A note on the total domination number of a tree

[1]Mustapha Chellali and [2]Teresa W. Haynes

[1]Department of Mathematics, University of Blida.
B.P. 270, Blida, Algeria.
E-mail: m_chellali@yahoo.com

[2] Department of Mathematics, East Tennessee State University
Johnson City, TN 37614 USA
E-mail: haynes@etsu.edu

Abstract

A set S of vertices is a total dominating set of a graph G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set is the total domination number $\gamma_t(G)$. We show that for a nontrivial tree T of order n and with ℓ leaves, $\gamma_t(T) \geqslant (n+2-\ell)/2$, and we characterize the trees attaining this lower bound.

1 Introduction

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood* is $N(v) \cup \{v\}$. The *degree* of a vertex v denoted by $\deg_G(v)$ is the cardinality of its open neighborhood. A *leaf* of a tree T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S and is a *total dominating set*, abbreviated *TDS*, if every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G. A total dominating set of

JCMCC **58** (2006), pp. 189-193

G with minimum cardinality is called a $\gamma_t(G)$-set. Total domination was introduced by Cockayne, Dawes and Hedetniemi [2]. For a comprehensive survey of domination in graphs and its variations, see [3, 4].

Recently, the authors [1] showed that every tree T of order $n \geqslant 3$ and with s support vertices satisfies $\gamma_t(T) \leqslant (n+s)/2$. In this note we give a lower bound on the total domination number of a tree T in terms of the order n and the number of leaves ℓ, namely, $\gamma_t(T) \geq (n+2-\ell)/2$, and we characterize the extremal trees. Note that Lemańska [5] proved that $\gamma(T) \geqslant (n+2-\ell)/3$ for every tree T of order at least three.

2 Main results

Before presenting our main results, we make a couple of straightforward observations.

Observation 1 *If v is a support vertex of a graph G, then v is in every $\gamma_t(G)$-set.*

Observation 2 *For any connected graph G with diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaves of G.*

We show that if T is a tree of order n with ℓ leaves, then $\gamma_t(T)$ is bounded below by $(n+2-\ell)/2$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a P_4 with support vertices labeled x and y, and let $A(T_1) = \{x, y\}$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations. Let H be a path P_4 with support vertices u and v.

- **Operation $\mathcal{O}_1$:** Attach a vertex by joining it to any vertex of $A(T_i)$. Let $A(T_{i+1}) = A(T_i)$.
- **Operation $\mathcal{O}_2$:** Attach a copy of H by joining one of its leaves to any leaf in T_i. Let $A(T_{i+1}) = A(T_i) \cup \{u, v\}$.
- **Operation $\mathcal{O}_3$:** Attach a copy of H and a new vertex w by joining w to u and any leaf of T_i. Let $A(T_{i+1}) = A(T_i) \cup \{u, v\}$.

Lemma 3 *If $T \in \mathcal{T}$, then $A(T)$ is a $\gamma_t(T)$-set of size $(n+2-\ell)/2$.*

Proof. We use the terminology of the construction for the tree $T = T_k$, the set $A(T)$, and the graph H with support vertices u and v. To show that $A(T)$ is a $\gamma_t(T)$-set of cardinality $(n+2-\ell)/2$, we use induction on the number of operations k performed to construct T. The property is true for $T_1 = P_4$. Suppose the property is true for all trees of $\mathcal{T}$ constructed with $k-1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, $T' = T_{k-1}$, and assume that T' has order n' and ℓ' leaves. Since $\mathrm{diam}(T) \geq 3$, let D be a $\gamma_t(T)$-set that contains no leaf of T and as few vertices of $T - T'$ as possible.

If T was obtained from T' by Operation $\mathcal{O}_1$, then $\gamma_{pr}(T) = \gamma_{pr}(T')$, $n = n'+1$, and $\ell = \ell'+1$. By induction on T', $A(T') = A(T)$ is a $\gamma_t(T)$-set of cardinality $(n+2-\ell)/2$.

Assume now that T was obtained from T' using Operation $\mathcal{O}_2$ or $\mathcal{O}_3$. Then we have $n = n'+4$ and $\ell = \ell'$ or $n = n'+5$ and $\ell = \ell'+1$, respectively. Since $A(T) = A(T') \cup \{u, v\}$ is a TDS of T, $\gamma_t(T) \leqslant |A(T)| = \gamma_t(T') + 2$. Now by Observations 1 and 2, D contains u and v. Also D contains no neighbor of u besides v, for otherwise it can be replaced by a vertex of T' which contradicts our choice of D. Thus, $D - \{u, v\}$ is a TDS of T' and $\gamma_t(T') \leqslant \gamma_t(T) - 2$. It follows that $\gamma_t(T) = \gamma_t(T') + 2$ and $A(T)$ is a $\gamma_t(T)$-set. By induction on T', it is a routine matter to check that $|A(T)| = (n+2-\ell)/2$. ■

We now are ready to establish our main result.

Theorem 4 *If T is a nontrivial tree of order n and with ℓ leaves, then $\gamma_t(T) \geqslant (n+2-\ell)/2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then by Lemma 3, $\gamma_t(T) = (n+2-\ell)/2$. To prove that if T is a tree of order $n \geq 2$, then $\gamma_t(T) \geqslant (n+2-\ell)/2$ with equality only if $T \in \mathcal{T}$, we proceed by induction on the order n. If $\mathrm{diam}(T) \in \{1, 2\}$, then $\gamma_t(T) = 2 > (n+2-\ell)/2$. If $\mathrm{diam}(T) = 3$, then T is a double star where $T \in \mathcal{T}$ and $\gamma_t(T) = (n+2-\ell)/2$. In this case if T is different from $T_1 = P_4$, then it can be obtained from T_1 by using Operation $\mathcal{O}_1$. This establishes the base cases.

Assume that every tree T' of order $2 \leqslant n' < n$ and with ℓ' leaves satisfies $\gamma_t(T') \geqslant (n'+2-\ell')/2$ with equality only if $T' \in \mathcal{T}$. Let T be a tree of order n and diameter at least four having ℓ leaves.

If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to x. Then $\gamma_t(T') = \gamma_t(T), n' = n-1$, and $\ell' = \ell-1$. Applying the inductive hypothesis to T', we obtain the desired inequality. Further if $\gamma_t(T) = (n+2-\ell)/2$, then $\gamma_t(T') = (n+2-\ell)/2 = (n'+2-\ell')/2$, and $T' \in \mathcal{T}$. Thus, $T \in \mathcal{T}$ and is obtained from T' by using Operation $\mathcal{O}_1$. Henceforth, we can assume that every support vertex of T is adjacent to exactly one leaf.

We now root T at a vertex r of maximum eccentricity $\mathrm{diam}(T) \geqslant 4$. Let v be a support vertex at maximum distance from r, u be the parent of v, and w be the parent of u in the rooted tree. Note that $\deg_T(w) \geq 2$. Let S be a $\gamma_t(T)$-set that contains no leaves. Denote by T_x the subtree induced by a vertex x and its descendants in the rooted tree T. We distinguish between two cases.

Case 1. $\deg_T(u) \geqslant 3$. Then either u has a child $b \neq v$ that is a support vertex or every child of u except v is a leaf.

Suppose first that u has a child $b \neq v$ that is a support vertex. Let $T' = T - T_v$. Then $n' = n - 2 \geqslant 4$ and $\ell' = \ell - 1$. By Observation 1, v and b are in S. Observation 2 and our choice of S imply that S contains u. Therefore $S - \{v\}$ is a TDS of T' and $\gamma_t(T') \leqslant \gamma_t(T) - 1$. Again by Observation 2, there is a $\gamma_t(T')$-set that contains b and u, and such a set can be extended to a TDS of T by adding v. Hence $\gamma_t(T) \leqslant \gamma_t(T') + 1$ implying that $\gamma_t(T') = \gamma_t(T) - 1$. By induction on T', we have $\gamma_t(T) = \gamma_t(T')+1 \geqslant (n'+2-\ell')/2+1 = (n+2-\ell+1)/2$. Thus $\gamma_t(T) > (n+2-\ell)/2$.

Now assume that every child of u except v is a leaf. Since u is adjacent to exactly one leaf, $\deg_T(u) = 3$. If $\deg_T(w) \geqslant 3$, then let $T' = T - T_u$. Then $n' = n-4 \geqslant 3$, $\ell' = \ell-2$, and $\gamma_t(T) \leqslant \gamma_t(T')+2$ since any $\gamma_t(T')$-set can be extended to a TDS of T by adding the set $\{u, v\}$. Also since $\deg_T(w) \geqslant 3$, w is a support vertex or w has a descendant $x \neq u$ that is a support vertex. By our choice of v, the vertex x is at distance at most two from w. In any case Observation 1 and our choice S imply that w is total dominated by $S - \{u, v\}$, and hence $\gamma_t(T') \leqslant \gamma_t(T) - 2$. It follows that $\gamma_t(T) = \gamma_t(T') - 2$. By induction on T', we obtain $\gamma_t(T) = \gamma_t(T') + 2 \geqslant (n' + 2 - \ell')/2 + 2 = (n + 2 - \ell + 2)/2 > (n + 2 - \ell)/2$.

If $\deg_T(w) = 2$, then let $T' = T - T_w$. Then $n' = n - 5 \geqslant 1$. If $n' = 1$, then T is a corona of P_3 and $\gamma_t(T) = 3 > (n + 2 - \ell)/2$. Thus we assume that $n' \geqslant 2$ and so $\ell - 1 \geqslant \ell'$. Then S contains v and u, and without loss of generality, $w \notin S$ (else substitute w by a vertex from the closed neighborhood of the parent of w). Hence, $S - \{u, v\}$ is a TDS of T' and $\gamma_t(T') \leqslant \gamma_t(T) - 2$. Also $\gamma_t(T) \leqslant \gamma_t(T') + 2$ since every $\gamma_t(T')$-set can be extended to a TDS of T by adding $\{u, v\}$. It follows that $\gamma_t(T') = \gamma_t(T) - 2$. Now by induction on T', we obtain $\gamma_t(T) = \gamma_t(T')+2 \geqslant (n'+2-\ell')/2+2 \geqslant (n + 2 - \ell)/2$.

Further if $\gamma_t(T) = (n+2-\ell)/2$, then we have equality throughout this inequality chain. In particular, $\gamma_t(T') = (n'+2-\ell')/2$ and $\ell - 1 = \ell'$, that is, the parent of w in T is a leaf in T'. Thus by the inductive hypothesis on T', $T' \in \mathcal{T}$. Since T is obtained from T' by using Operation $\mathcal{O}_3$, it follows that $T \in \mathcal{T}$.

Case 2. $\deg_T(u) = 2$. If $\deg_T(w) \geqslant 3$, then let $T' = T - T_u$. Clearly, $n' = n - 3$ and $\ell' = \ell - 1$. Using an argument similar to one in Case 1, it is straightforward to show that $\gamma_t(T) = \gamma_t(T') + 2$. By induction on T', we have $\gamma_t(T) = \gamma_t(T')+2 \geqslant (n'+2-\ell')/2+2 = (n+2-\ell+2)/2 > (n+2-\ell)/2$.

Assume now that $\deg_T(w) = 2$. Let $T' = T - T_w$. Then $n' = n - 4$ and $\ell' \leqslant \ell$. Further we assume that $n' \geqslant 2$ else T is path P_5 where $\gamma_t(T) = 3 > (n+2-\ell)/2$. Also, as before it is straightforward to show that $\gamma_t(T) = \gamma_t(T') + 2$. Applying the inductive hypothesis to T', it follows that $\gamma_t(T) = \gamma_t(T') + 2 \geqslant (n'+2-\ell')/2 + 2 = (n+2-\ell)/2$.

Further if $\gamma_t(T) = (n+2-\ell)/2$, then we have equality throughout this inequality chain. In particular, $\gamma_t(T') = (n'+2-\ell')/2$ and $\ell = \ell'$, that is, the parent of w is a leaf in T'. Thus by the inductive hypothesis, $T' \in \mathcal{T}$. Since T is obtained from T' using Operation $\mathcal{O}_2$, it follows that $T \in \mathcal{T}$. ∎

References

[1] M. Chellali and T.W. Haynes, Total and paired domination numbers of a tree. *AKCE J. Graphs. Combin.*, 1, No. 2 (2004), 69-75.

[2] E.J. Cockayne R.M. Dawes et S.T. Hedetniemi, Total domination in graphs. *Networks* 10 (1980), 211-219.

[3] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998).

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York (1998).

[5] M. Lemańska, Lower bound on the domination number of a tree. *Discuss. Math. Graph Theory* 24(2) (2004), 165-169.