Six-Point Circles from a Triangle

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Abstract

In the Euclidean plane, let A, B, C be noncollinear points and T be the union of the lines AB, BC, CA. It is shown that there is a point P such that if $\tilde{T}$ is the image of T by any nonrotating uniform expansion about P, then $T \cap \tilde{T}$ is generally a six-point set that lies on a circle.

1. INTRODUCTION. The medians of a triangle ABC in the Euclidean plane divide it into six smaller triangles. In [1] it was shown that the circumcentres of these triangles all lie on a conic. A problem published in the American Mathematical Monthly [2] invited readers to show that the conic is in fact always a circle. The problem editors of the AMM received a number of solutions to this problem all of which "involved lengthy calculations (some done with Maple or Mathematica)". Apparently disenchanted with all such solutions, the editors published their own purely geometrical solution [3] which they say "may help to shed some light on why the result is true".

Greater understanding is usually achieved by a broader view and the introduction of key concepts. In the present paper we establish a theorem, a trivial application of which provides a solution to the AMM problem. As indicated in the above abstract, the theorem involves a certain point P that is uniquely associated with a nondegenerate triangle ABC. We have called this the "orthocentroid" of the triangle. Our definition of an orthocentroid has generalizations to nondegenerate simplexes in finite dimensional Euclidean space. However it seems that our theorem is essentially two dimensional, so in §2 we just give a simple account of the planar case.

The detailed statement and proof of our theorem are facilitated by using the field $\mathbb{C}$ of complex numbers. A purely geometrical statement and proof would be more cumbersome and involve tedious considerations of several cases.

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2. ORTHOCENTROID. This unfamiliar notion is essential for our purpose. Let A, B, C be noncollinear points in the Euclidean plane $\mathbb{E}^2$ and T be the union of the lines BC, CA, AB.

Say that a point P in $\mathbb{E}^2$ is an *orthocentroid* of T if it is the centroid of the three points obtained by the orthogonal projections of P onto the lines BC, CA, AB.

We first show that an orthocentroid exists. Let n_A, n_B, n_C denote the unit outward normals to the edges BC, CA, AB respectively of the triangle ABC. Notice that $-n_C$ is in the non-reflex angle subtended by n_A and n_B, so zero is a linear combination with positive coefficients of n_A, n_B, n_C. Consequently there are points P, A', B', C' in $\mathbb{E}^2$ such that $\overrightarrow{PA'}$, $\overrightarrow{PB'}$, $\overrightarrow{PC'}$ are positive multiples of n_A, n_B, n_C respectively and $\overrightarrow{PA'}+\overrightarrow{PB'}+\overrightarrow{PC'}=0$. Let T_1 be the union of the lines through A', B', C' that are perpendicular to PA', PB', PC' respectively. Then T_1 is similar to T and has orthocentroid P. By change of scale and translation if necessary, we may suppose that $T_1 = T$ so existence is proved.

Now suppose that P is any orthocentroid of T. We show that P is inside the triangle ABC. Let A', B', C' be the orthogonal projections of P onto the lines BC, CA, AB respectively. Each of the n_A, n_B, n_C is a unique linear combination with negative coefficients of the other two, so any linear combination of them that represents 0 has coefficients either all positive, all negative or all zero. Now $\overrightarrow{PA'}+\overrightarrow{PB'}+\overrightarrow{PC'}$ is such a linear combination: $\overrightarrow{PA'}= x n_A$, $\overrightarrow{PB'} = y n_B$, $\overrightarrow{PC'}= z n_C$, say. Notice that $x>0$ if and only if P is on the triangle ABC side of the line BC and similarly for y, z. Since P must be on the triangle side of at least one of the lines BC, CA, AB it follows that x, y, z are all positive and therefore P is inside the triangle.

The uniqueness of the orthocentroid is now established by showing that

$$|PA'| = \kappa|BC|, \quad |PB'| = \kappa|CA|, \quad |PC'| = \kappa|AB|$$

where $\kappa = 2|ABC|\big/\left(|BC|^2+|CA|^2+|AB|^2\right)$ and the modulus notation is used for segment length or triangle area.

Since P is inside the triangle, resolving $\overrightarrow{PA'}+\overrightarrow{PB'}+\overrightarrow{PC'}=0$ in the BC direction gives $|PB'|\sin\hat{C}-|PC'|\sin\hat{B}=0$, so

$$\frac{|PB'|}{\sin\hat{B}} = \frac{|PC'|}{\sin\hat{C}}\,.$$

Similarly

$$\frac{|PC'|}{\sin\hat{C}} = \frac{|PA'|}{\sin\hat{A}}\,.$$

Hence, by the sine rule, for some κ_1

$$|PA'| = \kappa_1 |BC|, \quad |PB'| = \kappa_1 |CA|, \quad |PC'| = \kappa_1 |AB| .$$

Also

$$|PA'||BC| + |PB'||CA| + |PC'||AB| = 2|ABC|$$

so $\kappa_1 = \kappa$ as required.

3. THE SIX-POINT CIRCLES. Henceforth the field $\mathbb{C}$ of complex numbers is used as a model for $\mathbb{E}^2$ and zero will be our orthocentroid. For any complex number $a \neq 0$, define

$$L_a = \{(1+ix)a : x \in \mathbb{R}\} .$$

This plays the role of our line *BC* and *a* corresponds to A'.

Lemma. *Suppose that a, b are non-zero complex numbers and L_a, L_b are not parallel. Then*

$$p_{a,b} = -2ab(\bar{a}-\bar{b})/(a\bar{b}-\bar{a}b)$$

is the point in $L_a \cap L_b$.

Proof. For some real x, y

$$p_{a,b} = (1+ix)a = (1+iy)b ,$$

so
$$(1-ix)\bar{a} = (1-iy)\bar{b} .$$

These equations yield $1+ix = 2b(\bar{b}-\bar{a})/(a\bar{b}-\bar{a}b)$ and the result is immediate.

Theorem. *Suppose that a, b, c are non-zero complex numbers with $a + b + c = 0$ and no two of the lines L_a, L_b, L_c are parallel. Then $T = L_a \cup L_b \cup L_c$ has orthocentroid zero and circumcentre $z = -2abc\bar{\sigma}\Delta^{-2}$ where*

$$\sigma = a^2+b^2+c^2$$

and
$$\Delta = \begin{vmatrix} a & b \\ \bar{a} & \bar{b} \end{vmatrix} = \begin{vmatrix} b & c \\ \bar{b} & \bar{c} \end{vmatrix} = \begin{vmatrix} c & a \\ \bar{c} & \bar{a} \end{vmatrix} .$$

Also, for any real $\lambda \neq 0,1$ *the set* $\lambda T \cap T$ *lies on the circle with centre* $z_\lambda = \frac{1}{2}(\lambda+1)z$ *which is the point midway between the circumcentres of* λT *and* T. *Finally the radius of this circle is*

$$r_\lambda = 2\left|\frac{abc}{\Delta^2}(\lambda\delta+\bar{\delta})\right|$$

where

$$\delta = \begin{vmatrix} a & b \\ \bar{c} & \bar{a} \end{vmatrix} = \begin{vmatrix} b & c \\ \bar{a} & \bar{b} \end{vmatrix} = \begin{vmatrix} c & a \\ \bar{b} & \bar{c} \end{vmatrix} .$$

Proof. Of course, the *circumcircle* of T is the circle that contains $p_{a,b}$, $p_{b,c}$ and $p_{c,a}$. Also $\lambda T = \{\lambda t : t \in T\}$.

By the definitions, 0 is the orthocentroid of T and the determinant equations are simple consequences of $a+b+c=0$.

By the Lemma, for any real $\mu \neq 0$,

$$\begin{aligned} p_{a,\mu b} &= -2a\mu b(\bar{a}-\mu\bar{b})/(a\mu\bar{b}-\bar{a}\mu b) \\ &= -2ab(\bar{a}-\mu\bar{b})/\Delta . \end{aligned}$$

Since Δ is unchanged by cyclic permutation of a, b, c, we deduce

$$p_{b,\mu c} = -2bc(\bar{b}-\mu\bar{c})/\Delta, \quad p_{c,\mu a} = -2ca(\bar{c}-\mu\bar{a})/\Delta .$$

Similarly,

$$\begin{aligned} p_{\mu a,b} &= -2ab(\mu\bar{a}-\bar{b})/\Delta, \quad p_{\mu b,c} = -2bc(\mu\bar{b}-\bar{c})/\Delta , \\ p_{\mu c,a} &= -2ca(\mu\bar{c}-\bar{a})/\Delta . \end{aligned}$$

Recall the definitions: $\sigma = a^2+b^2+c^2$, $z = -2abc\bar{\sigma}\Delta^{-2}$, $z_\mu = \frac{1}{2}(\mu+1)z$. Using $a+b+c=0$ we obtain

$$\tfrac{1}{2}\sigma = a^2+b^2+ab = b^2+c^2+bc = c^2+a^2+ca .$$

Now calculate

$$\begin{aligned} & z_\mu - p_{a,\mu b} \\ &= -\tfrac{1}{2}(\mu+1)2abc\bar{\sigma}\Delta^{-2} + 2ab(\bar{a}-\mu\bar{b})\Delta^{-1} \\ &= -ab\Delta^{-2}\left[\mu(c\bar{\sigma}+2\bar{b}\Delta)+c\bar{\sigma}-2\bar{a}\Delta\right] \end{aligned}$$

$$= -2ab\Delta^{-2}\Big[\mu\big\{c\big(\bar{b}^2+\bar{c}^2+\bar{b}\bar{c}\big)+\bar{b}\,(b\bar{c}-\bar{b}c)\big\}$$
$$+c\big(\bar{c}^2+\bar{a}^2+\bar{c}\,\bar{a}\big)-\bar{a}\,(c\bar{a}-\bar{c}a)\Big]$$
$$= -2ab\bar{c}\,\Delta^{-2}\big[\mu\{c\bar{c}+c\bar{b}+b\bar{b}\}+c\bar{c}+c\bar{a}+a\bar{a}\big]$$
$$= -2ab\bar{c}\,\Delta^{-2}\big[\mu(b\bar{b}-c\bar{a})+c\bar{c}-\bar{a}b\big]$$
$$= -2ab\bar{c}\,\Delta^{-2}\big(\mu\delta+\bar{\delta}\big)\ .$$

Since z_μ, Δ and δ are unchanged by a cyclic permutation of a, b, c, we deduce

$$z_\mu - p_{b,\mu c} = -2bc\bar{a}\Delta^{-2}\big(\mu\delta+\bar{\delta}\big),\ \ z_\mu - p_{c,\mu a} = -2ca\bar{b}\Delta^{-2}\big(\mu\delta+\bar{\delta}\big)\ .$$

Similarly $\quad z_\mu - p_{\mu a,b} = -2ab\bar{c}\,\Delta^{-2}\big(\mu\bar{\delta}+\delta\big)\ ,$

$$z_\mu - p_{\mu b,c} = -2bc\bar{a}\,\Delta^{-2}\big(\mu\bar{\delta}+\delta\big),\ \ z_\mu - p_{\mu c,a} = -2ca\bar{b}\,\Delta^{-2}\big(\mu\bar{\delta}+\delta\big)\ .$$

Hence all points of the set

$$I_\mu = \big\{p_{a,\mu b},\ p_{b,\mu c},\ p_{c,\mu a},\ p_{\mu a,b},\ p_{\mu b,c},\ p_{\mu c,a}\big\}$$

are distance $r_\mu = 2\big|abc\,\Delta^{-2}\big(\mu\delta+\bar{\delta}\big)\big|$ from z_μ. The set of vertices of T is I_1, so $z_1 = z$ is the circumcentre of T and therefore also λz is the circumcentre of λT. All the statements of the theorem are now clear.

Remarks 1. Let $L_{a,0}$, $L_{b,0}$, $L_{c,0}$ be the lines through the orthocentroid 0 that are parallel to L_a, L_b, L_c respectively and define $T_0 = L_{a,0} \cup L_{b,0} \cup L_{c,0}$. View T_0 as a "limit" of μT as μ tends to 0. Then $T_0 \cap T$ is a six-point circle, i.e. a six-point set that lies on a circle. This is easily deduced from the theorem since z_μ, r_μ and the elements of I_μ, regarded as functions of μ, all have continuous extensions at $\mu = 0$. The circle that contains $T_0 \cap T$ has centre $-abc\,\bar{\sigma}\Delta^{-2}$ and radius $2\,|\,abc\,\delta\,\Delta^{-2}\,|$.

2. If $\mu = 1$, then $\mu T \cap T = T$ is an infinite set. There are at most six other non-zero values of μ such that $\mu T \cap T$ is not a six-point circle. They occur when any one of the sets

$$L_{\mu a} \cap T,\ \ L_{\mu b} \cap T,\ \ L_{\mu c} \cap T,\ \ \mu T \cap L_a,\ \ \mu T \cap L_b,\ \ \mu T \cap L_c$$

contains only one point and this can happen only if $\mu < 0$.

3. A simple calculation shows that the segments $p_{\lambda a,b}\, p_{\lambda b,a}$, $p_{\lambda b,c}\, p_{\lambda c,b}$, $p_{\lambda c,a}\, p_{\lambda a,c}$ all have length $|(\lambda-1)abc\Delta^{-1}|$. When $\lambda > 0$ and $\lambda \neq 1$, these segments are all edges of the convex cyclic hexagon with vertices $\lambda T \cap T$.

We conclude with the application mentioned in the Introduction: barycentric subdivision of a triangle yields six triangles with concyclic circumcentres.

Proof. Work in $\mathbb{C}$ and suppose that zero is the centroid of the triangle and its vertices are $2a$, $2b$, $2c$. Then $a + b + c = 0$. Define $T = L_a \cup L_b \cup L_c$ as in the Theorem and observe that $\left(-\frac{1}{2}T\right) \cap T$ is the set of circumcentres.

References

[1] Clark Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium* **129** (1998).

[2] Floor van Lamoen, Problem 10830, *American Mathematical Monthly* (2000), 863.

[3] The Problem Editors of the American Mathematical Monthly, *American Mathematical Monthly* **109** (2002), 396-7.