (4t-1)$, e^t is the transpose of e, I is the identity matrix of order $4t-1$ and P a skew matrix of order $4t-1$. One can then write $P = Q - Q^t$, where Q is a $(0,1)$-matrix. Q is the desired matrix. □

Theorem 2. *Let* $4t$ *be the order of a skew-type Hadamard matrix. Then there is a quasi-symmetric* $((4t-1)^2, (4t-1)(2t-1), t(4t-3))$ *design.*

Proof. Let Q be as in Lemma 1. Let

$$D = \begin{pmatrix} e^t \otimes Q & Q \otimes (I + Q^t) + Q^t \otimes Q \end{pmatrix}.$$

Using $Q + Q^t + I = J$ and $QQ^t = Q^tQ = (t-1)J + tI$ we have

$$\begin{aligned} DD^t = & \\ & \begin{pmatrix} e^t \otimes Q & Q \otimes (I + Q^t) + Q^t \otimes Q \end{pmatrix} \begin{pmatrix} e \otimes Q^t \\ Q^t \otimes (I + Q) + Q \otimes Q^t \end{pmatrix} \\ = & \; J \otimes ((t-1)J + tI) + ((t-1)J + tI) \otimes ((2t-1)J + 2tI) + \\ & \; t(2t-1)(J - I) \otimes (J - I) \\ = & \; (4t^2 - 3t)J \otimes J + 2t^2 I \otimes I + t(2t-1) I \otimes I \\ = & \; (4t^2 - 3t)J \otimes J + (4t^2 - t) I \otimes I \end{aligned}$$

So D is a BIBD$((4t-1)^2, 4t(4t-1), (2t-1)4t, (2t-1)(4t-1), 4t^2 - 3t)$. A tedious, but elementary calculation shows that

$$D^tD = \begin{pmatrix} (4t-1)((t-1)J + tI) & (2t-1)^2 e \otimes J \\ (2t-1)^2 e^t \otimes J & (2t-1)^2 J \otimes J + t(4t-1) I \otimes I - tJ \otimes I \end{pmatrix}.$$

It is now evident that the off-diagonal entries of D^tD are either $(2t-1)^2$ or $4t^2 - 5t + 1$. This shows that D is the incidence matrix of the desired quasi-symmetric design. □

Example 3. We illustrate the construction for the case $t = 2$. This was also obtained by S. S. Shrikhande and D. Raghavarao in [1] using a projective plane of order two and parallel classes of lines of an affine plane of order three.

Consider the skew-type Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Then

$$Q = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Then $DD^t = 10J + 14I$ and D^tD is square matrix of order 56 with 21 on the diagonal, and 336 sevens and 2744 nines off the diagonal.

The D we obtain is given on the next page.

Corollary 4. *The strongly regular graphs derived from the class of quasi-symmetric* $((4t-1)^2, (4t-1(2t-1), t(4t-3))$ *designs obtained in Theorem 2 are disconnected and trivial.*

Proof. It follows from computations in Theorem 2 that the derived graph consists of two components. One component is the complete graph on $4t-1$ vertices. The other component is a regular graph on $(4t-1)^2$ vertices of degree $4t-2$. □

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A quasi-symmetric (49,56,24,21,10)-BIBD

References

[1] S. S. Shrikhande and D. Raghavarao, Affine α-resolvable incomplete block designs, 1965 Contributions to Statistics pp. 471–480 Statist. Publ. Soc., Calcutta.

[2] M. S. Shrikhande and S. S. Sane, Quasi-Symmetric Designs, London Mathematical Society, Lecture Notes No. 164, Cambridge University Press (1991).