Group Magicness of Complete N–partite Graphs

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Abstract

Let A be a non-trivial abelian group. We call a graph $G = (V, E)$ A-magic if there exists a labeling $f : E(G) \to A \setminus \{0\}$ such that the induced vertex set labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where the sum is over all $(u, v) \in E(G)$, is a constant map. In this paper, we show that $K_{k_1,k_2,\dots,k_n}$ $(k_i \geq 2)$ is A-magic, for all A where $|A| \geq 3$.

1 Introduction.

Let $G = (V, E)$ be a connected, simple graph. For any nontrivial abelian group A (written additively), let $A^* = A \setminus \{0\}$. A function $f : E(G) \to A^*$ is called a *labeling* of G. Any such labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where the sum is over all $(u, v) \in E(G)$. If there exists a labeling f whose induced map on $V(G)$ is a constant map, we say that f is an *A-magic labeling* and that G is an *A-magic* graph. The *integer-magic spectrum* of a graph G is the set $\mathrm{IM}(G) = \{k : G \text{ is } \mathbb{Z}_k\text{-magic and } k \geq 1\}$. By convention, $\mathbb{Z}$-magic graphs are considered to be $\mathbb{Z}_1$-magic.

$\mathbb{Z}$-magic graphs were considered by Stanley [19,20], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [1,2,3] and others [7,8,14] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [4,6,9,10,11, 12,13].

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Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A-magic graph is due to J. Sedláček [15,16], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [5] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. The interested reader is directed to Wallis' [21] recent monograph on magic graphs.

2 Basic definitions and notation.

In the study of edge-magic labelings, Shiu, Lam and Lee [17] introduced the following notation. Suppose $f : E(G) \to X$ is a mapping (i.e., an edge labeling of G). The *labeling matrix* for f, denoted by $\mathcal{L}_f(G)$, is the matrix whose rows and columns are named by the vertices of G and defined in the following way: the (u, v)-entry is $f(uv)$ if $uv \in E$, and is $*$ otherwise. If f is an A-magic labeling of G, then $\mathcal{L}_f(G)$ is an *A-magic labeling matrix* of G. Note that the row sum of an A-magic labeling matrix is the A-magic value corresponding to the labeling f.

Thus, finding an A-magic labeling of G is equivalent to finding an A-magic labeling matrix $\mathcal{L}_f(G)$, where each row sum (as well as column sum) is the same constant value. In the context of row and column sums, entries with an $*$ are treated as 0.

A graph is called *fully magic* if it is A-magic, for every abelian group A. A graph is called *non-magic* if for every abelian group A, it is not A-magic.

In this paper, we analyze the group-magicness property for the class of complete n-partite graphs.

3 Main results.

First, let us make a few observations. They are straight-forward to verify and can be found in [14].

Observations:

1. A graph G is $\mathbb{Z}_2$-magic iff every vertex of G is of the same parity.
2. An eulerian graph G having an even number of edges is A-magic.
3. If A_1 is a subgroup of A and graph G is A_1-magic, then G is A-magic.

We now characterize the abelian groups A, for which $K_{m,n}$ is A-magic. Let f be a labeling of the complete bipartite graph $K_{m,n}$. Then, $\mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_m & B \\ B^T & \bigstar_n \end{pmatrix}$, where B is an $m \times n$ matrix, $\bigstar_m$ and $\bigstar_n$ are square matrices of order m and n respectively with all entries being $*$.

Theorem 1. *Let m and n be even. Then, $K_{m,n}$ has an A–magic labeling with magic value 0, for all A.*

Proof. Suppose $a \in A \setminus \{0\}$. Let S be an $m \times n$ matrix defined by $S_{i,j} = (-1)^{i+j}a$, where $S_{i,j}$ denotes the (i,j)–entry of S. Then, the row sums and the column sums of S are zero. Clearly, $\begin{pmatrix} \bigstar_m & S \\ S^T & \bigstar_n \end{pmatrix}$ is an A–magic labeling matrix of $K_{m,n}$, with A–magic value 0. □

Definition. The matrix S defined in the proof above is called an $m \times n$ *zero-sum $(a, -a)$-matrix.*

The integer–magic spectrum of $K_{1,n}$ has been found [11]. For convenience, we state the result here.

Theorem B. *$K_{1,1}$ is fully magic and $K_{1,2}$ is non–magic. For $n \geq 3$,*

$$\mathrm{IM}(K_{1,n}) = \bigcup_{p|(n-1)} p\mathbb{N}.$$

It is straight–forward to verify the following lemma.

Lemma 1. *For $n \geq 3$, $K_{1,n}$ is V_4–magic if and only if n is odd.*

So we may assume $m \geq 3$.

Lemma 2. *Let A be an abelian group of order at least 3. Then, there exist $a, b, c \in A \setminus \{0\}$ (not necessarily distinct) such that $a + b + c = 0$.*

Proof. It suffices to consider three cases, namely: $A = \mathbb{Z}$, $\mathbb{Z}_k$ for $k \geq 3$, or V_4.
If $A = \mathbb{Z}$, then it is obvious. If $A = \mathbb{Z}_k$, then choose $a = b = 1$ and $c = -2$. If $A = V_4$, then choose $a = (1,0)$, $b = (0,1)$ and $c = (1,1)$. □

Theorem 3. *Suppose m is odd, with $m \geq 3$ and $n \geq 2$. For any abelian group A where $|A| \geq 3$, $K_{m,n}$ has an A–magic labeling with magic value 0.*

Proof. Let $a, b, c \in A \setminus \{0\}$ be chosen in the same manner as discussed in the proof of Lemma 2.

CASE 1. n is even.
Let $B = \begin{pmatrix} C \\ D \end{pmatrix}$, where C is an $(m-3) \times n$ zero–sum $(a, -a)$–matrix and D is a $3 \times n$ matrix defined by

$$D_{i,j} = \begin{cases} (-1)^j a & \text{if } i = 1; \\ (-1)^j b & \text{if } i = 2; \\ (-1)^j c & \text{if } i = 3. \end{cases}$$

Note that if $m = 3$, then C does not appear. Then, $\mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_m & B \\ B^T & \bigstar_n \end{pmatrix}$ is an A–magic labeling matrix of $K_{m,n}$, for $A = \mathbb{Z}, \mathbb{Z}_k$ ($k \geq 3$), and V_4. By

Observation 3, $K_{m,n}$ is A–magic, for all A where $|A| \geq 3$.

CASE 2. n is odd.
Then, $n \geq 3$. Let B be a matrix of the following form:

$$B = \begin{pmatrix} C_1 & D_1^T \\ D_1 & E \end{pmatrix},$$

where C_1 is an $(m-3) \times (n-3)$ zero–sum $(a, -a)$–matrix, D_1 is a $3 \times (n-3)$ matrix defined similarly as in the proof of the previous case, and E is a Latin square of order 3 defined as follows:

$$E = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}.$$

Then, $\mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_m & B \\ B^T & \bigstar_n \end{pmatrix}$ is an A–magic labeling matrix of $K_{m,n}$, for $A = \mathbb{Z}, \mathbb{Z}_k$ $(k \geq 3)$, and V_4. By Observation 3, $K_{m,n}$ is A–magic, for all A where $|A| \geq 3$.

It is clear that the row sums and the column sums of these A–magic labeling matrices are zero. □

Here a few examples which illustrate Theorem 3.

EXAMPLE 1. $m = 3$ and $n = 4$. Then,

$$B = \begin{pmatrix} -a & a & -a & a \\ -b & b & -b & b \\ -c & c & -c & c \end{pmatrix}, \text{and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_3 & B \\ B^T & \bigstar_4 \end{pmatrix}.$$

EXAMPLE 2. $m = 3$ and $n = 3$. Then,

$$B = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}, \text{and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_3 & B \\ B^T & \bigstar_3 \end{pmatrix}.$$

EXAMPLE 3. $m = 5$ and $n = 4$. Then,

$$B = \begin{pmatrix} a & -a & a & -a \\ -a & a & -a & a \\ -a & a & -a & a \\ -b & b & -b & b \\ -c & c & -c & c \end{pmatrix}, \text{and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_5 & B \\ B^T & \bigstar_4 \end{pmatrix}.$$

EXAMPLE 4. $m = 5$ and $n = 2$. Then,

$$B = \begin{pmatrix} a & -a \\ -a & a \\ -a & a \\ -b & b \\ -c & c \end{pmatrix}, \text{and } \mathcal{L}_f(K_{m,n}) = \begin{pmatrix} \bigstar_5 & B \\ B^T & \bigstar_2 \end{pmatrix}.$$

We conclude by showing that $K_{k_1,k_2,...,k_n}$ ($k_i \geq 2$) is A–magic, for all A where $|A| \geq 3$. First, recall the following definitions and notation.

Definition. A graph G is *n–partite*, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, ..., V_n$ such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$.

Definition. A *complete n–partite graph* G is an n–partite graph with partite sets $V_1, V_2, ..., V_n$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$.

Notation. A complete n–partite graph G with partite sets $V_1, V_2, ..., V_n$, where $|V_i| = k_i$, is denoted by $K_{k_1,k_2,...,k_n}$.

We now establish the following result.

Theorem 4. *For $n \geq 2$, the complete n-partite graph $K_{k_1,k_2,...,k_n}$ with $k_i \geq 2$, is A-magic, for all A where $|A| \geq 3$.*

Proof. There are $\binom{n}{2}$ ways of choosing a pair from the partite sets $V_1, V_2, ..., V_n$. For each pair, apply a labeling on the corresponding edge set, using either Theorem 1 or Theorem 3. □

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