# Pair Covering Designs with Block Size 5 with Higher Index – The case of $v$ even

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**Abstract**

A $(v, k, \lambda)$ covering design is a set of $b$ blocks of size $k$ such that each pair of points occurs in at least $\lambda$ blocks, and the covering number $C(v, k, \lambda)$ is the minimum value of $b$ in any $(v, k, \lambda)$ covering design. For $k = 5$ and $v$ even, there are 24 open cases with $2 \leq \lambda \leq 21$, each of which is the start of an open series for $\lambda, \lambda+20, \lambda+40, \ldots$. In this article we solve 22 of these cases with $\lambda \leq 21$, leaving open $(v, 5, \lambda) = (44, 5, 13)$ and $(44, 5, 17)$ (and the series initiated for the former).

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# 1 Introduction

A $t$-$(v,k,\lambda)$ covering design, $(\mathcal{V},\mathcal{B})$, is a set of blocks, $\mathcal{B}$, of uniform size $k$ with the property that every $t$-tuple of the point set, $\mathcal{V}$, occurs in at least $\lambda$ blocks. We will only consider 2-$(v,5,\lambda)$ covers with $\lambda > 1$ here, i.e., pair covers by blocks of size 5, and restrict our study to the case $v$ even. Mills and Mullin [14] dealt with $v$ odd, apart from 4 open cases (see Theorem 1.4). We wish to establish the value of the covering number, $C_\lambda(v,5,2)$, which is the number of blocks in the smallest 2-$(v,5,\lambda)$ covering design. This number satisfies the Schönheim bound [18]:

$$C_\lambda(v,k,2) \geq L_\lambda(v,k,2) = \left\lceil \frac{v}{k} \left\lceil \frac{\lambda(v-1)}{k-1} \right\rceil \right\rceil .$$

However, it is known that the Schönheim bound cannot always be attained.

**Theorem 1.1** (Hanani [9]) *If $\lambda(v-1) \equiv 0 \pmod{(k-1)}$ and $\lambda v(v-1) \equiv (k-1)^2 \pmod{k(k-1)}$, then $C_\lambda(v,k,2) \geq L_\lambda(v,k,2) + 1$.*

**Definition 1.2** *Let $B_\lambda(v,k,2) = L_\lambda(v,k,2) + 1$ in the cases that the hypotheses of Theorem 1.1 are satisfied, and $B_\lambda(v,k,2) = L_\lambda(v,k,2)$ otherwise.*

Table 1: Cases where $B_\lambda(v,k,2) = L_\lambda(v,k,2) + 1$

| $v \pmod{20}$ | |
|---|---|
| 0, 1, 5, 6, 10, 11, 15, 16 | never |
| 2, 4, 12, 14 | $\lambda \equiv 8 \pmod{20}$ |
| 3 | $\lambda \equiv 6 \pmod{10}$ |
| 7, 19 | $\lambda \equiv 8 \pmod{10}$ |
| 8, 18 | $\lambda \equiv 16 \pmod{20}$ |
| 9, 17 | $\lambda \equiv 3 \pmod 5$ |
| 13 | $\lambda \equiv 1 \pmod 5$ |

In most cases, it is known that $C_\lambda(v,5,2) = B_\lambda(v,5,2)$. This is probably true of all $(v,\lambda)$ with the definite exception of a handful of small cases listed in Table 2. However, in spite of much work by a number of authors extending over at least thirty years, the spectrum is still not fully determined. For $\lambda = 1$, the earlier work [11, 13, 15, 16] completed the cases

$v \equiv 2,3 \pmod 4$ and limited $v \equiv 1 \pmod 4$. Recent work [1, 2, 6] has limited $v \equiv 0 \pmod 4$ and improved $v \equiv 1 \pmod 4$. We summarize the current status in Theorem 1.3 for $\lambda = 1$, and in Theorem 1.4 for $\lambda > 1$.

Table 2: Cases where $C_\lambda(v,k,2) > B_\lambda(v,k,2)$ is known

| $v$ | $\lambda$ | $C_\lambda(v,5,2)$ |
|---|---|---|
| 9,15 | 1,2 | $B_\lambda(v,5,2)+1$ |
| 12,20,24 | 1 | $B_\lambda(v,5,2)+1$ |
| 13 | 1 | $B_\lambda(13,5,2)+1 = L_\lambda(13,5,2)+2$ |
| 13 | 2 | $B_\lambda(13,5,2)+1$ |
| 16,17 | 1 | $B_\lambda(v,5,2)+2$ |
| 29 | 1 | $B_\lambda(29,5,2)+1$ or +2 |

**Theorem 1.3** *For $\lambda = 1$, $C_\lambda(v,5,2) = B_\lambda(v,5,2)$, except possibly when:*

*a.* $v = 15$;

*b.* $v \equiv 0 \pmod 4$ *and* $v =$ [12–24], [40–52], [96–108], 124, [132–144], [176–184], 220, 228, 252, 260, 280, 284, 340, 344;

*c.* $v \equiv 9,17 \pmod{20}$ *and* $v =$ [9–89], 109, [129–189], 209, [229–289], 309, [329–377], 429;

*d.* $v \equiv 13 \pmod{20}$ *and* $v \in \{13, 53, 73\}$.

**Theorem 1.4** ([5]) *For $\lambda > 1$, $C_\lambda(v,5,2) = B_\lambda(v,5,2)$, except possibly when:*

*a.* $\lambda = 2$ *and* $v \in \{9, 13, 15, 53, 63, 73, 83\}$;

*b.* $\lambda \equiv 3 \pmod{20}$ *and* $v \in \{18, 26, 122, 126, 138, 142, 146, 158, 162, 178, 186, 218, 226, 278\}$;

*c.* $\lambda \equiv 5 \pmod{20}$ *and* $v \in \{28, 56\}$;

*d.* $\lambda \equiv 7 \pmod{20}$ *and* $v \in \{22, 142, 162\}$;

*e.* $\lambda \equiv 9 \pmod{20}$ *and* $v \in \{28, 56\}$;

*f.* $\lambda \equiv 13 \pmod{20}$ *and* $v = 44$;

*g.* $\lambda \equiv 17 \pmod{20}$ *and* $v \in \{28, 44\}$.

Our main objective is to improve Theorem 1.4 and so establish the main result of this article. The result for $\lambda = 3$ is established in Theorem 2.10 (using Examples 2.8–2.9 and one design in the appendix); the result for $\lambda = 7$ is established in Theorem 2.12 (using Example 2.11 and one design in the appendix); the result for $v = 28$ is given by three designs in the appendix; the result for $v = 44$ is given in Remark 2.7; and the result for $v = 56$ is given in Corollary 2.5.

**Theorem 1.5** *For $\lambda > 1$, $C_\lambda(v,5,2) = B_\lambda(v,5,2)$, except possibly when:*

*a.* $\lambda = 2$ *and* $v \in \{9, 13, 15, 53, 63, 73, 83\}$;

*b.* $\lambda \equiv 13 \pmod{20}$ *and* $v = 44$;

*c.* $\lambda = 17$ *and* $v = 44$.

# 2 Constructions

Some of the terminology we will use is quite standard in design theory; see [7]. For clarification of our notation (specifically how we indicate the standard parameters), we refer to pairwise balanced designs (PBDs), (including BIBDs), as $(v, K, \lambda)$ designs, where $K$ is a list of block sizes that possibly occur. A group divisible design is referred to as a $(K, \lambda)$ GDD of group type $t_1^{g_1} \dots t_n^{g_n}$ if there are $g_i$ groups of size $t_i$ and transversal designs of order $n$ as $\mathrm{TD}_\lambda(k, n)$, dropping the subscript when $\lambda = 1$; note that a $\mathrm{TD}_\lambda(k, n)$ is a $(k, \lambda)$ GDD of group type $n^k$. The prefix "R" will denote a resolvable design.

**Theorem 2.1 (Hanani [9])** *The necessary conditions for the existence of a $(v, k, \lambda)$ BIBD are that $\lambda(v-1) \equiv 0 \pmod{(k-1)}$, $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $v \geq k$. These conditions are sufficient when $k = 5$ with the definite exception of $(v, k, \lambda) = (15, 5, 2)$.*

**Theorem 2.2 ([20])** *A 5-GDD of type $g^u$ exists when $u \geq 5$ and either:*

*a.* $g \equiv 0 \pmod{20}$; *or*

*b.* $g \equiv 0 \pmod{4}$ *and* $u \equiv 0, 1 \pmod{5}$.

**Theorem 2.3 ([2, 6, 8, 17, 19])** *A 5-GDD of type $g^5 m^1$ exists if $g \equiv 0 \pmod{4}$, $m \equiv 0 \pmod{4}$ and $m \leq 4g/3$, with the possible exceptions of $(g, m) = (12, 4)$ and $(12, 8)$.*

Next we have a couple of arithmetic results.

**Lemma 2.4** *If a $(v,k,\lambda)$ BIBD exists, then $C_\lambda(v,k,2) = L_\lambda(v,k,2)$ and $L_{\lambda+\mu}(v,k,2) = L_\lambda(v,k,2) + L_\mu(v,k,2)$.*

**Corollary 2.5** $C_\lambda(56,5,2) = B_\lambda(56,5,2)$ *for all* $\lambda \geq 1$.

*Proof:* In [1], it was established that $C_\lambda(56,5,2) = B_\lambda(56,5,2)$ for $\lambda = 1$, and since a $(56,5,4)$ BIBD exists, the result for $\lambda = 5$ and 9 then follows by Lemma 2.4. ∎

**Remark 2.6** Since a $(v,5,20)$ BIBD exists for all $v \geq 5$, if we have $C_\lambda(v,k,2) = B_\lambda(v,5,2)$ with $2 \leq \lambda \leq 21$, then we have $C_\mu(v,k,2) = B_\mu(v,5,2)$ for all $\mu \equiv \lambda \pmod{20}$ and $\mu \geq \lambda$, by repeatedly adjoining $(v,5,20)$ BIBDs to the minimum $(v,k,\lambda)$ cover.

**Remark 2.7** One possible construction of a $t$-$(v,k,\lambda+\mu)$ covering design is to take a $t$-$(v,k,\lambda)$ covering design and adjoin the blocks of a $t$-$(v,k,\mu)$ covering design. In particular, if $C_\alpha(v,k,2) = L_\alpha(v,k,2)$ for $\alpha = \lambda$ and $\alpha = \mu$, and $L_{\lambda+\mu}(v,k,2) = L_\lambda(v,k,2) + L_\mu(v,k,2)$, then $C_\alpha(v,k,2) = L_\alpha(v,k,2)$ for $\alpha = \lambda+\mu$. For the cases we leave open, we have $L_{\lambda+\mu}(v,5,2) = L_\lambda(v,5,2) + L_\mu(v,5,2)$ for $(v,\lambda,\mu) = (13 \pmod{20}, 1, 1)$, $(44,1,16)$, $(44,9,8)$ and $(44,13,4)$. Theorem 1.1 increases the Schönheim bound when $(v,\lambda) \equiv (13 \pmod{20}, 1)$ and $(44,8)$ where $\lambda$ is computed modulo 20. We do not know $C_\lambda(44,5,2)$ for $\lambda = 1$ or $\lambda \equiv 13 \pmod{20}$, but we do know $C_{21}(44,5,2) = L_{21}(44,5,2)$ and $C_\lambda(44,5,2) = L_\lambda(44,5,2)$ for $\lambda \equiv 16 \pmod{20}$, so $C_\lambda(44,5,2) = L_\lambda(44,5,2)$ for $\lambda \equiv 17 \pmod{20}$ with $\lambda \geq 37$.

We next look at $\lambda = 3$.

**Example 2.8** A 2-$(22,5,3)$ incomplete covering missing a 2-$(2,5,3)$ covering on $Z_{20} \cup \{\infty_1, \infty_2\}$ :

$$(0,2,3,12,19),\quad (0,1,6,8,12),\quad (1,12,14,18),$$
$$(0,5,10,15),\quad (0,5,10,15).$$

The first three orbits are full, while the last two are of length 5. The last orbit gives a parallel class on the finite points. For the other 5 parallel classes, we note the orbit of the third and fourth blocks is given by adding $0,1,\ldots,4$ to the following parallel class:

$$(1,12,14,18),\quad (6,17,19,3),\quad (11,2,4,8),$$
$$(16,7,9,13),\quad (0,5,10,15).$$

We add infinite points to the parallel classes on the short blocks and obtain our incomplete design. The missing subdesign is on the infinite points.

**Example 2.9** An 2-(26, 5, 3) incomplete cover missing a 2-(6, 5, 3) cover is constructed on $(\{a,b,c,d\} \times Z_5) \cup \{\infty_i : i = 0,1,\ldots,5\}$ and is given in three parts.

For the first part, we give a 3-resolvable 4-GDD of type $2^{10}$. Each row forms a 3-resolution set when developed over $Z_5$. The missing pairs are generated by $(a0, b0)$ and $(c0, d0)$. This design has an automorphism of order 720, and can be identified with Design 5 in [10] by labelling our points $a0, b0, c0, d0$ with their $0, 10, 2, 12$, and noting we use their automorphism $\beta$ to develop the design.

$$(a0,a4,c0,c3),\quad (b0,c2,d1,d3),\quad (a0,b2,b3,d2),$$
$$(a0,b1,b4,c2),\quad (b0,c0,c4,d2),\quad (a0,a2,d0,d1).$$

For the second part, we considered using Lamken et al.'s RC(20, 4) [12], but the excess graph of their design is three triangles and two squares, all disjoint, and this has no 1-factor, so we give a (20, 4, 1) cover which has three 2-resolution sets (generated by the first three pairs of base blocks) and a parallel class generated by the last base block. The repeated pairs are generated by $(a0, b0)$, $(a0, b2)$ and $(c0, d0)$, $(c0, d2)$.

$$(a0,a1,b0,b2),\quad (c0,c1,d0,d1),\quad (a0,a2,c0,d3),$$
$$(b0,b1,c0,d2),\quad (a0,b3,c1,c4),\quad (a0,b2,d0,d2),$$
$$(a0,b0,c2,d4).$$

The third part of our design is an RTD(4, 5) with the groups filled in.

The combined design has 90 4-blocks which can be partitioned into 6 3-resolution sets of 15 blocks, and a parallel class of 4 5-blocks. We add an infinite point to each 3-resolution set for our IC.

We have a (6, 5, 3) cover with 5 blocks (given by taking a (5, 4, 3) BIBD, and augmenting each block with an extra point), and adjoining this cover to the IC constructed above gives a (26, 5, 3) cover.

**Theorem 2.10** $C_3(v,5,2) = B_3(v,5,2)$ *for all* $v \geq 5$.

*Proof:* For $v$ odd, this was given by [4, 14] and, for $v \equiv 0 \pmod 4$, by [3]. Also, with our constructions for $v = 18$ and $v = 26$ in the Appendix and Example 2.9, designs for all $v < 100$ are known, see [5].

For the values $v > 100$, we can use our new designs and the previously known designs to give a more uniform presentation than was possible for Assaf and Singh [5]. For $v \equiv 2, 6 \pmod{20}$, with $v = 20n + e$ and $e =$ 2, 6, we can take a $(5,3)$ GDD of type $20^n$ and, using $e$ extra points, fill $n-1$ groups with a $(20+e, 5, 3)$ incomplete cover missing an $(e, 5, 3)$ subcover, then fill the final group with a $(20+e, 5, 3)$ cover. For $v \equiv 10 \pmod{20}$, we follow Assaf and Singh and fill a $(5,3)$ GDD of type $10^{v/10}$. For $v \equiv 14, 18 \pmod{20}$, we can take a $(5,3)$ GDD of type $(20n)^5u^1$ with $u < 100$ and $v \equiv 8, 12 \pmod{20}$ and, using 6 extra points, fill 5 groups with a $(20n+6, 5, 3)$ incomplete cover missing a $(6,5,3)$ subcover, then the final group with a $(u+6, 5, 3)$ cover. Assaf and Singh [5, Lemma 5.1] note that the incomplete cover exists for $2 \leq n \leq 4$, (as well as for 38 and 58 points) and we have shown existence for all the other values (noting our construction above always gives a $(20n+6, 5, 3)$ cover with a $(6,5,3)$ subcover). The required GDD is from Theorem 2.3, but this fails to give us constructions for $134 \leq v \leq 198$, $274 \leq v \leq 298$ and $394 \leq v \leq 398$. However, Theorem 2.3 does give us GDDs of types $32^5m^1$ and $52^5m^1$ for $m = u$ and $m = 20+u$, which deals with $174 \leq v \leq 198$ and $274 \leq v \leq 298$, and Hanani gives $(6,3)$ GDDs of types $5^7$ and $5^8$ (see [9, Lemma 4.22]), and we can also get a $\{5,6\}$ GDD of type $5^{17}(10+u/4)^1$ by removing a parallel class of an $(85,5,1)$ RBIBD and augmenting 12 or 13 other parallel classes with a new point. Truncating one group of the $(6,3)$ GDDs to size $u/4$ and giving all points of these and the augmented RBIBD a weight of 4 in Wilson's fundamental construction gives $(5,3)$ GDDs of types $20^mu^1$ for $m = 6$, 7, and of types $20^{17}(40+u)^1$. We can fill all but the exceptional group of these GDDs with an incomplete cover missing a $(6,5,3)$ subcover using 6 extra points, and then fill the final group with a known cover to get our result. ■

**Example 2.11** A 2-$(22,5,7)$ incomplete covering missing a 2-$(2,5,7)$ covering on $I_{20} \cup \{\infty_1, \infty_2\}$ : First we construct a $(22,5,4)$ PBD missing a $(2,5,4)$ subdesign, by taking a $(5,1)$ GDD of type $4^6$ with 5 groups spanning $I_{20}$ and the last group on $\{a_1, a_2, a_3, a_4\}$. Identify the points $a_i$ with $\infty_2$, and form a block on each other group with $\infty_1$, then form a $(21,5,3)$ BIBD on $I_{20} \cup \{\infty_1\}$. Finally adjoin a 2-$(22,5,3)$ incomplete cover missing a 2-$(2,5,3)$ subcover given in Example 2.8. (We also remark that a complete 2-$(22,5,7)$ cover is given in the appendix.)

**Theorem 2.12** *$C_7(v,5,2) = B_7(v,5,2)$ for all $v \geq 5$.*

*Proof:* The only cases we need deal with, by [5, 14], are $v = 22$ which is solved in the Appendix, and some $v \equiv 2 \pmod{20}$ with $v > 100$, and we can write these as $v = 20n + 2$ with $n \geq 5$, take a GDD of type $20^n$ and, using 2 extra points, fill $n - 1$ groups with a $(22, 5, 7)$ incomplete cover missing an $(2, 5, 7)$ subcover given in Example 2.11, then fill the final group with a $(22, 5, 7)$ cover. ∎

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# Appendix

The appendix contains the covers for (18, 5, 3), (22, 5, 7), (28, 5, 5), (28, 5, 9) and (28, 5, 17).

Let $X(v) = \{1, 2, ..., v\}$. Let $X^{(k)}(v)$ be the set of all $k$-subsets of $X(v)$. We use the following compressed notation. Suppose the k-subsets of $X(v)$ are arranged in lexicographical order (for example, let $v = 4$, $k = 3$, then the order is 123, 124, 134, 234). We present the blocks of a design by a sequence $a_1, a_2, \ldots, a_b$, such that the $n$-th block of the design is the $(\sum_{i=1}^{n} a_i)$-th $k$-set from the lexicographical arrangement of $X^{(k)}(v)$, where $1 \le n \le b$.

An (18, 5, 3) cover:
46, 9, 58, 185, 475, 74, 190, 281, 35, 204, 44, 149, 18, 365, 452, 283, 278, 119, 490, 38, 160, 113, 101, 129, 157, 48, 164, 21, 121, 611, 141, 335, 25, 232, 243, 211, 92, 345, 124, 231, 337, 71, 8, 68, 293, 309, 14.
The excess pairs are $(1, i)$ for $i = 2, 3, \ldots, 6$ and $(j, j+6)$ for $j = 7, 8, \ldots, 12$.

A $(22, 5, 7)$ cover:
512, 39, 82, 103, 21, 71, 94, 189, 186, 132, 176, 9, 121, 27, 38, 36, 383, 38, 2, 260, 74, 37, 109, 65, 212, 24, 204, 170, 158, 185, 430, 198, 248, 797, 126, 52, 239, 63, 139, 113, 18, 200, 107, 370, 93, 47, 96, 84, 241, 251, 60, 314, 151, 41, 32, 596, 63, 220, 152, 108, 365, 7, 72, 62, 74, 138, 137, 586, 17, 164, 8, 329, 165, 466, 573, 16, 164, 284, 54, 351, 144, 6, 514, 63, 84, 48, 280, 71, 43, 208, 185, 256, 529, 268, 176, 158, 60, 41, 335, 231, 1, 269, 48, 164, 36, 168, 54, 229, 63, 297, 8, 275, 150, 382, 195, 146, 23, 167, 7, 109, 111, 135, 221, 98, 178, 51, 2, 70, 193, 142, 67, 78, 345, 312, 25, 180, 48, 15, 109, 32, 179, 78, 91, 210, 63, 151, 279, 153, 58, 242, 194, 154, 153, 301, 236, 319, 172, 92, 282, 132, 36, 64, 250.
The excess pairs are $(1, i)$ for $i = 2, 3, \ldots, 6$ and $(j, j+8)$ for $j = 7, 8, \ldots, 14$.

A $(28, 5, 5)$ cover:
131, 298, 314, 242, 1307, 99, 380, 470, 296, 523, 100, 843, 404, 262, 339, 1340, 30, 290, 670, 446, 76, 184, 1040, 2450, 170, 222, 363, 616, 353, 438, 736, 381, 931, 162, 465, 483, 44, 1766, 24, 908, 44, 336, 633, 538, 175, 1564, 237, 186, 453, 758, 406, 53, 241, 620, 319, 869, 463, 263, 986, 912, 845, 377, 48, 471, 774, 259, 766, 274, 150, 4, 378, 1382, 479, 1707, 674, 749, 87, 62, 854, 679, 117, 649, 95, 142, 750, 454, 209, 952, 1511, 136, 17, 152, 618, 40, 52, 174, 7, 866, 46, 899, 1328, 529, 2035, 266, 390, 655, 132, 462, 1655, 65, 202, 584, 659, 413, 195, 1596, 153, 600, 667, 214, 109, 302, 342, 314, 726, 702, 719, 660, 417, 157, 945, 1227, 1392, 118, 93, 697, 60, 1003, 3, 106, 199, 951, 839, 216, 96, 247, 267, 1329, 8, 899, 278, 163, 472, 2206, 334, 412, 646, 446, 651, 165, 154, 484, 26, 712, 99, 184, 556, 41, 1372, 276, 163, 427, 346, 160, 609, 976, 73, 293, 301, 1338, 399, 533, 1017, 116, 811, 156, 1367, 865, 178, 409, 302.
The excess pairs are $(1, i)$ for $i = 2, 3, \ldots, 6$; $(2, i)$ for $i = 7, 8, 9, 10$; $(7, i)$ for $i = 11, 12, 13, 14$, and $(j, j + 7)$ for $j = 15, 16, \ldots, 21$.

A $(28, 5, 9)$ cover:
22, 96, 139, 98, 267, 217, 646, 418, 143, 446, 77, 182, 112, 199, 199, 260, 452, 807, 341, 158, 44, 639, 49, 444, 536, 295, 345, 523, 238, 90, 43, 418, 166, 130, 407, 674, 285, 292, 35, 215, 662, 164, 132, 161, 50, 354, 120, 8, 566, 69, 1203, 310, 386, 22, 120, 422, 809, 40, 157, 128, 73, 315, 504, 33, 102, 507, 266, 296, 1127, 280, 36, 375, 479, 13, 26, 588, 338, 33, 847, 283, 112, 35, 25, 298, 587, 204, 9, 1153, 84, 326, 81, 91, 245, 491, 624, 42, 351, 10, 298, 189, 273, 9, 307, 304, 230, 122, 72, 83, 429, 231, 1282, 35, 42, 677, 651, 116, 86, 21, 496, 439, 50, 705, 184, 496, 208, 650, 29, 320, 114, 38, 365, 544, 368, 71, 131, 84, 227, 131, 353, 68, 111, 260, 382, 24, 104, 181, 194, 379, 366, 251, 110, 1216, 479, 71, 52, 193, 521, 149, 292, 324, 88, 247, 24,

365, 543, 120, 312, 65, 153, 23, 1108, 128, 102, 420, 77, 289, 1213, 74, 33, 823, 261, 147, 13, 558, 90, 183, 301, 224, 55, 136, 413, 238, 118, 29, 710, 208, 433, 675, 330, 44, 168, 338, 245, 237, 516, 62, 251, 1033, 103, 61, 718, 271, 222, 109, 490, 90, 88, 265, 274, 23, 21, 528, 1076, 38, 61, 16, 861, 73, 129, 1071, 253, 216, 376, 304, 346, 135, 25, 145, 982, 229, 462, 40, 548, 192, 114, 493, 134, 147, 401, 48, 1, 68, 576, 290, 269, 244, 52, 256, 113, 718, 275, 691, 165, 358, 438, 215, 302, 10, 569, 8, 259, 280, 446, 147, 210, 329, 155, 65, 762, 115, 8, 236, 245, 5, 606, 301, 309, 215, 246, 596, 657, 102, 163, 35, 1096, 91, 241, 374, 761, 114, 53, 79, 444, 181, 23, 472, 319, 137, 432, 531, 78, 123, 70, 400, 462, 358, 378, 134, 288, 257, 526, 587, 169, 207, 184, 124, 169, 85, 95, 78, 334, 109, 467, 380, 157, 610, 172, 672, 431, 60, 194, 57.
The excess pairs are $(1,2)$ twice, $(1,i)$ for $i = 3,4,5$, $(2,i)$ for $i = 6,7,8$, and $(j,j+10)$ for $j = 9,10,\ldots,18$.

A (28,5,17) cover:
184, 271, 469, 69, 77, 90, 67, 17, 31, 368, 73, 3, 262, 149, 227, 202, 10, 275, 22, 13, 114, 5, 93, 111, 189, 248, 45, 350, 44, 291, 8, 101, 132, 305, 210, 37, 354, 35, 57, 290, 84, 366, 245, 64, 19, 101, 74, 158, 351, 155, 16, 292, 213, 7, 98, 101, 273, 48, 303, 63, 17, 28, 251, 326, 47, 94, 65, 112, 515, 118, 51, 253, 67, 400, 141, 265, 127, 500, 100, 39, 237, 126, 182, 57, 72, 68, 45, 59, 154, 451, 8, 91, 182, 114, 192, 34, 61, 88, 34, 67, 60, 60, 29, 259, 17, 589, 224, 502, 45, 10, 167, 383, 203, 19, 37, 188, 254, 88, 20, 55, 102, 354, 172, 64, 335, 16, 214, 218, 136, 113, 62, 116, 345, 121, 352, 123, 88, 2, 151, 129, 70, 261, 10, 20, 93, 244, 62, 52, 460, 313, 67, 33, 186, 100, 230, 164, 122, 337, 114, 18, 190, 63, 177, 26, 72, 82, 247, 331, 78, 65, 18, 259, 17, 320, 379, 80, 34, 175, 12, 122, 63, 62, 174, 600, 325, 215, 162, 2, 51, 417, 87, 232, 166, 398, 64, 259, 102, 120, 51, 173, 427, 155, 50, 37, 130, 10, 186, 147, 46, 106, 330, 73, 39, 282, 142, 16, 137, 40, 9, 190, 657, 248, 57, 201, 104, 48, 295, 164, 146, 88, 79, 380, 563, 106, 49, 52, 36, 306, 280, 14, 204, 335, 174, 113, 239, 115, 206, 59, 372, 2, 92, 243, 127, 70, 159, 35, 177, 3, 342, 264, 2, 85, 342, 197, 194, 124, 210, 154, 325, 117, 195, 46, 95, 34, 304, 13, 25, 236, 116, 16, 285, 28, 204, 19, 37, 59, 263, 48, 8, 104, 439, 6, 59, 373, 6, 296, 92, 111, 137, 69, 132, 59, 354, 95, 81, 192, 159, 41, 112, 66, 272, 375, 97, 38, 160, 357, 203, 219, 60, 65, 94, 182, 270, 484, 39, 3, 306, 335, 88, 70, 2, 224, 61, 25, 43, 15, 177, 231, 67, 124, 121, 66, 94, 189, 265, 593, 1, 178, 60, 93, 111, 288, 67, 42, 4, 180, 126, 205, 3, 225, 171, 63, 467, 35, 501, 43, 143, 148, 39, 341, 272, 7, 24, 151, 114, 128, 203, 84, 53, 101, 39, 88, 333, 197, 283, 68, 396, 4, 57, 119, 43, 105, 342, 29, 36, 4, 425, 160, 251, 38, 510, 35, 364, 164, 156, 253, 186, 92, 91, 41, 60, 137, 9, 303, 297, 148, 84, 2, 93, 22, 1, 98, 48, 330, 75, 590, 300, 414, 27, 120, 44, 63, 225, 17, 240, 29, 37, 264, 455, 101, 174, 315, 260, 68, 25, 201, 247, 17, 104, 45, 310, 28, 249, 4, 341, 71, 67, 454, 111, 90, 22, 31, 15, 231, 13, 43, 279, 304, 28, 214, 106, 25,