Pair-Resolvability and Tight Embeddings for Path Designs

Spencer P. Hurd, The Citadel
171 Moultrie St, MSC-25
Charleston, SC, 29409,
Dinesh G. Sarvate
The College of Charleston
Charleston, SC, 29424

Abstract

We define a new type of resolvability called α-pair-resolvability in which each point appears in each resolution class as a member of α-pairs. The concept is intended for path designs (or other designs) in which the role of points in blocks is not uniform or for designs which are not balanced. We determine the necessary conditions and show they are sufficient for k = 3 and $\alpha = 2, 3$ ($\alpha \geq 2$ is necessary in every case). We also consider near α-pair-resolvability and show the necessary conditions are suficient for $\alpha = 2, 4$. We consider under what conditions is it possible for the ordered blocks of a path design to be considered as unordered blocks and thereby create a triple system (a tight embedding) and there also we show the necesary conditions are sufficient. We show it is always possible to embed maximally unbalanced path designs PATH$(v, 3, 1)$ into PATH$(v + s, 3, 1)$ for admissible s, and to embed any PATH$(v, 3, 2\lambda)$ into a PATH$(v + s, 3, 2\lambda)$ for any $s \geq 1$.

1 Introduction

A decomposition of a graph T into isomorphic copies of a graph G is a classical problem in graph theory and combinatorics. The G-decomposition of T is balanced if every vertex (point) of T is in the same number of copies of G. These problems overlap into combinatorial designs when the pairs of points (edges) meet certain requirements. In this case, the copies of G are called blocks of the design. We use the notation λK_v to denote λ copies of the complete graph K_v on v vertices, and in this paper we decompose $T = \lambda K_v$ into simple paths of length $k - 1$ in such a way that the paths correspond to a system of blocks of a design. Such a path is a junction of $k - 1$ unordered pairs.

For $k = 3$, each block $\langle a, b, c\rangle$ of the design will correspond to two unordered pairs or edges $\{a, b\}$ and $\{b, c\}$. In the notation, then, the role of point b is different in that it appears in two pairs, but points a and c appear in only one pair. More generally, a *path design* PATH(v, k, λ) consists of a set V of size v and a collection B of k-element (linearly ordered) subsets called blocks. Each block $\langle a_1, a_2, ..., a_k\rangle$ will correspond to the unordered pairs $\{a_i, a_{i+1}\}$ for $i = 1, ..., k - 1$. Each pair of points must appear (consecutively) in exactly λ blocks. In this way, each block corresponds directly to a length $k - 1$ path of

λK_v, and the block $\langle a_1, a_2, ..., a_k \rangle$ is identified with the block which has the same points but with the reverse order.

A partition of the complete graph K_v (or of λK_v) into classes of unordered pairs (paths of length one) in which each point appears exactly once in the class is a one-factorization of K_v and the classes are called one-factors. One-factors have proved very useful in constructions of designs and we use them in Section 4 for this purpose. However, we concentrate in this note on paths of length two and three and always assume $k>2$ in any case.

In the $k = 3$ case there are two other types of ordered-block systems. For example, in a directed triple system (DTS) each ordered block (a, b, c) is a *transitive triple* corresponding to the ordered pairs (a, b), (b, c) and (a, c). Another type of triple system is that in which each ordered block is regarded as a *cyclic* triple (a, b, c) and which corresponds to the *ordered* pairs (a, b), (b, c) and (c, a). These are called Mendelson triple systems (MTS). See Chapters 24 and 25 of [6], and also [4] for another variation.

A PATH design is *balanced* if every point appears in the same number of blocks. Balanced path designs are called handcuffed designs since only adjacent points are viewed as pairs (as if they were handcuffed like prisoners linked by a chain) [9]. A BIBD(v, k, λ), a balanced incomplete block design, is a balanced G-decomposition of λK_v into copies of $G = K_k$. When $k = 3$, we call the BIBD a triple system and use the notation TS(v, λ) [6].

We may decompose two copies of K_3 to form PATH$(3, 3, 2)$ - the blocks are $\langle a, c, b \rangle$, $\langle a, b, c \rangle$, and $\langle b, a, c \rangle$. Put another way, we may form PATH$(3, 3, 2)$ by discarding one copy of K_3 from a set of three copies of K_3 which have been formed into triangles, by removing one edge from each triangle. PATH$(4, 3, 1)$, which decomposes K_4 also has three blocks, namely, $\langle a, d, b \rangle$, $\langle a, b, c \rangle$, and $\langle a, c, d \rangle$. PATH$(4, 3, 1)$ is *not* an H$(4, 3, 1)$, i.e. a handcuffed design, since the design is not balanced. Indeed, there is no H$(4, 3, 1)$.

In Section 2 we briefly highlight selected literature, give a few new embedding results, and seek to quantify just how unbalanced a path design can be. In Section 3 we consider the following main embedding problem. *For what parameters is it possible to use the blocks of a PATH(v, k, λ^*) so that the same block system regarded as unordered gives a BIBD(v, k, λ)?* We call this a *tight embedding* and solve this question completely for $k = 3$ by showing the necessary conditions are sufficient. We define in Section 4 a completely new type of resolvability, *α-pair-resolvablility*, and we show the necessary conditions are sufficient for $k = 3$ and $\alpha = 2, 3$. We show the necessary conditions are sufficient for *near* α-pair-resolvability for $k = 3$ and $\alpha = 2, 4$.

2 Embedding Maximally Unbalanced Designs

Since the number of blocks b in a PATH$(v, 3, \lambda)$ is half the number of unordered pairs of points, we have

$$|B| = b = \lambda \binom{v}{2}/2 = \lambda v(v-1)/4.$$

If λ is even, then this is not a restriction, but if the index λ is odd, then $v \equiv 0, 1 (\bmod 4)$ is necessary.

We define two numbers r_1 and r_2, where r_1 is the number of times a point is an end-point of a path (block) and r_2 is the number of times a point is an interior point of a path. Equivalently, r_i (for $i = 1, 2$) is the number of blocks in which a point is paired i times. It follows that for any point $r = r_1 + r_2$. When the design is balanced, r_1 and r_2 are constants for the design and we have [9]

$$r_2 = \lambda(v-1) - r \text{ and } r_1 = 2r - \lambda(v-1).$$

From this we can obtain

$$r_1 = 2r/k \text{ and } r_2 = r - 2r/k.$$

In a series of papers culminating in [13], Lawless [14],[15] and Hung and Mendelsohn [12],[13] have shown the following:

Theorem 1 *A balanced PATH$(v, 2h+1, 1)$ exists if and only if $v \equiv 1 (\bmod 4h)$, and a balanced PATH$(v, 2h, 1)$ exists if and only if $v \equiv 1 (\bmod 2h-1)$. More generally, there exists a balanced PATH(v, k, λ) if and only if (i) $\lambda v(v-1) \equiv 0 (\bmod 2(k-1))$; (ii) $\lambda(k-2)(v-1) \equiv 0 (\bmod 2(k-1))$; and (iii) $\lambda(v-1) \equiv 0 (\bmod k-1)$.*

In order to exploit a construction, we reprove the fact for k = 3 due to Tarsi [19] that there are (unbalanced) path designs whenever $\lambda v(v-1) \equiv 0 (\bmod 2(k-1))$ and v $\geq k$. See also [7] [8].

Lemma 2 *If $v \equiv 0, 1 (\bmod 4)$, then there is an unbalanced PATH$(v, 3, 1)$.*

Proof. We first prove by induction on n that an unbalanced PATH$(4n, 3, 1)$ always exists. When n = 1, the example with blocks $\langle 1, 2, 3 \rangle$, $\langle 1, 3, 4 \rangle$, and $\langle 1, 4, 2 \rangle$ is unbalanced as 1 appears in every block but 2 does not. Suppose an unbalanced X = PATH$(4n, 3, 1)$ exists. Then we create an unbalanced Y = PATH$(4n+4, 3, 1)$ by adding to the blocks of X the following blocks: $\langle 2i-1, 4n+j, 2i \rangle$ for $i = 1, 2, ..., 2n$ and $j = 1, ..., 4$. Finally add the blocks of a PATH$(4, 3, 1)$ on points $\{4n+1, ..., 4n+4\}$. The design Y is unbalanced since the blocks inherited from X were unbalanced. Now suppose X = PATH$(4n, 3, 1)$. We create Y = PATH$(4n+1, 3, 1)$ by adding blocks $\langle 2i-1, 4n+1, 2i \rangle$ for $i = 1, 2, ..., 2n$. ■

Theorem 3 *There is an embedding of any X = PATH$(v, 3, 1)$ into some Y = PATH$(v+s, 3, 1)$ for all admissible s.*

Proof. It follows from the proof of the previous Lemma that s may be any number (0 or 1 mod 4) when v is a multiple of 4 and s may be any multiple of 4 when v is 1 mod 4. ■

Lemma 4 *For every $v \geq 3$, there exists a PATH$(v, 3, 2)$.*

Proof. For $v = 3$, use blocks $\langle 1,2,3\rangle$, $\langle 2,3,1\rangle$, and $\langle 3,1,2\rangle$. Suppose X = PATH$(v,3,2)$ exists for some $v \geq 3$. Then we create a PATH$(v+1,3,2)$ by adding the following blocks to those of X: $\langle i, v+1, i+1\rangle$, for $i = 1, 2, ..., v-1$, and the block $\langle v, v+1, 1\rangle$. The result follows by induction on v. ∎

Theorem 5 *There is an embedding of any PATH$(v,3,2\lambda)$ into a PATH$(v+s,3,2\lambda)$ for $s \geq 1$.*

We would like to quantify how unbalanced a path design can get. To that end let X = PATH(v,k,λ). We define, for every x in X, r_x is the number of blocks in which x appears. We say $\mu(X) = \min\{r_x : x \in X\}$ and $\mu(v,k,\lambda) = \min\{\mu(X) : X\text{=PATH}(v,k,\lambda)\}$. Similarly, we define $M(X) = \max\{r_x : x \in X\}$ and $M(v,k,\lambda) = \max\{M(X) : X\text{=PATH}(v,k,\lambda)\}$.

Theorem 6 $\mu(v,3,1) = \lfloor v/2 \rfloor$ *and* $M(v,3,1) = v-1$.

Proof. By induction. For $v = 4$, the result is obvious. Suppose the result is true for any $v = 4n$. Then the construction above shows that in some PATH$(4n+4,3,1)$ there is a vertex, say $4n+1$, such that $4n+1$ appears in $2n+2$ blocks, and another vertex (e.g., vertex 1) which appears in $v-1$ blocks, always as an endpoint. So the theorem holds for all $v \equiv 0 (\text{mod}\, 4)$. But now the construction applied again to get a PATH$(4n+1,3,1)$ shows $4n+1$ appears in $2n$ blocks and 1 appears in $4n$ blocks. It is clear that r_x cannot be greater than $v-1$ or less than $v/2$. ∎

In the next sections we consider what we call tight embeddings and pair-resolvability. We mention here that standard embeddings for *balanced* path designs (handcuffed designs) require greater increase in the number of new points, as is true for other design embeddings. For $k = 3$, Yan has shown [20] that a balanced pure H$(v,3,\lambda)$ can be embedded into a balanced pure H$(u,3,\lambda)$ if and only if $\lambda(v-1) \equiv \lambda v(v-1) \equiv \lambda(u-1) \equiv \lambda u(u-1) \equiv 0(\text{mod}\, 4)$ and $u \geq 2v+1$, with $\lambda \leq 2(v-1)/3$.

Embeddings of balanced path designs into other graphs have been studied, for instance into the triangle with attached edge and the 4-cycle systems among others; see [16], [17] and [18]. Other results on path designs and related matters can be found in, for example, in [8] and [11].

3 Tight Embeddings

In this section we begin by considering the problem of embedding a path design X^* into a triple system X on the same point set by considering the ordered triple $\langle a,b,c\rangle$ as an unordered triple $\{a,b,c\}$. We call this the "natural correspondence." The embedding is *tight* if the correspondence is a surjection. For example, there is no tight embedding of PATH$(4,3,1)$ into any triple system. The natural embedding of PATH$(4,3,1)$ into TS$(4,2)$ is loose since, in addition to blocks $\{a,d,b\}$, $\{a,b,c\}$ and $\{a,c,d\}$ we must add block $\{b,c,d\}$. It is equivalent to consider the tight embedding problem as one of locating an ordering of the blocks of a given triple system so that they can be regarded as the ordered blocks of an appropriate path design.

In this section we prove the following main theorem:

Theorem 7 *The necessary conditions are sufficient for the existence of* $X^* = PATH(v,3,\lambda^*)$ *which, under the natural correspondence* $\langle a,b,c\rangle \longrightarrow \{a,b,c\}$, *can be tightly embedded into* $X = TS(v,\lambda)$.

Theorem 8 *Suppose the blocks of a* $PATH(v,k,\lambda^*)$ *correspond to the blocks of a* $BIBD(v,k,\lambda)$ *under the natural map* $\langle a_1,a_2,...a_k\rangle \rightarrow \{a_1,a_2,...a_k\}$. *Then* $\lambda^* = 2\lambda/k$. *In particular, for* $k=3$, $\lambda \equiv 0 (\mathrm{mod}\, 3)$ *is a necessary condition.*

Proof. For any BIBD(v,k,λ) we have $vr = bk$ and $\lambda(v-1) = r(k-1)$. Now, since there are $\lambda^* v(v-1)/2$ pairs of points in the PATH design, $k-1$ of them per block, we have $\lambda^* v(v-1)/2 = b(k-1)$. Now substituting for b and then for r we get $\lambda^* = \frac{2b(k-1)}{v(v-1)} = \frac{2vr}{k} \cdot \frac{k-1}{v(v-1)} = \frac{2\lambda(v-1)(k-1)}{(k-1)k(v-1)} = 2\lambda/k$. Hence, for $k=3$, $\lambda^* = 2\lambda/3$ and $\lambda \equiv 0(\mathrm{mod}\, 3)$ is a necessary condition. ■

We give two examples of this natural correspondence in Tables 1 and 2 below.

1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	3	4	5	1	2
3	4	5	1	2	5	1	2	3	4

Table 1: A PATH(5,3,2) and a TS(5,3)

2	2	3	1	1	3	1	1	2	1	1	2
1	1	1	2	2	2	3	3	3	4	4	4
3	4	4	3	4	4	2	4	4	2	3	3

Table 2: A PATH(4,3,4) and a TS(4,6)

Lemma 9 *From three copies of any triple system which gives* $X = TS(v,3x)$ *we can construct the blocks of* $X^* = PATH(v,3,2x)$ *so that* X^* *is tightly embedded into* X. *If two copies of a BIBD form* $Y = BIBD(v,4,2x)$, *then there is an* $Y^* = PATH(v,4,x)$ *with* Y^* *tightly embedded into* Y. *If five copies of a BIBD form* $Z = BIBD(v,5,5x)$, *then there is a* $Z^* = PATH(v,5,2x)$ *with* Z^* *tightly embedded into* Z.

Proof. Order the three copies of a block of X so that each of the three points is in the center once. This forms X^*. Form Y^* by ordering each pair of blocks of Y once as $\langle a,b,c,d\rangle$ and once as $\langle b,d,a,c\rangle$. Form Z^* by ordering the five copies of a block $\{a,b,c,d,e\}$ as $\langle a,b,c,d,e\rangle$, $\langle a,c,d,e,b\rangle$, $\langle c,a,d,b,e\rangle$, $\langle c,e,a,b,d\rangle$, $\langle b,c,e,a,d\rangle$. ■

For $v \equiv 0,4(\mathrm{mod}\, 6)$ there exist triple systems TS$(v,2)$, but none for index $\lambda = 3$. Therefore, the following theorem is the best possible in the sense that there is no tight embedding for smaller index than 6.

Theorem 10 *If $v \equiv 0,4 \pmod 6$, there exists $X^* = Path(v,3,4)$ which can be tightly embedded into a triple system $X = TS(v,6)$ under the natural correspondence.*

Proof. Use three copies of a TS with index 2 and apply Lemma 9. ■

Theorem 11 *For all $v \equiv 1,3 \pmod 6$ there exist $TS(v,3)$ whose blocks form a $PATH(v,3,2)$.*

Proof. The triple systems exist in this case with index 1 so Lemma 9 can be applied. ■

Let us consider a triple system whose points we take to be Z_v and whose blocks are generated by a difference family. Then the starter block $\{a,b,c\}$ generates a set of blocks $\{a+i, b+i, c+i\}_{i=0}^{v-1}$ whose first two elements will have a difference $\pm(a-b) \bmod v$ and whose second and third elements will have a difference $\pm(b-c) \bmod v$. If we consider as PATH blocks all those blocks cyclically generated originally for the triple system, then they will form a PATH design only when all these particular differences are all there are. The difference set $\{a,b,c,d\}$ corresponds to the (PATH) differences $\pm(a-b), \pm(b-c)$, and $\pm(d-c)$.

Lemma 12 *A difference family $\{\{a_{i1}, a_{i2}, a_{i3}\}\}_{i=1}^{i=t}$ generates blocks for a PATH $(v,3,\lambda^*)$ if the differences $\{\pm(a_{i1}-a_{i2}), \pm(a_{i2}-a_{i3})\}_{i=1}^{i=t}$ cover all non-zero (mod v) points exactly λ^* times. A difference family $\{\{a_{i1}, a_{i2}, a_{i3}, a_{i4}\}\}_{i=1}^{i=t}$ generates blocks for a PATH(v, 4, λ^*) if the differences $\pm(a_{i1}-a_{i2}), \pm(a_{i2}-a_{i3}), \pm(a_{i3}-a_{i4})$ cover all non-zero (mod v) points exactly λ^* times.*

There is a cyclic solution for the case for $v \equiv 5 \pmod 6$, and indeed for any odd v, given by the following theorem.

Theorem 13 *Suppose $v \equiv 5 \pmod 6$. Then there exists a $TS(v,3)$ whose blocks under the natural correspondence form a $PATH(v,3,2)$.*

Proof. A suitable difference family which applies the previous Lemma is $\{\{0,i,2i\} : i = 1,\dots,(v-1)/2\}$. It is well-known that this generates a TS$(v,3)$. ■

When $v \equiv 2 \pmod 6$, a TS$(v,6)$ exists, but none with smaller index. Moreover, when $v \equiv 2 \pmod 4$ there is another difficulty.

Lemma 14 *[6] When $v \equiv 2 \pmod 4$ and $\lambda \equiv 2 \pmod 4$ there is no cyclic $TS(v,\lambda)$. If $v \geq 14$ and $v \equiv 2 \pmod{12}$ there is a cyclic $TS(v,12)$. If $v \equiv 8 \pmod{12}$ there is a cyclic $TS(v,6)$.*

The previous lemma and the standard necessary conditions tell us that for any TS$(6t+2,x)$, the index must be a multiple of 6, and cyclic designs exist only for half the cases (t odd). We provide a non-cyclic construction which completes the case and introduce a new type of resolvability.

A BIBD or other design is *α-resolvable* if its blocks can be put into resolution classes or parallel classes in which each point occurs α times. A design is *near α-resolvable* if its blocks can be put into classes so that each class fails to contain exactly one point but contains each other point exactly α times; see for example, [1], [2], [3], [10], and [21]. When $\alpha = 1$, the design is just resolvable or near resolvable.

Resolvability for PATH designs is more subtle since the role of a point in a block is not uniform. We define a PATH(v, k, λ) to be *α -pair-resolvable* if its blocks can be put into classes such that, in every class, each point occurs in α pairs. A PATH(v, k, λ) is near α-pair-resolvable if each class fails to contain exactly one point and contains each other point in α pairs. Of course α will not necessarily be even although always $\alpha \geq 2$.

Example 15 *Two copies of the design Path$(4,3,1)$ give the two resolution classes of a 3-pair-resolvable PATH$(4,3,2)$ which is not resolvable and not balanced.*

We consider resolvability more fully in the next section. For our purposes with respect to tight embeddings, the next example using Table 3 below is helpful.

Example 16 *Three copies of blocks for A in Table 3 easily yield $Z = TS(6,6)$. Let us define classes of blocks $X_1, X_2, ..., X_6$ as follows. Put one copy of the five blocks of A which do not contain 1 in X_1, and in general, put a copy of block $\{a, b, c\}$ in each of the three classes X_t such that $t \neq a, b,$ or c. It is easy to see that X_i is a near resolution class missing i and containing each other point exactly 3 times. So Z is near 3-resolvable. But within each X_i, it is easy to arrange that the 5 blocks contain the 5 points in the class once each as a center point. Now, under the natural correspondence, we see there exists $Z^* =$ Path$(6,3,4)$ which is tightly embedded into Z. Moreover, Z^* is near 4—pair-resolvable.*

1	1	1	1	1	2	2	2	3	3
2	2	3	4	5	3	4	5	4	4
3	4	5	6	6	6	5	6	5	6

Table 3: The blocks for $A =$ TS$(6,2)$.

Example 17 *A 2-pair-resolvable PATH$(6,3,4)$. The ordered blocks for the ten 2-pair-parallel classes are given by columns below. The parameters are the same as for the previous example.*

213	*124*	*135*	*236*	*364*	*416*	*345*	*256*	*214*	*146*
345	*634*	*425*	*156*	*425*	*236*	*321*	*346*	*263*	*135*
562	*156*	*164*	*142*	*315*	*254*	*561*	*213*	*453*	*625*

A *latin square* is an n-by-n array of n elements arranged so that every element appears once in each row and once in each column. A latin square is equivalent to the multiplication table of a quasigroup, and we will refer to the $(i,j)^{th}$ entry as $i \circ j$. Two latin squares of the same order on sets S and T are *orthogonal* if every element in $S \times T$ occurs exactly once among the n^2 pairs $(s_{ij}, t_{ij}), 1 \leq i,j \leq n$. A latin square is *self-orthogonal* if it is orthogonal to its transpose.

Theorem 18 *For every $v \geq 4$, there is a near four-pair-resolvable PATH$(v,3,4)$ which can be tightly embedded into a near three-resolvable TS$(v,6)$.*

Proof. If $n = 6$, the result is in the previous example. For all other n we construct $X = \text{TS}(n,6)$ on the points $N = \{1,2,...,n\}$ as follows. First, we use L, an n-by-n idempotent ($i \circ i = i$) self-orthogonal latin square (SOLS). Such an L exists for all $n \neq 2,3,6$. [22]. The blocks of X are the sets $\{\{a,\ b,\ a{\circ}b\}$: $a \neq b\}$. It is easy to see X has index 6 (and is a triple system). Since equations are uniquely solvable in quasigroups, and by idempotence, $a \circ b \neq a, b$. Thus, for any points a and b, (a,b) occurs once as the first two points of a path block, and once as the second two points of a path block. This is true for (b,a) as well. Hence, the pair $\{a,b\}$ occurs four times in four path blocks. The near resolution classes are $\{R_x$: x in $N\}$, and are given by putting block $\{a,b,a \circ b\}$ in R_x if neither a nor b is x and if $x = b \circ a$. Consider the set of blocks $\{a,b,a \circ b\}$ in R_x. Since x occurs once in every row and column of the latin square, the first two points $\{a,b\}$ in those blocks (with $a \neq b$) cover the points in $V \backslash \{x\}$ exactly twice. We claim the corresponding set of points $a \circ b$ covers this set exactly once more. To see this, first note that $a \circ b$ is never x. If $a \circ b = x$ and $b \circ a = x$ then, when L and its transpose are superimposed, the $(a,b)^{th}$ position will have (x,x). But the ordered pair (x,x) should occur only on the diagonal since L is idempotent. Suppose $\{i_1, j_1, i_1 \circ j_1\}$ and $\{i_2, j_2, i_2 \circ j_2\}$ are blocks in R_x. Now superimpose L and its transpose. At the (j_1, i_1) location we have the pair $(x, i_1 \circ j_1)$ and at the j_2, i_2 location we have the pair $(x, i_2 \circ j_2)$, but by orthogonality, these are different pairs. Hence, $i_1 \circ j_1 \neq i_2 \circ j_2$ and thus the points $V \backslash \{x\}$ appear exactly once in the third position in each block and three times each in R_x, and X is near 3-resolvable. But the previous argument also shows that the path blocks in R_x contain every $a \neq x$ as a pair element exactly four times. ∎

The previous theorem also completes the proof of Theorem 7, the main theorem in this section. In the examples and in the proofs, we constructed tight embeddings only for balanced path designs. This is no accident.

Theorem 19 *If $X^* = PATH(v,k,\lambda^*)$ is tightly embedded in $X = BIBD(v,k,\lambda)$, then X^* is balanced.*

Proof. Since the embedding is tight, each point appears in the same number r of blocks since X is a BIBD. (This is enough.) Moreover, for any point x we have $r_1 + r_2 = r$. But also x is paired in blocks with $r_1 + 2r_2$ points, and this quantity is equal to $\lambda^*(v-1)$. It follows that $r_2 = \lambda^*(v-1) - r$. Thus r_2, and hence r_1, are independent of the choice of x and are constants for the design X^* ■

We now briefly consider tightly embedding a PATH$(v, 4, \lambda^*)$ into BIBD$(v, 4, \lambda)$. If the BIBD is cyclic, we want to array the difference sets so as to satisfy Lemma 12. For example, $\{0, 1, 2, 4\}$ is a difference set which generates blocks for a BIBD(7, 4, 2), but the resulting blocks, as ordered sets, do not give a PATH design. However, the "same" difference set $\{0, 1, 4, 2\}$ generates a BIBD(7, 4, 2), but now the blocks, under the natural correspondence, give a PATH(7, 4, 1). A cyclic balanced PATH(10, 4, 1) which is not a BIBD(10, 4, 2): develop $\langle 1, 10, 2, 9\rangle$ fully, and develop $\langle 1, 5, 10, 6\rangle$ as a short block. In the same vein, the set $\{0, 1, 3, 9\}$ generates a BIBD(13, 4, 1). The two sets $\{0, 1, 3, 9\}$ and $\{3, 0, 9, 1\}$ generate a BIBD(13, 4, 2) whose blocks under the natural correspondence give a PATH(13, 4, 1). We can give a cyclic variation of Lemma 9.

Lemma 20 *If the difference family $\{\{a_{i1}, a_{i2}, a_{i3}, a_{i4}\}\}_{i=1}^{i=t}$ generates blocks for a BIBD$(v, 4, \lambda)$, then there exists a cyclic PATH$(v, 4, \lambda)$ which can be tightly embedded into a cyclic BIBD$(v, 4, 2\lambda)$.*

Proof. Develop each difference set once as presented and once as $\{a_{i2}, a_{i4}, a_{i1}, a_{i3}\}$. ■

Theorem 21 *If $v \equiv 1, 4 \pmod{12}$, then there is a PATH$(v, 4, 1)$ which can be tightly embedded into a BIBD$(v, 4, 2)$.*

Proof. This follows immediately from Lemma 9 since BIBD$(v, 4, 1)$ are known to exist for v $\equiv 1, 4 \pmod{12}$. ■

Conjecture 22 *If a BIBD$(v, 4, 2)$ exists, then the points within the blocks can be ordered so as to create a PATH$(v, 4, 1)$, i.e., a tight embedding. The necessary condition would be that $v \equiv 1 \pmod 3$.*

We have already shown that the conjecture is true for $v = 7$.

Example 23 *The following blocks are those of a PATH(10, 4, 1) which can be tightly embedded into a BIBD(10, 4, 2):*
$\langle 0,1,2,3\rangle$, $\langle 0,7,9,5\rangle$, $\langle 0,9,8,6\rangle$, $\langle 4,0,5,1\rangle$, $\langle 6,0,2,4\rangle$,
$\langle 8,0,3,7\rangle$, $\langle 1,8,7,2\rangle$, $\langle 1,6,5,8\rangle$, $\langle 3,1,9,6\rangle$, $\langle 9,1,4,7\rangle$,
$\langle 2,6,7,5\rangle$, $\langle 9,2,8,4\rangle$, $\langle 9,3,5,2\rangle$, $\langle 3,6,4,7\rangle$, $\langle 5,4,3,8\rangle$.

Example 24 *The starter blocks $\langle 1, 13, 3, 7\rangle$ and $\langle 1, 3, 10, 2\rangle$ generate a cyclic PATH(13, 4, 1) but do not generate a BIBD.*

We conclude this section with a natural generalization of Lemmas 9 and 20.

Theorem 25 *(a) Suppose k is odd and there exists a BIBD(v,k,λ). Then there exists $Y^* =$ PATH$(v,k,2\lambda)$ which can be tightly embedded into $Y =$ BIBD$(v,k,k\lambda)$. (b) Suppose k is even and there exists a BIBD(v,k,λ). Then there exixts $X^* =$ PATH(v,k,λ) which can be tightly embedded into $X =$ BIBD$(v,k,k\lambda/2$*
.

Proof. Suppose that k is odd and that $G = \{a_1, a_2, ..., a_k\}$ is a block of a $Z =$ BIBD(v,k,λ). Identify the elements of the block G with the set $H = \{1, k, 2, k-1, ..., (k+1)/2)\}$. Cyclically develop the set H as a starter block (mod k) and create a set of k blocks. Under the identification, this corresponds to a set of k blocks of elements all from G. Use these in the order as developed to create Y^*, and do this for each block of Z. If k is even, use a short block for the development corresponding to the block $\{1, k, 2, k-1, ..., k/2, k/2+1\}$. ∎

4 α-Resolvability and α-Pair-Resolvability

It is pointed out in [9] that a balanced path design (a handcuffed design) is strictly resolvable only when each class contains v/k blocks. Thus, $v \equiv 0 (\text{mod}\, k)$ is a necessary condition for strict resolvability. Combined with earlier results, resolvable path designs for $k = 3$ require $v \equiv 9 (\text{mod}\, 12)$.

An example of a balanced (cyclic of order 3) resolvable Path$(9,3,1)$ is given in [9], but the cyclic PATH$(9,3,1)$ generated by $\langle 0,1,3\rangle$ and $\langle 0,6,1\rangle$ is not resolvable and is not a TS but it is 4-pair-resolvable.

In fact, any cyclic PATH$(v,3,\lambda)$ is automatically 4-pair-resolvable. The resolution classes are the sets generated by each starter block. More generally, if a resolution exists, then the number of pairs in a resolution class must divide the total number of pairs. From this Hell and Rosa [9] derived a key necessary condition and Horton [10] proved the sufficiency for $k = 3$ (and asymptotically for any k) and Bermond, Heinrich and Yu [2] completed the result:

Lemma 26 *There exists a strictly resolvable balanced PATH(v,k,λ) if and only if $v \equiv 0 (\text{mod}\, k)$ and $v \equiv k^2 (\text{mod}(2k-2)/\gcd(2k-2, \lambda k))$.*

Example 27 *We give $X =$ PATH$(4,3,2)$ which is unbalanced, 2-pair-resolvable, and has $\mu(X) = 3$. The columns are the parallel classes.*

142	*243*	*341*
132	*213*	*321*

Example 28 *An unbalanced 2-pair-resolvable PATH$(6,3,2)$. Note that no $H(6,3,2)$ exists.*

123	*124*	*152*	*615*	*614*
145	*135*	*134*	*623*	*635*
365	*465*	*264*	*345*	*524*

Example 29 *We give an example of an unbalanced $X =$ PATH$(8,3,2)$ with $\mu(X) = 7$ which is 2-pair-resolvable. The blocks are:*

073	*174*	*275*	*376*	*470*	*571*	*672*
013	*124*	*235*	*346*	*450*	*561*	*602*
526	*630*	*140*	*251*	*362*	*403*	*514*
546	*650*	*160*	*201*	*312*	*423*	*534*

Theorem 30 *If X = PATH($v, 3, \lambda$) is 2-pair-resolvable, then v and λ are even.*

Proof. Suppose X = PATH($v, 3, \lambda$) is 2-pair-resolvable. Then in each class, a point is an end-point of a block twice or a mid-point once. In particular, there are $v/2$ blocks per class. Thus v must be even. Now, there are $\lambda v(v-1)/4$ blocks and $v/2$ blocks per class. Hence there are $\lambda(v-1)/2$ classes forcing the index to be even also. ■

The theorem implies, interestingly, that no 2-pair-resolvable PATH(v, 3, 2) is tightly embeddable into any TS(v, 3) since necessarily the TS requires v to be odd.

We make very full use in the proofs that follow of a certain one-factorization of the complete graph K_{2n} and so put into Table 4 below three "consecutive" one-factors. The one-factorization is given by $\{F_i : i = 1, ..., 2n-1\}$, where F_i $= \{\{2n, i\}\} \cup \{\{i-j, i+j\}\}_{j=1}^{j=n-1}$

	1	2	3	4
F_{i+0}	{2n, i}	{i-1,i+1}	{i-2,i+2}	{i-3,i+3}
F_{i+1}	{2n, i+1}	{i, i+2}	{i-1,i+3}	{i-2,i+4}
F_{i+2}	{2n,i+2}	{i+1,i+3}	{i,i+4}	{i-1,i+5}
	5	6	7	8
F_{i+0}	{i-4,i+4}	{i-5,i+5}	{i-6,i+6}	{i-7,i+7}
F_{i+1}	{i-3,i+5}	{i-4, i+6}	{i-5,i+7}	{i-6,i+8}
F_{i+2}	{i-2,i+6}	{i-3,i+7}	{i-4,i+8}	{i-5,i+9}

		n-1	n
F_{i+0}	...	{i-n+2, i+n-2}	{i-n+1, i+n-1}
F_{i+1}	...	{i-n+3, i+n-1}	{i-n+2, i+n}
F_{i+2}	...	{i-n+4, i+n}	{i-n+3, i+n+1}

Table 4

Theorem 31 *The necessary conditions are sufficient for the existence of a 2-pair-resolvable PATH(v, 3, λ).*

Proof. The necessary conditions are that v and λ are even. Suppose $v = 2n$ and $\lambda = 2$. We decompose two copies of K_2n into two copies of the edges $\{F_i : i = 1, ..., 2n-1\}$, where $F_i = \{\{2n, i\}, \{i-1, i+1\}, \{i-2, i+2\}, ..., \{i-(n-1), i+(n-1)\}$. For $i = 1, ..., 2n-1$, we create a closed path listing the edges used alternately from F_i and F_{i+1}. It is necessary that one point be the same in adjacent edges. The edges are referred to as (x, y) where $x = 0$ or 1 to indicate the row in Table 4 and $y = 1, ..., n$ is the column number. The desired path for even n is: $(0,1),(1,2),(0,3),(1,4),\dots,(0,n-1),(1,n),(0,n),(1,n-1),\dots,(0,2),(1,1)$. (The method for odd n is similar.) Indices in blocks are positive and evaluated mod $2n-1$. When subscript $i = 2n-1$, take $2n-1+1$ to mean 1. We may "cut" these closed paths into subpaths of length two and the collection of paths from each forms a resolution class in which each point appears corresponding to two pairs. ■

Theorem 32 *$\mu(v,3,2) = v-1$ and $M(v,3,2) = 2(v-1)$, with or without 2-pair-resolvability.*

Proof. The numbers are clearly optimal for (unbalanced) path designs, and it can easily be arranged in the cutting of subpaths in the construction for 2-pair-resolvable designs that the same vertex is always an endpoint or always a midpoint of some block. ■

Theorem 33 *For any $2n \geq 4$, there exists a $PATH(2n,3,4)$ which is 3-resolvable and 4-pair-resolvable for the same resolution classes.*

Proof. Use two copies of a PATH$(2n,3,2)$ each of which is 2-pair-resolvable formed as above. (This is also 2-pair-resolvable with twice as many classes.) In the second copy of each class, by shifting the cut in forming each resolution class by one segment, arrange that the complimentary vertices are midpoints. Then, forming one class from the two copies, the resulting classes are 3-resolvable and 4-pair resolvable at the same time since each point will appear once as a midpoint and twice as an endpoint. Notice the design is balanced. ■

Suppose $X =$ PATH$(v,3,\lambda)$ is 3-pair resolvable. Then in each class, a point appears 3 times as an end-point of a block or appears once as an end-point and once as a midpoint. PATH$(4,3,1)$ is 3-pair-resolvable (with one class). In general, there are $3v/2$ pairs per class, and thus there are 3v/4 blocks per class. Since the number of blocks is $\lambda v(v-1)/4$, there are $\lambda(v-1)/3$ classes. This is quite general, and we can say:

Theorem 34 *If $X = Path(v,3,\lambda)$ is α-pair-resolvable, then it is necessary that $\lambda(v-1) \equiv 0 (\bmod\, \alpha)$ and $\alpha v \equiv \lambda v(v-1) \equiv 0 (\bmod\, 4)$. In particular, if $\lambda \equiv 1,2 (\bmod\, 3)$, and if $X = PATH(v,3,\lambda)$ is 3-pair-resolvable, then $v \equiv 4 (\bmod\, 12)$. If $\lambda \equiv 0 (\bmod\, 3)$, and if $X = PATH(v,3,3)$ is 3-pair-resolvable, then $v \equiv 0 (\bmod\, 4)$.*

Theorem 35 *The necessary conditions are sufficient for the existence of 3-pair resolvable PATH(v, 3, λ).*

Proof. For the first case $\lambda \equiv 1,2 (\bmod\, 3)$, and it will suffice to consider index 1. Suppose $v = 12n+4$, and X is a resolvable BIBD$(12n+4,4,1)$ [1]. Let C be a resolution class for X. There are $3n+1$ blocks in C. For each block $\{a,b,c,d\}$ in C, form a copy of the PATH$(4,3,1)$ in Section 1. This forms a resolution class C^* for a path design X^*. Do this for each class of X, and

X^* is 3-pair-resolvable. The construction can be doubled for index two. Now, for case 2, we have $\lambda = 3$ and assume $v = 2n$ for n even. We decompose the edges of three copies of K_{2n} into the same one-factors as previously (see Table 4). For $i = 1, \ldots, n$ we make a resolution class from F_i, F_{i+1}, and F_{i+2}. (The subscripts on the F's are to be taken mod n.) Instead of having one long cycle, however, we make path blocks from the edges in four columns at a time. The first four columns of edges in Table 4 will make the edges: $\langle(0,1),(1,2)\rangle$, $\langle(2,3),(1,4)\rangle$, $\langle(0,3),(2,1)\rangle$, $\langle(1,1),(2,2)\rangle$, $\langle(1,3),(0,4)\rangle$, $\langle(0,2),(2,4)\rangle$. The next four columns will also make six path blocks using exactly the same sequence of edges (adding 4 to each column index). If the even n is a multiple of 4, then there are $4c$ columns for some c and we obtain $6c$ blocks in the class by repeating the pattern of new blocks. When n is 2 mod 4, the last two columns must be used to make 3 blocks, and this can be done as follows: $\langle(0,n-1),(1,n)\rangle$, $\langle(1,n-1),(2,n)\rangle$, $\langle(2,n-1),(0,n)\rangle$. For this last block (see Table 4), recall $i+n \equiv i-n+1 (\bmod 2n-1)$. ∎

Theorem 36 *If $X = PATH(v,3,\lambda)$ is near α– pair-resolvable then it is necessary that $\alpha = \lambda$ and $\lambda v(v-1) \equiv \alpha(v-1) \equiv 0 (\bmod 4)$.*

Proof. There are α pairs per point in a near resolution class which has v-1 points. Consequently there are $\alpha(v-1)/2$ pairs per class and $\alpha(v-1)/4$ blocks per class. This shows $\alpha(v-1) \equiv 0(\bmod 4)$. But there are v classes and hence $\alpha v(v-1)/4$ blocks in all. Since we know there are $\lambda v(v-1)/4$ blocks, $\lambda = \alpha$. ∎

In particular, if $X = \mathrm{PATH}(v,3,\lambda)$ is near 2-pair-resolvable, then $\lambda = 2$ and v is odd.

Example 37 *A near 2-pair-resolvable $PATH(5,3,2)$ such that class X_i misses point i.*

X_1	X_2	X_3	X_4	X_5
425	*134*	*125*	*123*	*132*
534	*451*	*541*	*351*	*241*

Example 38 *A near 2-pair-resolvable $PATH(7,3,2)$ such that class X_i misses i.*

X_1	X_2	X_3	X_4	X_5	X_6	X_7
526	*435*	*457*	*132*	*173*	*135*	*215*
637	*164*	*214*	*156*	*164*	*147*	*243*
547	*175*	*267*	*276*	*324*	*527*	*365*

We define a near-hamilton cycle in a graph to be a cycle which misses exactly one vertex.

Theorem 39 *Suppose v is odd. The necessary conditions are sufficient for the existence of near 2-pair-resolvable $PATH(v,3,\lambda)$.*

Proof. Let $v = 2m + 1$. Decompose $2K_v$ into near-hamilton cycles. This is always possible [5]. Appropriate cycles are given by $C_{2m} = \langle 0, 1, 2, ..., 2m - 1 \rangle$, and $C_{i+m} = \langle i, 1 + i, 2m - 1 + i, 2 + i, 2m - 2 + i, 3 + i, ..., m - 2 + i, m + 2 + i, m + 1 + i, 2m \rangle$, for $i = 0, 1, 2, ..., 2m - 1$. The subscripts and indices with i are calculated mod 2m. The near resolution classes are the sets of blocks made by cutting the cycles into paths. Class C_j misses point j for $j = 0, 1, 2, ..., 2m$. ∎

Theorem 40 *The necessary conditions are sufficient for the existence of a near 4-pair-resolvable PATH$(v, 3, \lambda)$.*

Proof. This follows from the necessary conditions (any v and $\lambda = 4$) and the proof of Theorem 18. ∎

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