On $Q(a)P(b)$-Super Edge-graceful Graphs

Dharam Chopra
Department of Mathematics and Statistics
Wichita State University
Wichita, Kansas 67260

Sin-Min Lee Department of Computer Science
San Jose State University
San Jose, California 95192

Dedicated to Professor R. Stanton

Abstract

Let a, b be two positive integers, for the graph G with vertex set $V(G)$ and edge set $E(G)$ with $p = |V(G)|$ and $q = |E(G)|$, we define two sets $Q(a)$ and $P(b)$ as follows:

$Q(a) = \{\pm a, \pm (a+1), ..., \pm(a + \frac{q-2}{2})\}$ if q is even

$Q(a) = \{0\} \cup \{\pm a, \pm(a+1), ..., \pm(a + (q-3)/2\}$ if q is odd.

$P(b) = \{\pm b, \pm(b+1), ..., \pm(b + (p-2)/2)\}$ if p is even

$P(b) = \{0\} \cup \{\pm b, \pm(b+1), ..., \pm(b + (\frac{p-3}{2})/2\}$ if p is odd

For the graph G with $p = |V(G)|$ and $q = |E(G)|$, G is said to be $Q(a)P(b)$-super edge-graceful (in short $Q(a)P(b)$-SEG), if there exists a function pair (f, f^+) which assigns interger labels to the vertices and edges; that is, $f^+ : V(G) \to P(b)$, and $f : E(G) \to Q(a)$ such that $f^+$is onto P and f is onto Q, and $f^+(u) = \sum\{f(u,v) : (u,v) \in E(G)\}$. We investigate $Q(a)P(b)$ super edge-graceful graphs.

1 Introduction

All graphs in this paper are finite simple graphs with no loops or multiple edges. A graph G with p vertices and q edges is graceful if there is an injective mapping $f : V \to \{0, 1, ..., q\}$ such that $f^* : E(G) \to \{1, 2, ..., q\}$ defined by $f^*(e) = |f(u) - f(v)|$ where $e = (u, v)$, is surjective.

Graceful graph labelings were first introduced by Alex Rosa (around 1967) as means of attacking the problem of cyclically decomposing the complete graph into other graphs. A well-known conjecture of Ringel and Kotzig is that all trees are graceful. Since Rosa's original article, more than six hundred papers have been written on graph labelings (see [2]).

Another dual concept of graceful labeling on graphs, edge-graceful labeling, was introduced by S.P. Lo [18] in 1985. G is said to be edge-graceful if the edges are labeled by $1, 2, 3...q$ so that the vertex sums are distinct, mod p.

A necessary condition of edge-gracefulness is (Lo[18])

$$q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p} \tag{1}$$

Lee [8] has proposed the following tantalizing conjecture:

Conjecture 1: The Lo condition (1.1) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of the above (Lee[7]) has also not yet been proved:

Conjecture 2: All odd-order trees are edge-graceful.

In [1,4,7,11,19] several classes of trees of odd order are proved to be edge-graceful. In [5], the following conjecture is proposed.

Conjecture 3: All odd-order unicyclic graphs are edge-graceful.

In [19] in order to work on Conjecture 2, Mitchem and Simoson introduced the concept of super edge-graceful graphs. Next, we want to generalize this concept in more general context.

Let a, b be two positive integers, for the graph G with vertex set $V(G)$ and edge set $E(G)$ with $p = |V(G)|$ and $q = |E(G)|$, we define two sets $Q(a)$ and $P(b)$ as follows:

$Q(a) = \{\pm a, \pm(a+1), ..., \pm(a+(q-2)/2)\}$ if q is even
$Q(a) = \{0\} \cup \{\pm a, \pm(a+1), ..., \pm(a+(q-3)/2\}$ if q is odd.
$P(b) = \{\pm b, \pm(b+1), ..., \pm(b+(p-2)/2)\}$ if p is even
$P(b) = \{0\} \cup \{\pm b, \pm(b+1), ..., \pm(b+(p-3)/2)\}$ if p is odd

Definition 1 *A (p,q)-graph G is said to be a $Q(a)P(b)$ super edge-graceful graph if there exists a function pair (f, f^+) which assigns interger labels to the vertices and edges; that is, $f : E(G) \to Q(a)$ and $f^+ : V(G) \to P(b)$, such that f^+ and f are bijective, and $f^+(u) = \sum\{f(u,v) : (u,v) \in E(G)\}$.*

When $a = b = 1$, the $Q(1)P(1)$-super edge graceful graphs are identical to the concept of super edge-graceful graphs which were introduced by J. Mitchem and A. Simoson [19]. We illustrate the above concepts with several examples

Example 1. The path P_4 and the ring worm $U_3(1,0,0)$ are not $Q(a)P(b)$-super edge-graceful for any intergers $a, b \geq 1$.

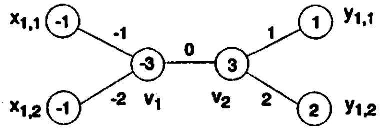
Example 2. Figure 1 shows that the path P_5 is $Q(1)P(1)$-super edge-graceful.

Figure 1:

The double star $D(m, n)$ is a tree of diameter three such that there are m appended edges on one end of P_2 and n appended edges on the other end

Example 3. The double star $D(m, 1)$ is not $Q(a)P(b)$-SEG for any $a, b \geq 1$ if m is odd. For $D(m, 1)$ has even number of vertices, its number of edges is odd. Then the edge (v_1, v_2) will have label 0. No matter what label $(v_2, y_{1,1})$ will receive, it will contribute identical labels on v_2 and $y_{1,1}$.

The double star $D(2, 2)$ is $Q(1)P(1)$-super edge-graceful (see [10]) but not $Q(a)P(b)$-super edge-graceful for any $a, b \geq 2$. See Figure 2.



D(2,2) is Q(1)P(1)-SEG

Figure 2:

The double star $D(3, 1)$ is not $Q(a)P(b)$-super edge-graceful for any $a, b \geq 1$.

The double star $D(3, 2)$ is $Q(a)P(a)$-super edge-graceful for any $a \geq 1$. (For the proof see Theorem 3.7.)

Example 4. The cycle C_3 is $Q(a)P(a)$-SEG for any $a \geq 1$. However, C_5 is $Q(a)P(1)$-SEG for $a = 1, 2$. See Figure 3.

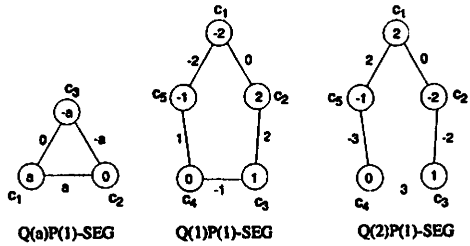


Figure 3:

Example 5. The graph in Figure 4 is $Q(1)P(1), Q(1)P(2)$ and $Q(2)P(2)$-SEG.

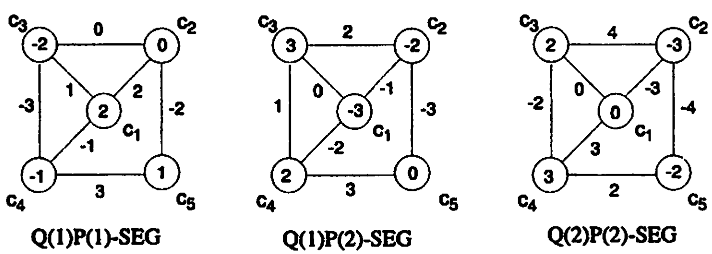


Figure 4:

2 Graphs which are $Q(a)$-super edge-graceful for all a.

In this section we show that there are infinitely many graphs which are $Q(a)P(1)$-SEG for all a.

Theorem 1 *The Eulerian* $(5, 6)$*-graph displayed in Figure 5 is* $Q(a)P(1)$ *for all* $a \geq 1$.

Theorem 2 *The* $(5, 7)$*-graph depicted in Figure 6 is* $Q(a)P(1)$*-SEG for all* $a \geq 2$.

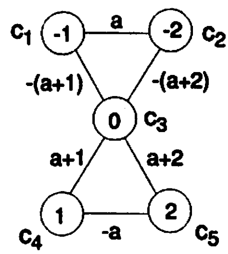


Figure 5:

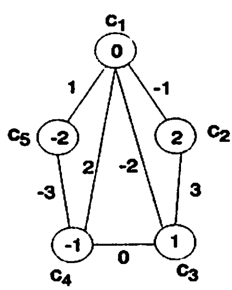


Figure 6:

Proof. If we replace ± 1 by $\pm a, \pm 2$ by $\pm(a+1)$ and ± 3 by $\pm(a+2)$ in the above labeling, we have a $Q(a)P(1)$-SEG labeling. ■

Theorem 3 *The* $(7,6)$*-graph displayed in Figure 7 is* $Q(a)P(a-1)$*-SEG for all* $a \geq 1$.

3 Strongly Super edge-graceful graphs

We introduce the following concept.

Definition 2 *A graph* G *is called strongly super edge-graceful if it is* $Q(a)P(a)$*-SEG for all* $a \geq 1$.

In this section, we provide several families of strongly super edge-graceful graphs.

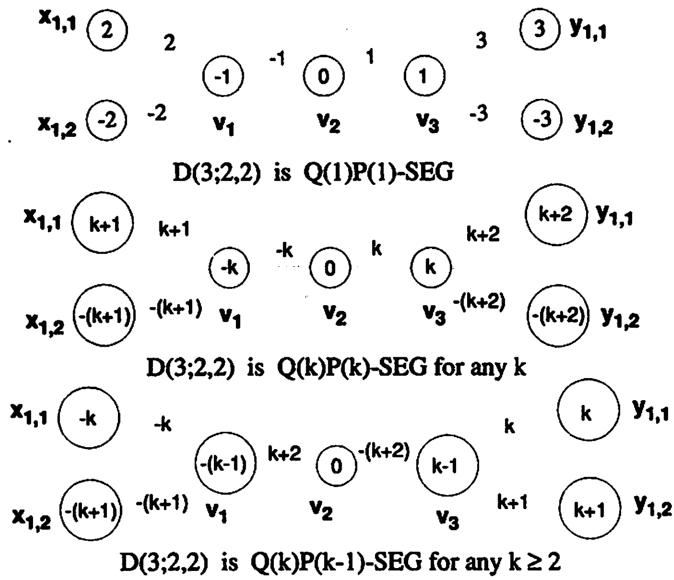


Figure 7:

Theorem 4 *A star $St(n)$ is strongly super edge-graceful if and only if n is even.*

Proof. If n is even then $Q(a)$ has even number of elements for any $a \geq 1$. Thus any bijection $f : E(St(n)) \to Q(a)$ will induce a bijection $f^+ : V(St(n)) \to P(a)$. Thus $St(n)$ is strongly super edge-graceful.

If n is odd, say $n = 2s+1$ then for any $a, b \geq 1$, we see that any bijection $f : E(ST(n)) \to \{0, a, a+1, ..., a+s-1, -a, -(a+1), ..., -(a+s-1)\}$ is not a $Q(a)P(b)$-SEG labeling for there are two vertices with labels 0. ∎

For $n > 3$, the wheel on n vertices, W_n is a graph with n vertices $x_1x_2, ...x_n, x_1$ having degree $n-1$ and all the other vertices having degree 3. The vertex x_1 is adjacent to all the other vertices, and for $i = 2, ..., n-1, x_i$ is adjacent to x_{i+1}, and $x_{n-1}$is adjacent to x_n. The edges of a wheel which include the hub are called spokes. In a wheel graph, the hub has degree $n-1$, and other nodes have degree 3. The edges in $\{(c_i, c_1) : i = 2, ..., n-1\}$ are spoke edges. The edges in $\{(c_i, c_{i+1}) : i = 2, ..., n-2\} \cup \{(c_{n-1}, c_2)\}$ are rim edges.

Theorem 5 *The wheel W_n is strongly super edge-graceful if n is odd.*

Proof. Suppose n is odd, we see that W_n has n vertices and $2n-2$ edges. We label the spoke edges of the wheel by $1, -1, 2, -2, ..., (n-1)/2, -(n-1)/2$ and the rim edges $(c_{n-1}, c_2), (c_2, c_3), (c_3, c_4), ..., (c_{n-2}, c_{n-1}), (c_{n-1}, c_n)$ by $(n-1)/2+1, -[(n-1)/2+1], (n-1)/2+2, -[(n-1)/2+2], ..., (n-1), -(n-1)$. We have a $Q(1)P(1)$-SEG labeling. See Figure 8 for $n = 5$ and 7.

Now for any $k \geq 1$, we replace any positive label x in the above labeling by $k+x$ and any negative label $-x$ by $-(k+x)$, we obtain a $Q(k+1)P(k+1)$-SEQ labeling. Hence the wheel W_n is strongly super edge-graceful. ■

Example 6. Figure 8 shows W_5 is $Q(1)P(1)$ and $Q(2)P(2)$-SEG, and W_7 is $Q(1)P(1)$ and $Q(5)P(5)$-SEG.

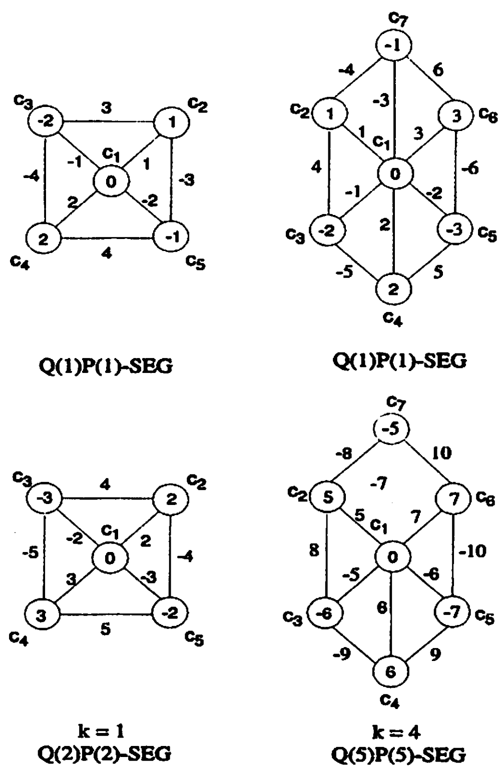


Figure 8:

Remark. The wheel W_4 is not $Q(a)P(b)$-SEG for any $a, b \geq 1$.

Theorem 6 *The corona $C_n \copyright K_1$ of a cycle C_n is stronly super edge-graceful for any $n \geq 3$.*

Proof. Let C_n be a cycle with the vertex set $\{x_1, x_2, ..., x_n\}$ and assume $V(C_n \copyright K_1) = \{x_1, x_2, ..., x_n\} \cup \{y_1, y_2, ..., y_n\}$.

Suppose the edge set $E(C_n \copyright K_1) = \{(x_i, x_{i+1}) : i = 1, 2, ..., n-1\} \cup \{(x_n, x_1)\} \cup \{(x_i, y_i) : I = 1, 2, ..., n\}$ Consider the following labeling

$f((x_i, x_{i+1})) = i$, for $i = 1, 2, ..., n-1$

$f((x_n, x_1)) = n$ and

$f((x_i, y_i)) = -1$, for $i = 1, 2, ..., n$..

We can see that the vertices in C_n have labels $\{n, 1, 2, 3, ..., n-1\}$ and the vertices in $\{y_1, y_2, ..., y_n\}$ have labels $\{-1, -2, ..., -n\}$. Figure 9 depicts the super edge-graceful labeling for $C_n \copyright K_1, n = 3$ and 4.

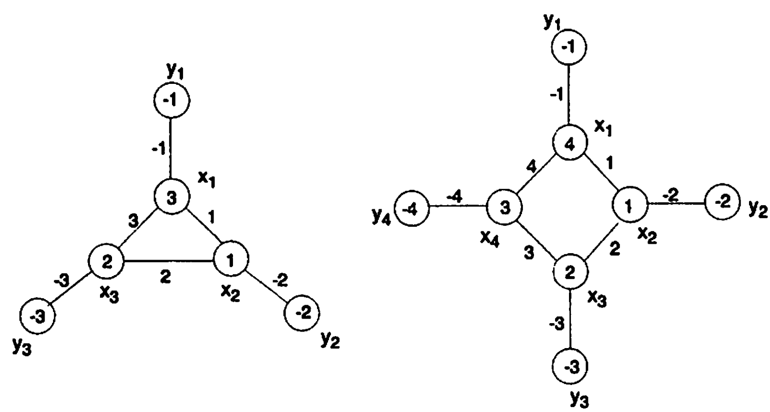


Figure 9:

Now for any $k \geq 1$, we replace each positive edge label x by $x + k$ and each negative edge label by $-(x + k)$ in the above edge labeling, we observe that the resulting and its induced labeling form a $Q(k+1)P(k+1)$-SEG labeling. ■

Theorem 7 *Among three trees of order 5, only two of them are strongly SEG.*

Proof. The path P_5 is $Q(1)P(1)$-super edge-graceful but not $Q(a)P(a)$-SEG for a ≥ 2. The star $St(4)$ is strongly SEG by Theorem 3. Another tree Y is strongly SEG follows from the labelings depicted in Figure 10.

Theorem 8 *Among five unicyclic graphs of order 5, only four of them are strongly SEG.*

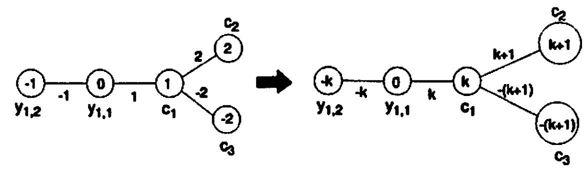


Figure 10:

Proof. The cycle C_5 is $Q(1)P(1)$-super edge-graceful with the edge labels $0, -2, 1, -1, 2$ along the cycles. However, it is not $Q(a)P(a)$-SEG for $a \geq 2$. The other four unicyclic graphs are strongly SEG are shown in Figure 11.

Theorem 9 *The double star $D(m,n)$ is strongly super edge-graceful for (1) all odd numbers $m \geq 3$ and even number $n \geq 2$ or $(2) m, n$ are odd and $m \geq 3$.*

Proof. (1) Suppose $m = 2t+1$ and $n = 2s$. We label the edges of $D(1,1)$ by -1 and 1. Then we amalgamate $D(1,1)$ with $(St(2t), c)$ on v_1, with the edge labels $\{\pm 2, \pm 3, ..., \pm(t+1)\}$. Then we amalgamate the resulting tree with $(St(2s), c)$ on v_2, with the edge labels $\{\pm(t+2), \pm(t+3), ..., \pm(t+s+2)\}$. We observe this is a $Q(1)P(1)$-SEG labeling for $D(m,n)$. For any $k \geq 2$, we replace the above edge label $+x$ by $x+k$ and negative edge label $-x$ by $-(x+k)$, we obtain a $Q(k)P(k)$-SEG labeling. Thus $D(m,n)$ is strongly SEG.

(2) If $m = 2t+1$ and $n = 2s+1$, then we see that $D(m,n)$ has even number of vertices. We observe that $Q(1) = \{0, \pm 1, \pm 2, ..., \pm(t+s+1)\}$. We label the edge (v_1, v_2) by 0. Then find a subset $\Im$ of $\{\pm 1, \pm 2, ..., \pm(t+s+1)\}$ consisting of m numbers with sum $t+s+2$. Then the complement $\Im^*$ of $\Im$ *in* $\{\pm 1, \pm 2, ..., \pm(t+s+1)\}$ will have sum $-(t+s+2)$. We label m pendant edges of v_1 by numbers in $\Im$ and label n pendant edges of v_2 by numbers in $\Im^*$, it will produce a $Q(1)P(1)$-SEG labeling. Using the same argument as in (1), we can show that $D(m,n)$ is $Q(a)P(a)$-SEG for any $a \geq 1$.

Example 7. $D(3,5)$ is $Q(k)P(k)$-SEG for all $k \geq 1$, see Figure 12.

Remark. Mitchem and Simoson [19] showed that $D(n,n)$ is $Q(1)O(1)$-SEG. We see that $D(2,2)$ is not $Q(2)P(2)$-SEG. Furthermore, $D(2,4)$ is $Q(1)P(1)$ and $Q(2)P(2)$-SEG but not $Q(3)P(3)$-SEG.

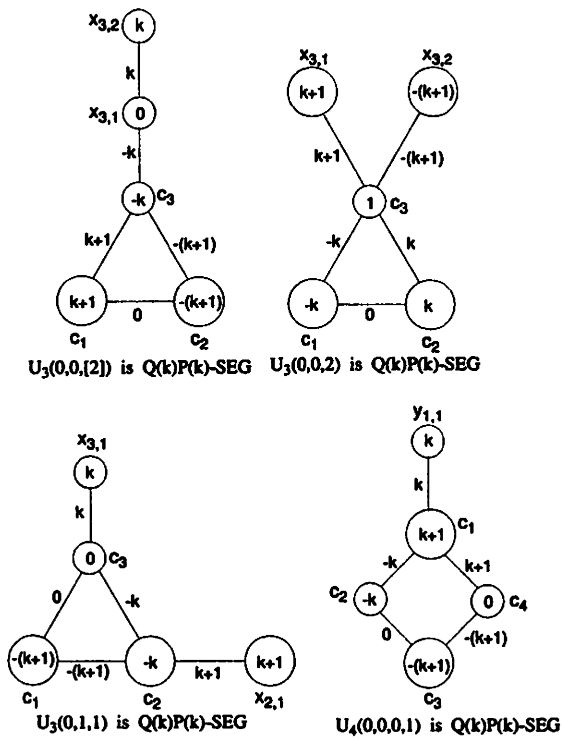


Figure 11:

4 Construction of $Q(a)P(b)$-Super Edge-graceful Graphs

Staton and Zarnke in [23], used the amalgamation method to construct new graceful tree from two given graceful trees. We will use a similar construction to obtain new $Q(a)P(b)$-SEG graphs.

We denote the class of all $Q(a)P(b)$-super edge-graceful graphs by the notation $SEG(a, b)$. Then $\cap\{SEG(k, k) : k = 1, 2, ...\}$ is the class of all strongly SEG graphs

Let $\mathfrak{R}$ be the class of all strongly SEG graphs of odd orders. For each G in $\mathfrak{R}$ and any $k > 1$, there exists a $Q(k)P(k)$-SEG labeling ℓ and a unique vertex u with $\ell(u) = 0$.

Construction 1. Assume $H \in SEG(a, b)$ and $|V(H)| = |E(H)| = 2t$. We also assume max $Q(b)$ =max $P(a) = t$.

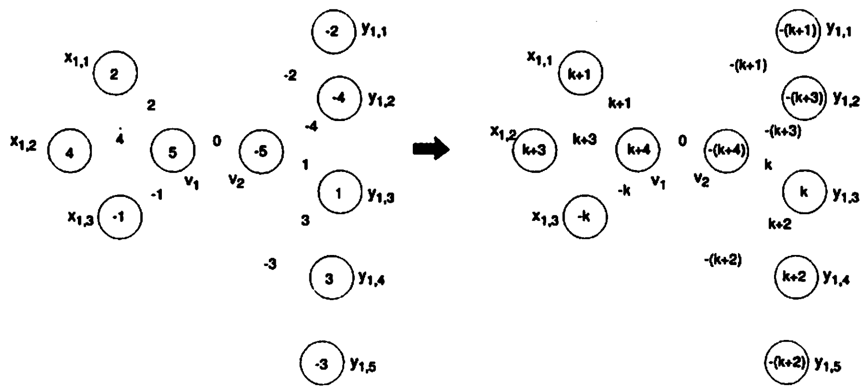


Figure 12:

Let $G \in \mathfrak{R}$ which is $(G, \ell) \in Q(t+1)P(t+1)$ is SEG with labeling (ℓ, ℓ^+) such that $\ell(w) = 0$ for some $w \in V(G)$. For any $v \in V(H)$, we form the union of H and G, add a new edge (v, w) and label the new edge 0. We denote the final graph by $\sum(H, G, T(v, w))$.

Now take the one point union of H and $\{G_v : v \in S\}$ and then identify v with w_v. We denote the final graph by $Amal(H, S, \phi)$.

Theorem 10 *The graph $Amal(H, S, \phi)$ is $Q(a)P(b)$-SEG.*

Example 8. Figure 13 illustrates two such constructions.

Construction 2. $p = 2t, q = 2t + 1$.

Assume $H \in SEG(a, b)$ and $p = |V(H)| = 2t, q = |E(H)| = 2t + 1$. We also assume max $Q(b)$ =max $P(a) = t$. For any $v \in V(H)$, and any $G \in \mathfrak{R}$, we will construct a $Q(a)P(b)$-super edge-graceful graphs from H and ϕ as follows:

We have $G \in \mathfrak{R}$, w is the unique vertex in G with $\ell(w) = 0$. Now take the one point union of (H, v) and (G, w) by moving the graph G to H by submerging vertex w into v. We denote the final graph by $Amal((H, v), (G, w))$.

Theorem 11 *The graph $Amal((H, v), (G, w_v))$ is $Q(a)P(b)$-SEG.*

Example 9. The graph G in Figure 14 has 6 vertices and 7 edges, which is $Q(1)P(1)$-SEG. The wheel H is strongly SEG with 5 vertices, with $\ell(c_1) = 0$. We form the one-point union and obtain another $Q(1)P(1)$-SEG graph.

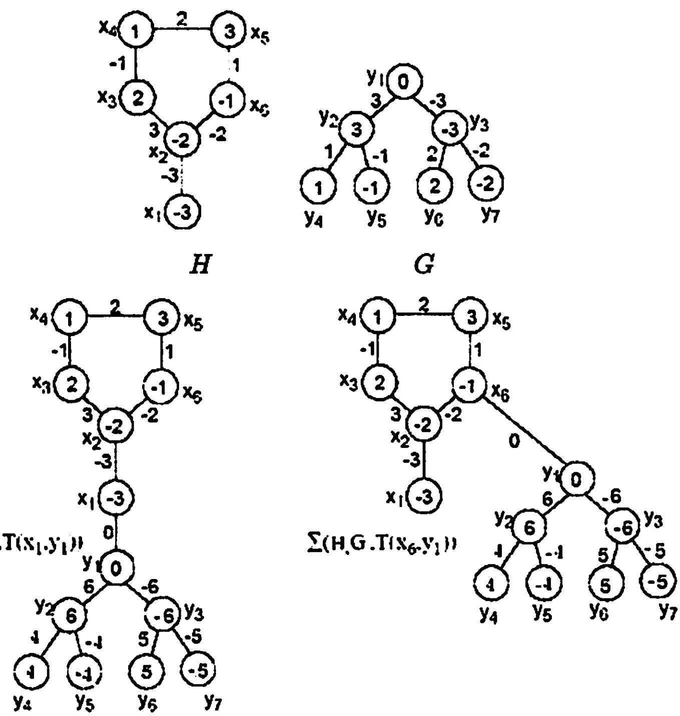


Figure 13:

Construction 3. $p = 2t + 1, q = 2t$.

Assume $H \in SEG(a, b)$ with $p = 2t + 1$ and $q = 2t$. We also assume $\max Q(t) = \max P(t) = t$. Let (ℓ, ℓ^+) be a $Q(a)P(b)$-SEG mapping with $\ell(v) = 0$ for some v in $V(H)$. Let $G \in \mathfrak{R}$ which is $Q(t+1)P(t+1)$-SEG with (ℓ, ℓ^+) labeling. Assume $u \in V(G)$ and $\ell(u) = 0$. Then $Amal(H, G, (v, u))$ is $Q(a)P(b)$-SEG.

Example 10. See Figure 15 for examples of such a construction.

Construction 4. $p = 2t + 1, q = 2t + 1$.

Assume $H \in SEG(a, b)$ and $p = |V(H)| = 2t + 1, q = |E(H)| = 2t + 1$. We also assume max $Q(b)$ =max $P(a) = t$. For any $S \subseteq V(H)$, a mapping ϕ:S$\rightarrow \mathfrak{R}$, we will construct a $Q(a)P(b)$-SEG graph from H and ϕ as follows:

For each v in S, let $G_v = \phi(v)$ where $(\phi(v), \ell_v) \in \mathfrak{R}$ and $w_v$is the distinct vertex with $\ell_v(w_v) = 0$. Assume $S = \{v_1, v_2, \ldots, v_k\}$. We form successively

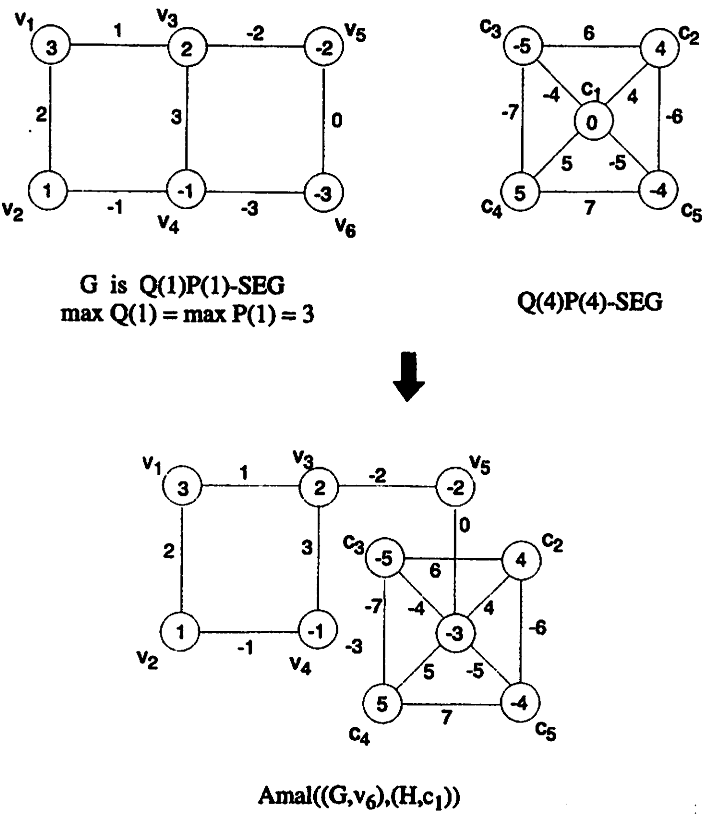


Figure 14:

$H_1, H_2, \ldots, H_k$ as follows:

$$\begin{aligned} H_1 &= Amal((H, v_1), (\phi(v_1), w_{v_1})), \\ H_2 &= Amal((H_1, v_2), (\phi(v_2), w_{v_2})), \\ &\vdots \\ H_{k-1} &= Amal((H_{k-2}, v_{k-1}), (\phi(v_{k-1}), w_{v_{k-1}})), \\ H_k &= Amal((H_{k-1}, v_k), (\phi(v_k), w_{v_k})). \end{aligned}$$

We denote the final graph by $Amal(H, S, \phi)$.

Theorem 12 *The graph $Amal(H, S, \phi)$ is $Q(a)P(b)$-SEG.*

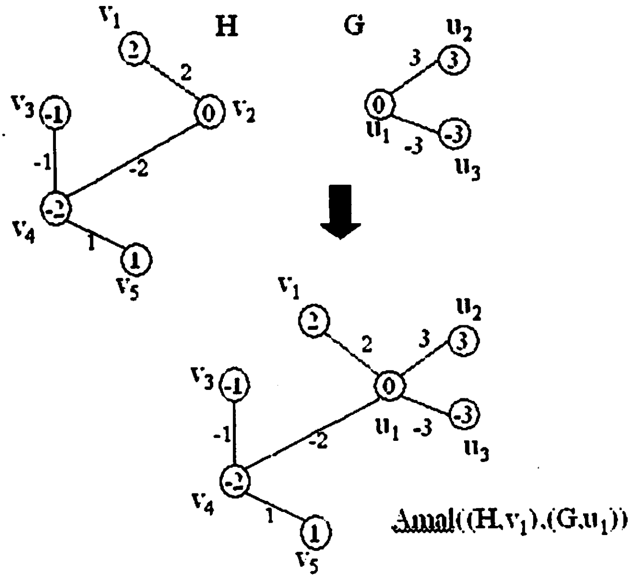


Figure 15:

We will illustrate this construction by the following example.

Example 11. The unicyclic graph in Figure 16 has max$Q(1)$ =max$P(1)$ = 2.

Example 12. The $Q(1)P(1)$-SEG graph $U_4(3,2,0,4)$ can be obtained from $U_4(1,0,0,0)$ by construction 4. See Figure 17.

Remark. The requirement that $\mathfrak{R}$ be the class of all strongly SEG graphs of odd orders is essential. The star $St(3)$ is not strongly SEG. Let c be the center of $St(3)$, we observe that $Amal(C_3,(St(3),c))$ is not $Q(1)P(1)$-SEG. However, the other amalgamation $Amal(C_3,(St(3),x_1))$ is $Q(1)P(1)$-SEG. See Figure 18.

In particular, we have

Corollary 1 *For any strongly SEG graph H with $S \subseteq V(H)$, and a mapping $\phi : S \to \mathfrak{R}$, the graph $Amal(H,S,\phi)$ is strongly SEG.*

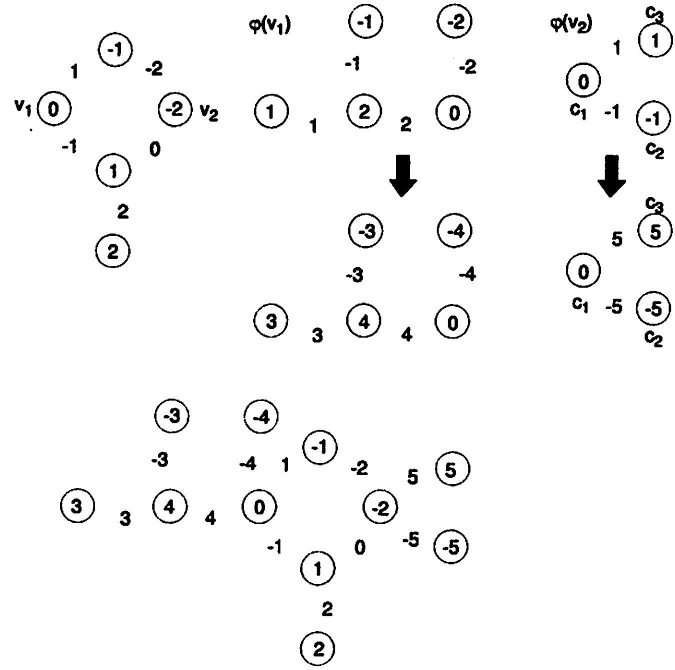


Figure 16:

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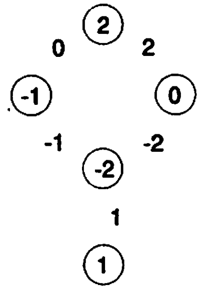

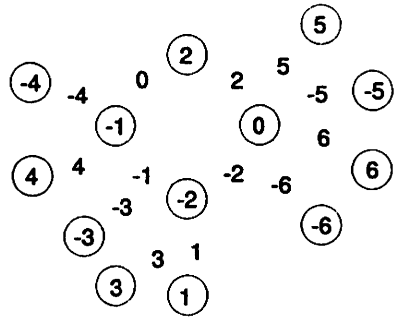

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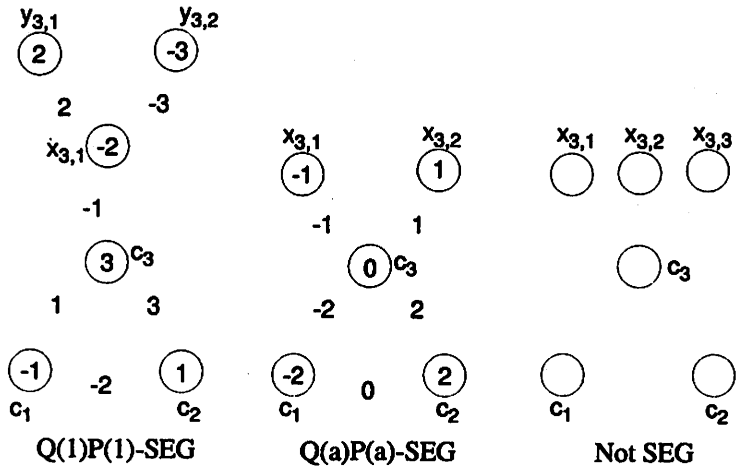

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