The basis number of the strong product of trees and cycles with some graphs

M.M.M. Jaradat
Yarmouk University
Department of Mathematics
Irbid-Jordan
mmjst4@yu.edu.jo

Abstract

The basis number of a graph G is defined to be the least integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. MacLane [16] proved that a graph, G, is planar if and only if the basis number of G is less than or equal to 2. Ali and Marougi [3] proved that the basis number of the strong product of two cycles and a path with a star is less than or equal to 4. In this work, (1) we prove that the basis number of the strong product of two cycles is 3. (2) We give the exact basis number of a path with a tree containing no subgraph isomorphic to a 3-special star of order 7. (3) We investigate the basis number of a cycle with a tree containing no subgraph isomorphic to a 3-special star of order 7. The results in (1) and (2) improve the upper bound of the basis number of the strong product of two cycles and a star with a path which were obtained by Ali and Marougi.

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1 Introduction.

The basis number of graphs has been studied by many authors, dating to the investigations of planarity of graphs by MacLane [16] who proved that a graph is planar if and only if its basis number is less than or equal to 2.

Formally, the basis number was introduced by Schmeichel [17] who proved that the basis number of a complete graph is less than or equal to 3.

Unless otherwise specified, all graphs considered here are connected, finite, undirected and simple. Most of the notations that follow can be found in [7]. Given a graph G, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which corresponds to the cycles in G generate a subspace called the cycle space of G and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that if G is a connected graph, then $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$.

Definition 1.1. *A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a d–fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number, $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The required basis for $\mathcal{C}(G)$ is a basis that is $b(G)$-fold.*

Studying the basis number of graph products has been done by many authors. Schmeichel [17] and Ali [2] gave an upper bound for the semi-strong product, $\bullet$, of some special graphs. They proved the following results:

Theorem 1.1. *(Schmeichel) For each $n \geq 7, b(K_n \bullet P_2) = 4$.*

Theorem 1.2. *(Ali) For each integers $n, m, b(K_m \bullet K_n) \leq 9$.*

Schmeichel proved a more general case, in fact, he proved that $b(K_{n,m}) = 4$ for each $n, m \geq 5$ except possibly for $K_{6,10}, K_{5,n}$ and $K_{6,n} (n = 5, 6, 7, 8)$. The basis number of the cartesian product, $\times$, of two cycles was obtained by Ali and Marougi [4] who proved the following:

Theorem1.3. *(Ali and Marougi) For any two cycles C_n and C_m* with *$n, m \geq 3$, we have $b(C_n \times C_m) = 3$.*

Alsardary and Wojciechowski [5] gave the following result:

Theorem 1.4. *(Alsardary and Wojciechowski) For every $d \geq 1$ and $n \geq 2$, we have $b(K_n^d) \leq 9$ where K_n^d is a d times cartesian product of K_n.*

The direct product, $\wedge$, was studied by Ali [1] and Jaradat [11]. They gave the following results.

Theorem 1.5. *(Ali) For any two cycles C_n and C_m* with *$n, m \geq 3$, $b(C_n \wedge C_m) = 3$.*

Theorem 1.6. *(Jaradat) For each bipartite graphs G and H, $b(G \wedge H) \leq 5 + b(G) + b(H)$.*

Theorem 1.7. *(Jaradat) For each bipartite graph G and cycle C, $b(G \wedge C) \leq 3 + b(G)$.*

Jaradat [11] classified trees with respect to the basis number of their direct product with paths of order greater than or equal to 5. Many other papers appeared to investigate the basis number of graph products, see [6], [9], [12], [13] and [14].

Our aim, as suggested in the title, is to investigate the basis number of the strong product, $\otimes$, of some classes of graphs. Ali and Marougi [3] gave the following results of the basis number of strong product.

Theorem 1.8. *(Ali and Marougi) For each two paths P_m, P_n* and for each cycle C_k with $n, m, k \geq 4$, $b(P_n \otimes P_m) = b(P_n \otimes C_k) = 3$.

Theorem 1.9. *(Ali and Marougi) For each two cycles C_m, C_n* and for each star S_k with $n, k \geq 4$ and $m \geq 3$, $3 \leq b(C_n \otimes C_m) \leq 4$ and $3 \leq b(P_n \otimes S_k) \leq 4$.

In view of results of Ali and Marougi, one is naturally led to the following question:

Problem: *Can we give exact values for the basis number of the strong product of two cycles and a path with a star?*

The main focus of this paper is to obtain a solution to the above problem. In fact, we give the exact basis number of the strong product of more general classes of graphs. We prove that, under some restrictions on the order of graphs, the basis number of the strong product of two cycles and a path (cycle) with a tree contains no 3-special star of order 7 is equal to 3.

For completeness, let us recall the definition of the following two products. Let G and H be two graphs The *cartesian product* $G^* = G \times H$ has vertex-set $V(G^*) = V(G) \times V(H)$ and edge-set $E(G^*) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)\}$. The *strong product* $G^* = G \otimes H$ has vertex-set $V(G^*) = V(G) \times V(H)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$ or $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$. It is clear to see that the above operations are commutative and $G_1 \times G_2 \subset G_1 \otimes G_2$.

In the rest of this work $f_B(e)$ stands for the number of cycles in $B \subseteq \mathcal{C}(G)$ containing e and $E(B) = \cup_{b \in B} E(b)$.

2 The basis number of the strong product of trees with paths and cycles.

The main idea in this section is to divide the strong product of a tree with a path (cycle) into familiar subgraphs (not necessary pairwise edge disjoint) that can be done by using a certain decomposition which decomposes trees into paths of order 2 and stars. Then we use linearly independent sets of the cycle spaces of those subgraphs to construct a basis for the strong product of a tree with a path (cycle).

Let S_n be a star of order n such that $V(S_n) = \{v_1, v_2, \ldots, v_n\}$ and $d(v_1) = n - 1$. Set the following set of cycles of $C(ab \otimes S_n)$:

$$\begin{aligned}
\mathcal{J}_a &= \{\mathcal{J}_a^{(l)} = (a, v_1)(b, v_l)(a, v_l)(a, v_1) | l = 2, 3, \cdots, n\} \\
\mathcal{J}_b &= \{\mathcal{J}_b^{(l)} = (b, v_1)(b, v_l)(a, v_l)(b, v_1) | l = 2, 3, \cdots, n\} \\
\mathcal{J}_{ab} &= \{\mathcal{J}_{ab}^{(2)} = (b, v_1)(a, v_2)(a, v_1)(b, v_1)\} \cup \\
\{\mathcal{J}_{ab}^{(l)} &= (a, v_1)(a, v_l)(b, v_1)(a, v_{l-1})(b, v_{l-1})(a, v_1) | l = 3, 4, \cdots, n\}
\end{aligned}$$

Let

$$\mathcal{J}_S^{(ab)} = \mathcal{J}_a \cup \mathcal{J}_b \cup \mathcal{J}_{ab}$$

Lemma 2.1. *$\mathcal{J}_S^{(ab)}$ is a linearly independent set of cycles.*

Proof. For each $l = 2, 3, \ldots, n$, set $\mathcal{D}_l = \{\mathcal{J}_a^{(l)}, \mathcal{J}_b^{(l)}, \mathcal{J}_{ab}^{(l)}\}$. Since $\mathcal{J}_a^{(l)}$ contains $(a, v_l)(a, v_1)$ which is not an edge of $\mathcal{J}_b^{(l)}$, $\{\mathcal{J}_a^{(l)}, \mathcal{J}_b^{(l)}\}$ is linearly independent. Since $\mathcal{J}_a^{(l)} + \mathcal{J}_b^{(l)} = (a, v_1)(b, v_l)(b, v_1)(a, v_l)(a, v_1) \pmod 2$ which is not $\mathcal{J}_{ab}^{(2)}$, as a result $\mathcal{D}_l$ is linearly independent for each $l = 2, 3, \ldots n$. Note that $\mathcal{J}_S^{(ab)} = \cup_{l=2}^{n} \mathcal{D}_l$. Thus, to show that $\mathcal{J}_S^{(ab)}$ is linearly independent we proceed by using induction on n. If $n = 2$, then $\mathcal{J}_S^{(ab)} = \mathcal{D}_2$ which is linearly independent, by the above argument. Assume $n > 2$ and the result holds for less than n. By the above argument and inductive step both of $\mathcal{D}_n$ and $\cup_{l=2}^{n-1} \mathcal{D}_l$ are linearly independent. Clearly, $E(\mathcal{D}_n) \cap E(\cup_{l=2}^{n-1} \mathcal{D}_l) = \{(a, v_1)(b, v_{l-1}), (a, v_{l-1})(b, v_{l-1}), (a, v_{l-1})(b, v_1)\}$ which is an edge set of a path. Thus any linear combination of $\mathcal{D}_n$ must contain at least one edge of $(a, v_1)(a, v_n), (b, v_1)(b, v_n), (a, v_1)(b, v_n), (b, v_1)(a, v_n)$ and $(a, v_n)(b, v_n)$ which is not in any cycle of $\cup_{l=2}^{n-1} \mathcal{D}_l$. Therefore, $\cup_{l=2}^{n} \mathcal{D}_l$ is linearly independent. The proof is completed.

Remark 2.1. *(1) If $e = (a, v_1)(a, v_l)$, then $f_{\mathcal{J}_S^{(ab)}}(e) \leq 2$. (2) If $e = (b, v_1)(b, v_l)$, then $f_{\mathcal{J}_S^{(ab)}}(e) = 1$. (3) If $e = (a, v_l)(b, v_l)$ where $l \neq 1, n$, then $f_{\mathcal{J}_S^{(ab)}}(e) \leq 3$. (4) If $e = (a, v_1)(b, v_1)$, then $f_{\mathcal{J}_S^{(ab)}}(e) = 1$. (5) If*

$e = (a, v_n)(b, v_n)$, *then* $f_{\mathcal{J}_S^{(ab)}}(e) = 2$. *(6) If* $e = (a, v_1)(b, v_l)$ such that $l \geq 2$, *then* $f_{\mathcal{J}_S^{(ab)}}(e) \leq 2$. *(7) If* $e = (b, v_1)(a, v_l)$ such that $l \geq 2$, *then* $f_{\mathcal{J}_S^{(ab)}}(e) \leq 3$.

The following fact will be used frequently in the sequel.

Fact 2.1. *Any non-trivial linear combination of cycles of linearly independent set of cycles is a cycle or an edge disjoint union of cycles.*

It should be mentioning that finding the basis number of the strong product of a cycle (path) with a tree can not be found using the direct method as in [3] because trees have no uniform forms. Therefore, we shall recall a certain decomposition which decomposes a tree into stars and paths of order 2.

Proposition 2.2 *(Jaradat [15]) Let* T *be a tree of order* ≥ 2. *Then* T *can be decomposed into pairwise edge disjoint of subgraphs* $S_1, S_2, \ldots, S_r$ *for some integer* r *such that the following holds:*

(i) For each $i \geq 1, S_i$ *is either a star or a path of order* 2 *and* S_1 *is a path incident with an end vertex.*

(ii) For each $v \in V(T)$, *if* $d_T(v) \geq 2$, *then* $|\{i : v \in V(S_i)\}| = 2$, *and if* $d_T(v) = 1$, *then* $|\{i : v \in V(S_i)\}| = 1$.

(iii) $V(S_i) \cap (\cup_{j=1}^{i-1} V(S_j)) = v_1^{(i)}$ *where* $d_{S_i}(v_1^{(i)}) = \max_{v \in V(S_i)} d_{S_i}(v)$, $d_{\cup_{j=1}^{i-1} S_j}(v_1^{(i)}) = 1$ *for each* $i = 2, 3, \ldots, r$, *and* $v_1^{(i)} \neq v_1^{(j)}$ *for each* $i \neq j$.

Let T be a tree and $T = \cup_{j=1}^{r} S_j$ as in Proposition 2.2. Let $V(S_j) = \{v_1^{(j)}, v_2^{(j)}, \ldots, v_{n_j}^{(j)}\}$ be the vertex set of S_j with $d_{S_j}(v_1^{(j)}) = n_j - 1$. We define

$$\mathcal{B}(ab \otimes T) = \cup_{j=1}^{r} \mathcal{J}_{S_j}^{(a_i a_{i+1})},$$

where $\mathcal{J}_{S_j}^{(a_i a_{i+1})}$ is the linearly independent subset of $\mathcal{C}(ab \otimes S_j)$ as in Lemma 2.1.

Lemma 2.3. $\mathcal{B}(ab \otimes T)$ *is a linearly independent subset of* $\mathcal{C}(ab \otimes T)$.

Proof. We now prove that $\mathcal{B}(ab \otimes T) = \cup_{j=1}^{r} \mathcal{J}_{S_j}^{(a_i a_{i+1})}$ is linearly independent by using induction on r. If $r = 1$, then the result is true, by Lemma 2.1. Assume $r \geq 2$ and the result holds for less than r. Note that $\mathcal{B}(ab \otimes T) = \cup_{j=1}^{r-1} \mathcal{J}_{S_j}^{(ab)} \cup \mathcal{J}_{S_r}^{(ab)}$. By the inductive step and Lemma 2.1, both of $\cup_{j=1}^{r-1} \mathcal{J}_{S_j}^{(ab)}$ and $\mathcal{J}_{S_r}^{(ab)}$ are linearly independent. Since $V(S_r) \cap V(\cup_{j=1}^{r-1} S_j) = v_1^{(r)}$, we have

$$E(\cup_{j=1}^{r-1} \mathcal{J}_{S_j}^{(ab)}) \cap E(\mathcal{J}_{S_r}^{(ab)}) = \{(a, v_1^{(r)})(b, v_1^{(r)})\}$$

which is an edge. Let C be a non-trivial linear combination of cycles of $\mathcal{J}_{S_r}^{(ab)}$. Assume that

$$C = \sum_{i=1}^{f} d_i \pmod 2$$

where $d_i \in \cup_{j=1}^{r-1} \mathcal{J}_{S_j}^{(ab)}$. Then

$$E(C) = E(d_1 \oplus \cdots \oplus d_f).$$

where $\oplus$ is the ring sum. Thus,

$$E(C) \subseteq E(\cup_{j=1}^{r-1} \mathcal{J}_{S_j}^{(ab)}) \cap E(\mathcal{J}_{S_r}^{(ab)}) = \{(a, v_1^{(r)})(b, v_1^{(r)})\}.$$

This contradicts Fact 2.1. Thus, $\mathcal{B}(ab \otimes T)$ is linearly independent. The proof is completed.

A tree T consisting of n equal order paths $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ is called an n-special star if there is a vertex, say v_1, such that v_1 is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v_1\}$ for each $i \neq j$ (see [11]).

In the following work we assume $P_m = a_1 a_2 \ldots a_m$, $P_n = v_1 v_2 \ldots v_n$, $C_m = a_1 a_2 \ldots a_m a_1$ and $C_n = v_1 v_2 \ldots v_n v_1$.

Theorem 2.4. *Let T be a tree of order n containing no subgraph isomorphic to a 3-special star of order 7 and P_m be a path. Then $b(P_m \otimes T) \leq 3$. Moreover, the equality holds for $n, m \geq 4$.*

Proof. By Theorems 1.8 and 1.9, it suffices to show that $b(P_m \otimes T) \leq 3$. Let $T = \cup_{j=1}^{r} S_j$ as in Proposition 2.2. Let $V(S_j) = \{v_1^{(j)}, v_2^{(j)}, \ldots, v_{n_j}^{(j)}\}$ be the vertex set of S_j with $d_{S_j}(v_1^{(j)}) = n_j - 1$. Since T is a tree containing no subgraph isomorphic to a 3-special star of order 7, we can choose S_j's and label their vertices in such away that $v_n^{(j-1)} = v_1^{(j)}$ for each $j = 2, 3, \ldots, r$. Define $\mathcal{B}(P_m \otimes T) = \cup_{i=1}^{m-1} \mathcal{B}(a_i a_{i+1} \otimes T)$ where $\mathcal{B}(a_i a_{i+1} \otimes T)$ is the linearly independent subset of $\mathcal{C}(a_i a_{i+1} \otimes T)$ as defined in Lemma 2.3. To show that $\mathcal{B}(P_m \otimes T)$ is linearly independent, we use induction on m and argue more or less as in the proof of Lemma 2.3 by taking in account that

$$E(\cup_{i=1}^{m-2} \mathcal{B}(a_i a_{i+1} \otimes T)) \cap E(\mathcal{B}(a_{m-1} a_m \otimes T)) = E(a_m \times T)$$

which is an edge set of a tree. Since

$$\sum_{j=1}^{r} n_j = n + r - 1,$$

as a result

$$\begin{aligned}
|\mathcal{B}(P_m \otimes T)| &= \sum_{i=1}^{m-1}\sum_{j=1}^{r} |\mathcal{J}_{S_j}^{(a_i a_{i+1})}| \\
&= \sum_{i=1}^{m-1}\sum_{j=1}^{r} 3(n_j - 1) \\
&= \sum_{i=1}^{m-1} 3(\sum_{j=1}^{r} n_j - \sum_{j=1}^{r} 1) \\
&= \sum_{i=1}^{m-1} 3(n + r - 1 - r) \\
&= 3(n-1)(m-1) \\
&= \dim \mathcal{C}(P_m \otimes T),
\end{aligned}$$

Thus, $\mathcal{B}(P_m \otimes T)$ is a basis for $\mathcal{C}(P_m \otimes T)$. We now show that $\mathcal{B}(P_m \otimes T)$ is a 3- fold basis. Let $e \in E(P_m \otimes T)$. Then (1) if $e = (a_i, v_k^{(j)})(a_{i+1}, v_k^{(j)})$ for some $1 \leq j \leq r$ and $k \neq 1$, then

$$f_{\mathcal{B}(P_m \otimes T)}(e) = f_{\mathcal{B}(a_i a_{i+1} \otimes T)}(e) = f_{\mathcal{J}_{S_j}^{(a_i a_{i+1})}} \leq 3.$$

(2) If $e = (a_i, v_1^{(1)})(a_{i+1}, v_1^{(1)})$, then

$$f_{\mathcal{B}(P_m \otimes T)}(e) = f_{\mathcal{B}(a_i a_{i+1} \otimes T)}(e) = f_{\mathcal{J}_{S_1}^{(a_i a_{i+1})}}(e) = 1.$$

(3) If $e = (a_i, v_1^{(j)})(a_{i+1}, v_1^{(j)})$ where $j \geq 2$, then

$$f_{\mathcal{B}(P_m \otimes T)}(e) = f_{\mathcal{B}(a_i a_{i+1} \otimes T)}(e) = f_{\mathcal{J}_{S_{j-1}}^{(a_i a_{i+1})}}(e) + f_{\mathcal{J}_{S_j}^{(a_i a_{i+1})}}(e) = 2 + 1 = 3.$$

(4) If $e = (a_i, v_1^{(j)})(a_{i+1}, v_l^{(j)})$ or $(a_i, v_l^{(j)})(a_{i+1}, v_1^{(j)})$, then

$$f_{\mathcal{B}(P_m \otimes T)}(e) = f_{\mathcal{B}(a_i a_{i+1} \otimes T)}(e) = f_{\mathcal{J}_{S_j}^{(a_i a_{i+1})}} \leq 3.$$

(5) If $e = (a_i, v_1^{(j)})(a_i, v_l^{(j)})$, then

$$\begin{aligned}
f_{\mathcal{B}(P_m \otimes T)}(e) &= f_{\mathcal{B}(a_{i-1} a_i \otimes T)}(e) + f_{\mathcal{B}(a_i a_{i+1} \otimes T)}(e) = f_{\mathcal{J}_{S}^{(a_{i-1} a_i)}}(e) \\
&\quad + f_{\mathcal{J}_{S_j}^{(a_i a_{i+1})}}(e) \\
&\leq 2 + 1 = 3.
\end{aligned}$$

The proof is completed.

By specializing T in Theorem 2.4 into a star, we have the following result which improve Ali and Marougi's result.

Corollary 2.5. *Let S_n be a star of order n and P_m is a path of order m. Then $b(P_m \otimes S_n) \leq 3$. Moreover, the equality holds for $n, m \geq 4$.*

The following result of Ali and Marougi is an immediate consequence of Theorem 2.4.

Corollary 2.6. *(Ali and Marougi) Let P_m and P_n be two paths with $m, n \geq 4$. Then $b(P_m \otimes P_n) = 3$.*

Theorem 2.7. *Let T be a tree of order $n \geq 3$ containing no subgraph isomorphic to 3-special star of order 7 and C_m be a cycle of order $m \geq 3$. Then $b(C_m \otimes T) = 3$.*

Proof. Define $\mathcal{B}(C_m \otimes T) = \mathcal{B}(P_m \otimes T) \cup \mathcal{B}(a_m a_1 \otimes T) \cup \{C_m \times v_1^{(1)}\}$ where $\mathcal{B}(P_m \otimes T)$ and $\mathcal{B}(a_m a_1 \otimes T)$ are as in Theorem 2.4 and Lemma 2.3. Thus, both of $\mathcal{B}(P_m \otimes T)$ and $\mathcal{B}(a_m a_1 \otimes T)$ are linearly independent. Since

$$E(\mathcal{B}(P_m \otimes T)) \cap E(\mathcal{B}(a_m a_1 \otimes T)) = E(a_1 \times T) \cup E(a_m \times T)$$

which is an edge set of a forest, any linear combination of $\mathcal{B}(a_m a_1 \otimes T)$ can not be written as a linear combination of $\mathcal{B}(P_m \otimes T)$. Thus $\mathcal{B}(P_m \otimes T) \cup \mathcal{B}(a_m a_1 \otimes T)$ is linearly independent. We now show that $C_m \times v_1^{(1)}$ is linearly independent of cycles of $\mathcal{B}(P_m \otimes T) \cup \mathcal{B}(a_m a_1 \otimes T)$. Assume that C is a linear combination of cycles of $\mathcal{B}(P_m \otimes T) \cup \mathcal{B}(a_m a_1 \otimes T)$. Since

$$\begin{aligned} E(C_m \times v_1^{(1)}) \cap E(\mathcal{B}(a_i a_{i+1} \otimes T)) &= \left\{(a_i \times v_1^{(1)})(a_{i+1} \times v_1^{(1)})\right\}, \text{ and} \\ E(C_m \times v_1^{(1)}) \cap E(\mathcal{B}(a_m a_1 \otimes T)) &= \left\{(a_m \times v_1^{(1)})(a_1 \times v_1^{(1)})\right\}, \end{aligned}$$

we have that

$$C_m \times v_1^{(1)} = \sum_{i=1}^{m} R_i \pmod 2,$$

where R_i is a non-trivial linear combination of cycles of $\mathcal{B}(a_i a_{i+1} \otimes T)$ for each $i = 1, 2, \ldots, m-1$ and R_m is a non-trivial linear combination of cycles of $\mathcal{B}(a_m a_1 \otimes T)$. Thus,

$$R_m = (C_m \times v_1^{(1)}) \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_{m-1}$$

where $\oplus$ is the ring sum. Hence

$$\begin{aligned} E(R_m) &= E\left(C_m \times v_1^{(1)} \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_{m-1}\right) \subseteq (E(\mathcal{B}(P_m \otimes T) \cup \\ &\quad E(C_m \times v_1^{(1)})\right) \cap E\left(\mathcal{B}(a_m a_1 \otimes T)\right). \end{aligned}$$

But,

$$\begin{aligned} &\left(E(\mathcal{B}(P_m \otimes T) \cup E(C_m \times v_1^{(1)})\right) \cap E\left(\mathcal{B}(a_m a_1 \otimes T)\right) \\ = \quad & E(a_1 \times T) \cup E(a_m \times T) \cup \{(a_1, v_1^{(1)})(a_m, v_1^{(1)})\} \end{aligned}$$

which is an edge set of a path. This contradicts Fact 1.1. Thus, $\mathcal{B}(C_m \otimes T)$ is linearly independent. Since

$$\begin{aligned} |\mathcal{B}(C_m \otimes T)| &= |\mathcal{B}(P_m \otimes T)| + |\mathcal{B}(a_m a_1 \otimes T)| + 1 \\ &= 3(n-1)(m-1) + 3(n-1) + 1 \\ &= 3m(n-1) + 1 \\ &= \dim \mathcal{C}(C_m \otimes T), \end{aligned}$$

$\mathcal{B}(C_m \otimes T)$ is a basis. It is easy to show that $\mathcal{B}(C_m \otimes T)$ is a 3-fold basis. On the other hand, Assume that $n \geq 3$ and $\mathcal{B}$ is a 2-fold basis for $\mathcal{C}(C_m \otimes T)$. Since the girth of $C_m \otimes T$ is 3, we have that

$$\begin{aligned} 3|\mathcal{B}| &\leq 2|E(C_m \otimes T)| \\ 9mn - 9m + 3 &\leq 8mn - 6m \\ mn - 3m + 3 &\leq 0 \\ m(n-3) + 3 &\leq 0, \end{aligned}$$

which implies that $n \leq 2$. This is a contradiction. The proof is completed.

The following results are immediate corollaries of the above result, that is by specializing the tree T into path and star.

Corollary 2.8. *(Ali and Marougi) Let P_n be a path of order $n \geq 3$ and C_m be a cycle of order $m \geq 3$. Then $b(C_m \otimes P_n) = 3$.*

Corollary 2.9. *Let S_n be a star of order $n \geq 3$ and C_m be a cycle of order $m \geq 3$. Then $b(C_m \otimes S_n) = 3$.*

3 The basis number of strong product of two cycles.

We now turn our attention to deal with the basis number of the strong product of two cycles. As we mentioned above Ali and Marougi [3] proved that $3 \leq b(C_m \otimes C_n) \leq 4$ for each $m \geq 3$ and $n \geq 4$. We shall prove that $b(C_m \otimes C_n) = 3$ for each $m, n \geq 3$. We set the following three sets of cycles:

$$\begin{aligned}
\mathcal{L}_{ab} &= \left\{\mathcal{L}^{(j)} = (a,v_j)(b,v_{j+1})(a,v_{j+1})(a,v_j) | j = 1,2,3,\cdots,n-1\right\} \\
&\cup \left\{\mathcal{L}^{(n)} = (a,v_n)(b,v_1)(a,v_1)(a,v_n)\right\} \\
\mathcal{A}_{ab} &= \left\{\mathcal{A}^{(j)} = (a,v_j)(a,v_{j+1})(b,v_j)(a,v_j) | j = 1,2,3,\cdots,n-1\right\} \\
&\cup \left\{\mathcal{A}^{(n)} = (a,v_n)(a,v_1)(b,v_n)(a,v_n)\right\} \\
\mathcal{S}_{ab} &= \left\{\mathcal{S}^{(j)} = (a,v_{j+1})(b,v_j)(b,v_{j+1})(a,v_{j+1}) | j = 1,2,3,\cdots,n-1\right\} \\
&\cup \left\{\mathcal{S}^{(n)} = (a,v_1)(b,v_n)(b,v_1)(a,v_1)\right\}
\end{aligned}$$

Let

$$\mathcal{B}_{ab} = \mathcal{L}_{ab} \cup \mathcal{A}_{ab} \cup \mathcal{S}_{ab}$$

Lemma 3.1. *$\mathcal{B}_{ab}$ is a linearly independent set of cycles. Moreover, any linear combination of cycles of $\mathcal{B}_{ab}$ is either $(a \times C_n) \cup (b \times C_n)$ or contains at least two edges of $\{(a,v_j)(b,v_l) | 1 \leq j,l \leq n\}$.*

Proof. Let $\mathcal{B}_{ab}^{(j)} = \{\mathcal{L}^{(j)}, \mathcal{A}^{(j)}, \mathcal{S}^{(j)}\}$ for each $j = 1,2,\ldots,n$. Then by using the same arguments as in Lemma 2.1 we prove that $\mathcal{B}_{ab} = \cup_{j=1}^{n} \mathcal{B}_{ab}^{(j)}$ is linearly independent for each $j = 1,2,\ldots n$. We now show the second part of the lemma. Assume that O is a linear combination of cycles of $\mathcal{B}_{ab}$, say $T = \{T_1, T_2, \ldots, T_k\}$. To this end, either O contains no edges of $\{(a,v_j)(b,v_l) | 1 \leq j,l \leq n\}$ or contains only one edge of $\{(a,v_j)(b,v_l) | 1 \leq j,l \leq n\}$ or contains at least two edges of $\{(a,v_j)(b,v_l) | 1 \leq j,l \leq n\}$.

Case 1. O contains no edges of $\{(a,v_j)(b,v_l) | 1 \leq j,l \leq n\}$. Since $\mathcal{L}^{(j)}$ is the only cycle of $\mathcal{B}_{ab}$ containing $(a,v_j)(b,v_{j+1})$ and since $\mathcal{L}_{ab}$ is an edge pairwise disjoint set of cycles, $\mathcal{L}^{(j)} \notin T$ for each $j = 1,2,\ldots n$. Thus, T consists only of cycles of $\mathcal{A}_{ab} \cup \mathcal{S}_{ab}$. Since any linear combination of $\mathcal{A}_{ab}$ (or $\mathcal{S}_{ab}$) must contain at least one edge of the form $(a,v_{j+1})(b,v_j)$ for some j, as a result T must consist of cycles of both of $\mathcal{A}_{ab}$ and $\mathcal{S}_{ab}$. Thus, we may assume that $T_1 \in \mathcal{A}_{ab}$, say $T_1 = \mathcal{A}^{(j)}$ for some j. Since $(a,v_{j+1})(b,v_j) \in E(\mathcal{A}^{(j)})$ and $(a,v_{j+1})(b,v_j) \notin E(O)$ and since the only

cycles of $\mathcal{A}_{ab} \cup \mathcal{S}_{ab}$ which contain such an edge are $\mathcal{A}^{(j)}, \mathcal{S}^{(j)}$, as a result $\mathcal{S}^{(j)} \in T$. Thus, $(a, v_j)(b, v_j), (a, v_{j+1})(b, v_{j+1}) \in E(\mathcal{A}^{(j)} \oplus \mathcal{S}^{(j)})$ where $\oplus$ is the ring sum. Since $(a, v_{j+1})(b, v_{j+1}) \notin E(O)$ and $\mathcal{A}^{(j+1)}$ is the only cycle of $\mathcal{A}_{ab} \cup \mathcal{S}_{ab}$ which contains $(a, v_{j+1})(b, v_{j+1})$, it implies that $\mathcal{A}^{(j+1)} \in T$. Thus, $(a, v_j)(b, v_j), (a, v_{j+1})(b, v_{j+2}) \in E(\mathcal{A}^{(j)} \oplus \mathcal{S}^{(j)} \oplus \mathcal{A}^{(j+1)})$. Since $(a, v_{j+1})(b, v_{j+2}) \notin E(O)$ and the only cycles of $\mathcal{A}_{ab} \cup \mathcal{S}_{ab}$ which contain such an edge are $\mathcal{A}^{(j+1)}, \mathcal{S}^{(j+1)}$, as a result $\mathcal{S}^{(j+1)} \in T$. By continuing in this process, we have that $\mathcal{A}^{(j)}, \mathcal{S}^{(j)}, \mathcal{A}^{(j+1)}, \mathcal{S}^{(j+1)}, \ldots, \mathcal{A}^{(n)}, \mathcal{S}^{(n)} \in T$ and $(a, v_j)(b, v_j), (a, v_1)(b, v_1) \in E(\mathcal{A}^{(j)} \oplus \mathcal{S}^{(j)} \oplus \mathcal{A}^{(j+1)} \oplus \mathcal{S}^{(j+1)} \oplus \ldots \oplus \mathcal{A}^{(n)} \oplus \mathcal{S}^{(n)})$. Since $(a, v_1)(b, v_1) \notin E(O)$ and the only cycle of $\mathcal{A}_{ab} \cup \mathcal{S}_{ab}$ which contains such an edge is $\mathcal{A}^{(1)}$, it implies that $\mathcal{A}^{(1)} \in T$. Hence, $(a, v_j)(b, v_j), (a, v_2)(b, v_2) \in E(\mathcal{A}^{(1)} \oplus \mathcal{A}^{(j)} \oplus \mathcal{S}^{(j)} \oplus \mathcal{A}^{(j+1)} \oplus \mathcal{S}^{(j+1)} \oplus \ldots \oplus \mathcal{A}^{(n)} \oplus \mathcal{S}^{(n)})$. Since $(a, v_2)(b, v_2) \notin E(O)$ and the only cycles of $\mathcal{A}_{ab} \cup \mathcal{S}_{ab}$ which contain such an edge are $\mathcal{A}^{(1)}, \mathcal{S}^{(1)}$, as a result $\mathcal{S}^{(1)} \in T$. By continuing in this process we have that $\mathcal{A}^{(1)}, \mathcal{S}^{(1)}, \ldots, \mathcal{A}^{(n)}, \mathcal{S}^{(n)} \in T$ and so $\{\mathcal{A}^{(1)}, \mathcal{S}^{(1)}, \ldots, \mathcal{A}^{(n)}, \mathcal{S}^{(n)}\} = T$. But it is easy to see that $\mathcal{A}^{(1)} \oplus \mathcal{S}^{(1)} \oplus \ldots \oplus \mathcal{A}^{(n)} \oplus \mathcal{S}^{(n)} = (a \times C_n) \cup (b \times C_n)$.
Case 2. O contains only one edge of $\{(a, v_j)(b, v_l) | 1 \leq j, l \leq n\}$, say $(a, v_{j_0})(b, v_{l_0})$ for some j_0, l_0. Then O is subgraph of $(a, v_{j_0})(b, v_{l_0}) \cup (a \times C_n) \cup (b \times C_n)$. Since the only cycles or edge disjoint union of cycles of $(a, v_{j_0})(b, v_{l_0}) \cup (a \times C_n) \cup (b \times C_n)$ are $(a \times C_n)$, $(b \times C_n)$ and $(a \times C_n) \cup (b \times C_n)$, as a result by Fact 1.1, O must be either $(a \times C_n)$ or $(b \times C_n)$ or $(a \times C_n) \cup (b \times C_n)$. This is contradiction because non of which contains $(a, v_{j_0})(b, v_{l_0})$. Therefore, this case cannot be happened.
Case 3. O contains at least two cycles of $\{(a, v_j)(b, v_l) | 1 \leq j, l \leq n\}$. Then the result is done. The proof is completed.

Remark 3.1. *(1) If $e = (a, v_i)(b, v_i)$, then $f_{\mathcal{B}_{ab}}(e) \leq 3$. (2) If $e = (a, v_i)(b, v_{i+1})$ or $(a, v_{i+1})(b, v_i)$* or *$(a, v_1)(b, v_n)$* or *$(a, v_n)(b, v_1)$, then $f_{\mathcal{B}_{ab}}$ $(e) \leq 2$. (3) if $e = (a, v_i)(a, v_{i+1})$* or *$(a, v_1)(a, v_n)$, then $f_{\mathcal{B}_{ab}}(e) \leq 2$. (2) If $e = (b, v_i)(b, v_{i+1})$* or *$(b, v_1)(b, v_n)$, then $f_{\mathcal{B}_{ab}}(e) = 1$.*

We now define the following cycle of $C_m \otimes C_n$:

$$\mathcal{F}_n = \begin{cases} (a, v_1)(b, v_2)(a, v_3)(b, v_4) \ldots (a, v_{n-1})(b, v_n)(a, v_1) & \text{if } n \text{ is even} \\ (a, v_1)(b, v_1)(a, v_2)(b, v_3) \ldots (a, v_{n-1})(b, v_n)(a, v_1) & \text{if } n \text{ is odd} \end{cases}$$

Lemma 3.2. *$\mathcal{B}^*_{ab} = \mathcal{B}_{ab} - \{\mathcal{S}^{(n)}\} \cup \{\mathcal{F}_n\}$ is linearly independent set of cycles. Moreover, any linear combination of cycles of $\mathcal{B}^*_{ab}$ either contains at least one edge of the form $(a, v_j)(b, v_l)$ for some j, l or contains only one copy of C_n, in fact of $a \times C_n$.*

Proof. By Lemma 3.1, $\mathcal{B}_{ab} - \{\mathcal{L}^{(n)}, \mathcal{A}^{(n)}, \mathcal{S}^{(n)}\}$ is linearly independent. Since $\mathcal{F}_n$ contains $(b, v_n)(a, v_1)$ which is not in any cycle of $\mathcal{B}_{ab} - \{\mathcal{L}^{(n)}, \mathcal{A}^{(n)},$

$\mathcal{S}^{(n)}\}$, $\mathcal{B}_{ab} - \{\mathcal{L}^{(n)}, \mathcal{A}^{(n)}, \mathcal{S}^{(n)}\} \cup \{\mathcal{F}_n\}$ is linearly independent. Since $\mathcal{A}^{(n)}$ contains $(a, v_n)(a, v_1)$ which is not in any cycle of $\mathcal{B}_{ab} - \{\mathcal{L}^{(n)}, \mathcal{A}^{(n)}, \mathcal{S}^{(n)}\} \cup \{\mathcal{F}_n\}$, $\mathcal{B}_{ab} - \{\mathcal{L}^{(n)}, \mathcal{S}^{(n)}\} \cup \{\mathcal{F}_n\}$ is linearly independent. Similarly, since $\mathcal{L}^{(n)}$ contains $(a, v_n)(b, v_1)$ which is not in any cycle of $\mathcal{B}_{ab} - \{\mathcal{L}^{(n)}, \mathcal{S}^{(n)}\} \cup \{\mathcal{F}_n\}$, $\mathcal{B}_{ab} - \{\mathcal{S}^{(n)}\} \cup \{\mathcal{F}_n\}$ is linearly independent. The second part follows easily from noting that the edge $(b, v_1)(b, v_n)$ appears in no cycle of $\mathcal{B}^*_{ab}$.

Remark 3.2. *(1) if* $e = (a, v_1)(b, v_2)$ or $(a, v_2)(b, v_1)$ or $(a, v_n)(b, v_1)$ or $(a, v_1)(b, v_n)$, *then* $f_{\mathcal{B}^*_{ab}}(e) \leq 2$. *(2) If* $e = (a, v_i)(b, v_{i+1})$ *or* $(a, v_{i+1})(b, v_i)$ *and not of the above forms, then* $f_{\mathcal{B}^*_{ab}}(e) \leq 3$. *(3) If* $e = (a, v_i)(a, v_{i+1})$ or $(a, v_1)(a, v_n)$, *then* $f_{\mathcal{B}^*_{ab}}(e) \leq 2$. *(2) If* $e = (b, v_i)(b, v_{i+1})$, *then* $f_{\mathcal{B}^*_{ab}}(e) = 1$. *(4) If* $e = (b, v_1)(b, v_n)$, *then* $f_{\mathcal{B}^*_{ab}}(e) = 0$.

Fact 3.1. *If* G and H *are two isomorphic graphs, then* $b(H) = b(G)$.

Theorem 3.3. *For any two cycles* C_m, C_n with $m, n \geq 3$, we have $b(C_m \otimes C_n) = 3$.

Proof. To prove that $b(C_m \otimes C_n) \leq 3$, it suffices to exhibit a 3-fold basis. We now consider the following cases:
Case 1. m is even (no matter wether n is even or is odd). Define

$$\mathcal{B}(C_m \otimes C_n) = (\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}) \cup \mathcal{B}^*_{a_m a_1} \cup \{C\}$$

where

$$C = (a_1, v_1)(a_2, v_2)(a_3, v_1)(a_4, v_2) \dots (a_{m-1}, v_1)(a_m, v_2)(a_1, v_1).$$

We now use induction on m to show that $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}$ is linearly independent. If $m = 3$, then $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}} = \mathcal{B}_{a_1 a_2} \cup \mathcal{B}_{a_2 a_3}$. By Lemma 3.1, each linear combination of cycles of $\mathcal{B}_{a_1 a_2}$ contains an edge of $\{(a_1, v_j)(a_2, v_k) | 1 \leq j, k \leq n\} \cup E(\ a_1 \times C_n)$ which is not in any linear combination of $\mathcal{B}_{a_2 a_3}$. Thus, $\mathcal{B}_{a_1 a_2} \cup \mathcal{B}_{a_2 a_3}$ is linearly independent. Assume $m \geq 4$ and it is true for less than n. Now, $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}} = (\cup_{i=1}^{m-2} \mathcal{B}_{a_i a_{i+1}}) \cup \mathcal{B}_{a_{m-1} a_m}$. Thus, by the inductive step and Lemma 3.1, both of $\cup_{i=1}^{m-2} \mathcal{B}_{a_i a_{i+1}}$ and $\mathcal{B}_{a_{m-1} a_m}$ are linearly independent. Similarly, by Lemma 3.1, any linear combination of cycles of $\mathcal{B}_{a_{m-1} a_m}$ contains an edge of $\{(a_{m-1}, v_j)(a_m, v_k) | 1 \leq j, k \leq n\} \cup E(\ a_m \times C_n)$ which is not in any cycle of $\cup_{i=1}^{m-2} \mathcal{B}_{a_i a_{i+1}}$. Thus $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}$ is linearly independent. Now, by Lemma 3.1 and noting that

$$E(\mathcal{B}_{a_i a_{i+1}}) \cap E(\mathcal{B}_{a_j a_{j+1}}) = \begin{cases} E(a_j \times C_n), & \text{if } j = i + 1, \\ E(a_i \times C_n), & \text{if } i = j + 1, \\ \phi, & \text{otherwise,} \end{cases}$$

we have that any linear combinations of cycles of $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}$ must contain either at least two edges of the form $\{(a_i, v_j)(a_{i+1}, v_l) | 1 \leq j, l \leq n\}$ for some

$1 \leq i \leq m-1$ which are not in $\mathcal{B}^*_{a_m a_1}$ or contain at least two copies of C_n. On the other hand, by Lemma 3.2, any linear combination of cycles of $\mathcal{B}^*_{a_m a_1}$ either contains an edge of $\{(a_1, v_j)(a_m, v_k) | 1 \leq j, k \leq n\}$ which is not in any cycle of $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}$ or contains only one copy of C_n, in fact $a_m \times C_n$. Thus any linear combination of $\mathcal{B}^*_{a_m a_1}$ cannot be written as a linear combination of cycles of $\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}$. Hence, $(\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}) \cup \mathcal{B}^*_{a_m a_1}$ is linearly independent. We now show that C is independent of cycles of $(\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}) \cup \mathcal{B}^*_{a_m a_1}$. Assume that C is a linear combination modulo 2 of cycles of $(\cup_{i=1}^{m-1} \mathcal{B}_{a_i a_{i+1}}) \cup \mathcal{B}^*_{a_m a_1}$. By a similar argument as in Theorem 2.7,

$$C = \sum_{i=1}^{m} D_i \pmod 2$$

where D_i is a linear combination of $\mathcal{B}_{a_i a_{i+1}}$ for each $i = 1, 2, \ldots, m-1$ and D_m is a linear combination of $\mathcal{B}^*_{a_1 a_m}$. Thus,

$$D_1 = C + \sum_{i=2}^{m} D_i \pmod 2$$

So,

$$\begin{aligned} E(D_1) &= E(C \oplus D_2 \oplus \cdots \oplus D_m) \\ &\subseteq E(\mathcal{B}_{a_1 a_2}) \cap \left(E(\cup_{i=2}^{m} \mathcal{B}_{a_i a_{i+1}}) \cup (E\mathcal{B}^*_{a_m a_1}) \cup E(C)\right). \end{aligned}$$

But

$$\begin{aligned} & E(\mathcal{B}_{a_1 a_2}) \cap \left(E(\cup_{i=2}^{m} \mathcal{B}_{a_i a_{i+1}}) \cup E(\mathcal{B}^*_{a_m a_1}) \cup E(C)\right) \\ \subseteq\ & E(a_m \times C_n) \cup E(a_1 \times P_n) \cup \{(a_1, v_1)(a_2, v_2)\}. \end{aligned}$$

This contradicts Lemma 3.1. We now show that $\mathcal{B}(C_m \otimes C_n)$ is a basis. Note that

$$|\mathcal{B}_{a_i a_{i+1}}| = |\mathcal{B}^*_{a_m a_1}| = 3n$$

Thus,

$$\begin{aligned} |\mathcal{B}(C_m \otimes C_n)| &= \sum_{i=1}^{m-1} |\mathcal{B}_{a_i a_{i+1}}| + |\mathcal{B}^*_{a_m a_1}| + 1 \\ &= 3mn + 1 \\ &= \dim \mathcal{C}(C_m \otimes C_n). \end{aligned}$$

Therefore, $\mathcal{B}(C_m \otimes C_n)$ is a basis for $\mathcal{C}(C_m \otimes C_n)$. Let $e \in E(C_m \otimes C_n)$. (1) If $e = (a_i, v_1)(a_{i+1}, v_2)$ and i is odd, or $(a_i, v_2)(a_{i+1}, v_1)$ and i is even, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B_{a_i a_{i+1}}}(e) + f_{\{C\}}(e) \leq 2 + 1 = 3$. (2) If $e = (a_1, v_1)(a_m, v_2)$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B^*_{a_m a_1}}(e) + f_{\{C\}}(e) \leq 2 + 1 = 3$. (3) If $e = (a_i, v_j)(a_{i+1}, v_{j+1})$ or $(a_i, v_{j+1})(a_{i+1}, v_j)$ or $(a_i, v_1)(a_{i+1}, v_m)$ or $(a_i, v_m)(a_{i+1}, v_1)$ and not of the above form, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B_{a_i a_{i+1}}}(e) \leq 3$. (4) If $e = (a_1, v_1)(a_m, v_2)$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B^*_{a_m a_1}}(e) + f_{\{C\}}(e) \leq 2 + 1 = 3$. (5) If $e = (a_i, v_j)(a_i, v_{j+1})$ or $(a_i, v_1)(a_i, v_n)$ and $i \neq 1, n$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B_{a_{i-1} a_i}}(e) + f_{B_{a_i a_{i+1}}}(e) \leq 2 + 1 = 3$. (6) If $e = (a_1, v_j)(a_1, v_{j+1})$ or $(a_1, v_1)(a_1, v_n)$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B_{a_1 a_2}}(e) + f_{B^*_{a_m a_1}}(e) \leq 2 + 1 = 3$. (7) If $e = (a_m, v_j)(a_m, v_{j+1})$ or $(a_m, v_1)(a_m, v_n)$ and $i \neq 1$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B_{a_{m-1} a_m}}(e) + f_{B^*_{a_m a_1}}(e) \leq 2 + 1 = 3$. (8) If $e = (a_1, v_1)(a_m, v_2)$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B^*_{a_m a_1}}(e) + f_{\{C\}}(e) \leq 2 + 1 = 3$. (9) If $e = (a_i, v_j)(a_{i+1}, v_j)$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B_{a_{i-1} a_i}}(e) \leq 3$ (9) if $e = (a_1, v_j)(a_m, v_j)$, then $f_{\mathcal{B}(C_m \otimes C_n)} = f_{B^*_{a_m a_1}}(e) \leq 3$. Therefore, $b(C_m \otimes C_n) \leq 3$

Case 2. m is odd. Then we have two subcases to consider.

Subcase 1. n is even. Since $C_m \otimes C_n$ is isomorphic to $C_n \otimes C_m$, by Fact 3.1 and Case 1, $b(C_m \otimes C_n) \leq 3$.

Subcase 2. n is odd. Define

$$\mathcal{B}(C_m \otimes C_n) = (\cup_{i=1}^{m-1} B_{a_i a_{i+1}}) \cup B^*_{a_m a_1} \cup \{T\}$$

where

$$\begin{aligned} T \ = \ & (a_1, v_1)(a_2, v_2)(a_3, v_1)(a_4, v_2) \ldots (a_{m-2}, v_1)(a_{m-1}, v_2)(a_{m-2}, v_3) \\ & (a_{m-1}, v_4) \ldots (a_{m-2}, v_{n-2})(a_{m-1}, v_{n-1})(a_m, v_n). \end{aligned}$$

By using the same arguments as in Case 1. We prove that $\mathcal{B}(C_m \otimes C_n)$ is a 3-fold basis. And so $b(C_m \otimes C_n) \leq 3$.

To show that $b(C_m \otimes C_n) \geq 3$ for any $m, n \geq 3$, we argue more or less as in the arguments of Theorem 2.7.

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