A Note On the Ramsey Numbers $R(C_4, B_n)$

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Abstract

The Ramsey number $R(C_4, B_n)$ is the smallest positive integer m such that for every graph F of order m, either F contains C_4 (a quadrilateral) or $\overline{F}$ contains B_n (a book graph $K_2 + \overline{K_n}$ of order $n+2$). Previously, we computed $R(C_4, B_n) = n+9$ for $8 \leq n \leq 12$. In this continuing work, we find that $R(C_4, B_{13}) = 22$ and surprisingly $R(C_4, B_{14}) = 24$, showing that their values are not incremented by one, as one might have suspected. The results are based on computer algorithms.

1 Introduction

For graphs G and H, a (G, H)-graph is a graph F that does not contain G, and is such that the complement $\overline{F}$ does not contain H. A $(G, H; n)$-graph is a (G, H)-graph of order n. The Ramsey number $R(G, H)$ is defined to be the least integer $n > 0$ such that there is no $(G, H; n)$-graph.

A regularly updated survey by Radziszowski [4] includes the most recent results on Ramsey numbers $R(G, H)$, for different graphs G and H. In this paper we consider the case where G is a quadrilateral C_4 (cycle of order 4) and H is a book graph B_n.

In Section 2, we present known results for Ramsey numbers $R(C_4, B_n)$. Section 3 presents the statistics of $R(C_4, B_n)$ for $n = 13$ and 14.

A general utility program for graph isomorph rejection, *nauty* [3], written by Brendan McKay, was used extensively. The graphs themselves are available from the author.

2 Results

In [2], Faudree, Rousseau and Sheehan gave the bounds for $R(C_4, B_n)$:

Theorem (Faudree, Rousseau and Sheehan [2]).
(i). If q is a prime power, then $q^2+q+2 \le R(C_4, B_{q^2-q+1}) \le q^2+q+4$. In particular, $22 \le R(C_4, B_{13}) \le 24$.
(ii). Let g be the real valued function defined by $g(x) = x + \sqrt{x-1} + 2$ and $f(x) = g(g(x))$, then $R(C_4, B_n) \le f(n)$.
(iii). $R(C_4, B_n) = 7, 9, 11, 12, 13$ and 16, for $2 \le n \le 7$ respectively.

Theorem (Tse [6]). *$R(C_4, B_n) = 17, 18, 19, 20$ and 21, for $8 \le n \le 12$ respectively.*

Note that $R(C_4, B_8) = 17$ not 16 as claimed in [2] and $R(C_4, B_n)$ is not incremented by 1.

In Table I below, we give a list of known $R(C_4, B_n)$ and the upper bound $f(n)$.

n	$R(C_4, B_n)$	$f(n)$
2	7	9
3	9	10
4	11	11
5	12	13
6	13	15
7	16	16
8	17	17
9	18	18
10	19	20
11	20	21
12	21	23
13	**22**	24
14	**24**	25

Table I. Known $R(C_4, B_n)$ and upper bound $f(n)$.

3 Enumerations of $R(C_4, B_n)$

The algorithm we employed is the same as in [6] and similar to the one in computing $R(B_3, K_5)$ [1], and $R(C_4, K_7)$, $R(C_4, K_8)$ [5], thus we omit the details. We will use the same definitions and notations as in [6]. It is computationally infeasible to generate all (C_4, B_n)-graphs, for $n = 13$ and 14. We only enumerate (C_4, B_n)-graphs on $R(C_4, B_n) - 1$ vertices, and their statistics are presented in Table II. Finally, We give the adjacency matrix of the only $(C_4, B_{14}; 23)$-graph in Figure III.

n	m	$(C_4, B_n; m)$-graphs
13	21	11357443
14	23	1

Table II. Number of $(C_4, B_n; R(C_4, B_n) - 1)$-graphs, for $n = 13$ and 14.

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 1 0 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 1 0 1 0
 2 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 0 0 1 1
 3 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0 0 1 0 0 1
 4 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 1 1 0 0 0 0 0
 5 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 1 0 0 0 0 1 0 1
 6 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 1 0 0 0 0
 7 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 1 0
 8 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 0 0
 9 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 1 0 0
10 1 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 0 0
11 0 0 0 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0
12 0 1 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 1
13 1 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1
14 0 0 1 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 1 1 0 0 0
15 0 1 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1 0
16 0 0 0 0 1 0 1 1 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0
17 0 0 1 1 0 0 1 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0
18 0 0 0 1 0 1 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0
19 0 0 0 0 0 1 1 0 0 0 0 1 0 1 0 1 0 0 0 0 0 0 0
20 1 0 1 0 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0
21 0 0 0 0 1 0 0 0 1 1 0 0 0 0 1 1 0 0 0 0 0 0 0
22 1 1 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0
23 0 1 1 0 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
```

Figure III. Adjacency matrix of the only $(C_4, B_{14}; 23)$-graph.

References

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