

On the Complexity of Finding Optimal Global Alliances

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Abstract

A *defensive alliance* in a graph $G(V, E)$ is a set of vertices $S \subseteq V$ such that for every vertex $v \in S$, the closed neighborhood $N_G[v]$ of v has at least as many vertices in S as it has in $V - S$. An *offensive alliance* is a set of vertices $S \subseteq V$, such that for every vertex v in the boundary $\partial(S)$ of S the number of neighbors that v has in S is greater than or equal to the number of neighbors it has in $V - S$. A subset of vertices which is both an offensive and a defensive alliance is called a *powerful alliance*. An alliance which is also a dominating set is called a *global alliance*. In this paper, we show that finding an optimal defensive (offensive, powerful) global alliance is an NP-hard problem.

1 Introduction

Initiated by Kristiansen, Hedetniemi and Hedetniemi [9], the study of alliance as a graph-theoretic concept has recently attracted a great deal of attention due to some interesting applications in a variety of areas ranging from computer networks to national defense [7][9]. Our main interest lies in understanding the extent to which alliances characterize *Web communities*—groups of web pages which share a common theme such as content, geographic location, domain, etc. A number of authors have adopted the definition of Web community proposed recently by Flake, Lawrence, and Giles [4]: “A Web community is a set of web pages having more hyperlinks (in either direction) to members of the set than to non-members”. This definition coincides with that of a defensive alliance given in [9]. Recently, several heuristic algorithms for mining Web communities (i.e., defensive alliances) have been proposed [5][10]. However, the computational complexity of alliance-mining problems remains largely unexplored. In this paper we investigate the complexity of three such problems.

We begin with some terminology and definitions. Let $G(V, E)$ be a graph with vertex set V and edge set E . For any vertex $v \in V$, the *open*

neighborhood of v , $N_G(v)$, is defined as the set of all vertices adjacent to v , while the *closed neighborhood* of v , $N_G[v]$, is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *boundary* of a set $S \subseteq V$ is defined as the set of vertices at distance one from S , i.e., $\partial(S) = \bigcup_{v \in S} N_G(v) - S$.

Next, we present the definitions of the three types of alliance that concern this paper:

Defensive Alliance [7]: A non-empty set of vertices $S \subseteq V$ is called a defensive alliance if and only if for every $v \in S$, $|N_G[v] \cap S| \geq |N_G[v] - S|$. One could say that every vertex in S is defended from possible attack by vertices in $V - S$.

Offensive Alliance [3]: A non-empty set of vertices $S \subseteq V$ is called an offensive alliance if and only if for every $v \in \partial(S)$, $|N_G[v] \cap S| \geq |N_G[v] - S|$. In this case it is said that every vertex in $\partial(S)$ is vulnerable to possible attack by vertices in S .

Powerful Alliance [1]: An alliance which is both offensive and defensive is called a powerful alliance and it may be succinctly characterized as a non-empty set of vertices S such that for every $v \in N_G[S]$, $|N_G[v] \cap S| \geq |N_G[v] - S|$.

An alliance S is called *global* if it forms a *dominating set* (i.e., a set $S \subseteq V$ such that every vertex in $V - S$ is adjacent to at least one vertex in S).

In the paper that introduced the concept of alliance, Kristiansen et al. [9] proved several results on the minimum cardinality of a defensive alliance. Fricke et al. [6] proved a couple of conjectures posed earlier by Kristiansen et al. [9]. Haynes, Hedetniemi and Henning [7] initiated the study of the properties of global defensive alliances, while Favaron et al. [3] introduced offensive alliances. Trees that have equal domination and global strong alliance numbers were studied by Haynes, Hedetniemi and Henning [8], whereas maximum alliance-free and minimum alliance-cover sets were investigated by Shafique and Dutton [11].

In this paper we are concerned with the computational complexity of three optimization problems on *global* alliances whose decision versions are defined as follows:

Global Defensive Alliance (GDA):

Given: A graph $G(V, E)$ and a positive integer $K \leq |V|$.

Question: Is there a global defensive alliance in G of size K or less?

Global Offensive Alliance (GOA):

Given: A graph $G(V, E)$ and a positive integer $K \leq |V|$.

Question: Is there a global offensive alliance in G of size K or less?

Global Powerful Alliance (GPA):

Given: A graph $G(V, E)$ and a positive integer $K \leq |V|$.

Question: Is there a global powerful alliance in G of size K or less?

In all subsequent proofs we shall employ transformations from the Dominating Set problem defined as:

Dominating Set (DS):

Given: A graph $G(V, E)$ and a positive integer $K \leq |V|$.

Question: Is there a dominating set in G of size K or less?

2 The complexity of the GDA, GOA, and GPA problems

The GDA, GOA and GPA problems are clearly in the set NP. Therefore, we only discuss the transformations from instances of the DS problem to instances of these three problems.

Theorem 1. GDA is NP-Complete.

Proof. Let $I = [G(V, E), K]$ be any instance of DS. We must construct an instance $I' = [G'(V', E'), K']$ of GDA such that G has a dominating set of size K or less if and only if G' has a global defensive alliance of size K' or less.

First, we describe a procedure to construct the graph G' . Initially let $G' = G$. Then, for each non-isolated vertex $v_i \in V$ add $d_G(v_i) - 1$ components $C_{i,1}, \dots, C_{i,d_G(v_i)-1}$ to the graph G' . We say that the components $C_{i,j}$, $1 \leq j \leq d_G(v_i) - 1$, are "rooted" at vertex v_i . Each component $C_{i,j}$ consists of two vertices and two edges, as follows:

$$C_{i,j} = (\{a_{i,j}, b_{i,j}\}, \{(v_i, a_{i,j}), (a_{i,j}, b_{i,j})\}). \quad (1)$$

In other words, the vertex $a_{i,j}$ of the component $C_{i,j}$ is connected to the "root" v_i , as well as to the other vertex $b_{i,j}$ of this component. Letting

$$A_{v_i} = \{a_{i,j} \mid 1 \leq j \leq d(v_i) - 1\}, \quad B_{v_i} = \{b_{i,j} \mid 1 \leq j \leq d(v_i) - 1\}, \quad (2)$$

$$A_S = \bigcup_{v_i \in S} A_{v_i}, \quad B_S = \bigcup_{v_i \in S} B_{v_i}, \quad (3)$$

and

$$A = A_V, \quad B = B_V, \quad (4)$$

the graph G' is completely specified by:

$$V' = V \cup A \cup B, \quad (5)$$

$$E' = E \cup \left(\bigcup_{v_i \in V} \{(v_i, a_{i,j}), (a_{i,j}, b_{i,j}) \mid a_{i,j} \in A_{v_i}, b_{i,j} \in B_{v_i}\} \right). \quad (6)$$

In the remainder of the paper, we shall refer to the vertices (edges) of components $C_{i,j}$ as "component" vertices (edges) to distinguish them from the vertices (edges) of G .

Let Q be the total number of components $C_{i,j}$. To complete the

construction of the instance I' we let $K' = K + Q$. Figure 1 shows an example of the graph obtained by this construction procedure.

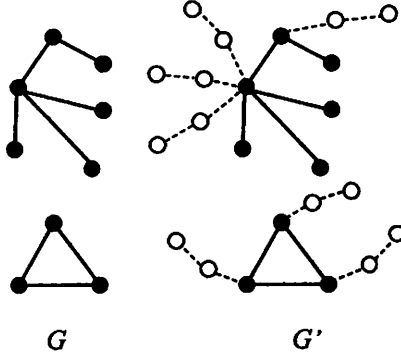


Figure 1. Construction of an instance of GDA from an instance of DS. The “component” vertices are represented by empty circles and the “component” edges are represented by dotted lines.

Note that $Q = \sum_{v_i \in V} (d_G(v_i) - 1) = 2|E| - |V|$. Therefore, the construction of G' can be accomplished in linear time.

It remains to show that G has a dominating set of size K or less if and only if G' has a global defensive alliance of size K' or less. First, suppose that $S \subseteq V$ is a dominating set in G with $|S| \leq K$. Let

$$S' = S \cup A_S \cup B_{V-S}. \quad (7)$$

Note that S is a subset of S' . Furthermore, for each vertex $v_i \in S$, S' contains all the vertices $a_{i,1}, \dots, a_{i,d_G(v_i)-1}$. Finally, for each vertex $v_j \notin S$, S' contains all the vertices $b_{j,1}, \dots, b_{j,d_G(v_j)-1}$. These observations together with the following lemma imply that I' is a YES instance of GDA.

Lemma 1. S' is a global defensive alliance in G' with size K' or less.

Proof. S' contains all vertices of S as well as one vertex from each component $C_{i,j}$. Therefore,

$$|S'| = |S| + Q \leq K + Q = K'. \quad (8)$$

Furthermore, S' dominates V (because S dominates V and $S \subseteq S'$), and it also dominates all the components $C_{i,j}$ (because S' contains exactly one vertex from every such component). Thus, S' is a dominating set in G' .

Finally, S' is a defensive alliance in G' . To see this, first note that every vertex $v_i \in S' \cap V$, has exactly $d_G(v_i) - 1$ neighbors $a_{i,1}, \dots, a_{i,d_G(v_i)-1}$ in the set

S' . Since v_i can have at most $d_G(v_i)$ neighbors outside S' (which happens only if all the neighbors of v_i in V are outside S'), the defensive alliance property is satisfied for v_i . Furthermore, each vertex $a_{i,j} \in S'$ has exactly one neighbor inside S' (the “root” vertex v_i) and exactly one neighbor outside (the vertex $b_{i,j}$), and thus it satisfies the defensive alliance property. Finally, each vertex $b_{i,j} \in S'$ has degree one in G' , therefore it satisfies the defensive alliance property. \square

Conversely, suppose that S' is a global defensive alliance in G' with K' or less vertices. We need to find a set $S \subseteq V$ of size K or less that forms a dominating set in G . Let us begin with the following simple observation:

Observation 1. S' contains at least Q “component” vertices.

Proof. S' is a dominating set in G' , hence it contains at least one vertex from each component $C_{i,j}$. \square

As an immediate corollary of Observation 1 we get:

$$|S' \cap V| \leq K' - Q = K. \tag{9}$$

Since the set $S' \cap V$ has size at most K , this set may be considered as a first candidate for a dominating set in G . However, it is not hard to see that $S' \cap V$ does not necessarily form a dominating set in G . The problem is that there might exist a vertex $v_i \in (V' - S') \cap V$ which has no neighbor in $S' \cap V$. Figure 2 shows an example of such a vertex.

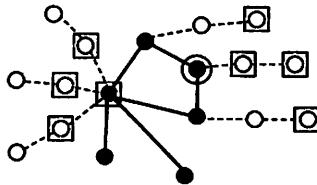


Figure 2. A graph G' , a global defensive alliance S' in G' (vertices surrounded by squares), and a “non-component” vertex (surrounded by a circle) that has only one neighbor in S' which is a “component” vertex.

Such a vertex, v_i wouldn't be dominated by $S' \cap V$. In that case we say that v_i is a “component-dominated” vertex. Now, let D' be the set of all “component-dominated” vertices, that is:

$$D' = \{v_i \in (V' - S') \cap V \mid v_i \text{ has no neighbor in } V \cap S'\}. \tag{10}$$

Note that the vertices of D' are the only ones among the vertices of V that

will *not* be dominated by $S' \cap V$. Hence, the set $(S' \cup D') \cap V$ must form a dominating set in G . The next lemma, which is a strengthened version of Observation 1, implies that $|S'| \leq K$.

Lemma 2. S' contains at least $Q + |D'|$ “component” vertices.

Proof. Consider an arbitrary vertex $v_i \in D'$. There must be a vertex $a_{i,j}$ such that $a_{i,j} \in S'$ (because S' is a dominating set and v_i does not have any neighbor in $S' \cap V$). Now, the vertex $b_{i,j}$ must also be in S' , because otherwise the defensive alliance property would be violated for $a_{i,j}$. Hence, it follows that for every vertex $v_i \in D'$ there exists at least one component $C_{i,j}$ with both vertices in S' . Thus, in total, there are $|D'|$ components with both vertices contained in S' . The remaining $Q - |D'|$ components, must each have at least one vertex in S' because S' is a dominating set. Therefore, the number of “component” vertices in S' is at least $2|D'| + Q - |D'|$, i.e., $Q + |D'|$. \square

Now, let

$$S = (S' \cup D') \cap V. \quad (11)$$

From Lemma 2 it follows that $|S| \leq K$. As argued earlier, S is also a dominating set in G . Hence, I is a YES instance of DS. This completes the proof of Theorem 1. \square

Theorem 2. GOA is NP-Complete.

Proof. The transformation that maps instances $I = [G(V, E), K]$ of DS into instances $I' = [G'(V', E'), K']$ of GOA is exactly the same as the one employed for the GDA problem. We prove that G has a dominating set with K or less vertices if and only if G' has a global offensive alliance with K' or less vertices.

Suppose that $S \subseteq V$ is a dominating set in G with $|S| \leq K$. Let

$$S' = S \cup A. \quad (12)$$

We show that S' forms a global offensive alliance in G' and that the size of S' is K' or less. Indeed, as in the proof of Lemma 1, it can be shown that S' is a dominating set in G' and that $|S'| \leq K'$. It remains to show that S' is an offensive alliance in G' . Since S' is a dominating set in G' , we get:

$$\partial(S') = V' - S' = (V - S) \cup B. \quad (13)$$

Let $v' \in \partial(S')$. If $v' \in B$, then $|N_{G'}[v'] \cap S'| = |N_{G'}[v'] \cap (V' - S')| = 1$, so the offensive alliance condition is satisfied. Otherwise, if $v' \in (V - S)$, there must exist a vertex $w \in S$ such that $(v', w) \in E$ (because S is a dominating set in G). This guarantees that $|N_{G'}[v'] \cap S'| \geq d_G(v')$ and

$|N_G[v] \cap (V' - S')| \leq d_G(v)$, i.e., the offensive alliance property is again satisfied.

Conversely, suppose that S' is a global offensive alliance in G' with K' or less vertices. Let

$$S = S' \cap V. \quad (14)$$

Since S' is a dominating set in G' , we get (as in the proof of Lemma 2) that $|S| \leq K$. In this case, S is also a dominating set in G . Indeed, let v be a vertex from $V - S$. We show that there must exist another vertex $w \in S$ such that $(v, w) \in E$. For, assume there is no such vertex. Then, $|N_G[v] \cap (V' - S')| \geq d_G(v) + 1$, whereas $|N_G[v] \cap S'| \leq d_G(v) - 1$ (with equality holding only if $A_v \subseteq S'$). Hence, the offensive alliance property would be violated for v . \square

Theorem 3. GPA is NP-Complete.

Proof. Let $I = [G(V, E), K]$ be any instance of DS. The procedure employed to construct an instance $I' = [G'(V', E'), K']$ of GPA is similar to (but different from) the one used in the previous two cases.

Again, initially let $G' = G$. Then, for every vertex $v_i \in V$ with $d_G(v_i) \geq 2$ add $d_G(v_i)$ components $C_{i,1}, \dots, C_{i,d(v_i)}$ to G' (the following arguments would remain valid even if we add $d_G(v_i)$ components only for all vertices v_i with $d_G(v_i) = 2$ and $d_G(v_i) - 1$ components for all vertices v_i with $d_G(v_i) \geq 3$). Each component $C_{i,j}$ is exactly the same as in the preceding construction. However, this time for every vertex v_i with $d_G(v_i) \geq 2$, the vertices of A_{v_i} are joined together by a path $P_i = \langle a_{i,1}, a_{i,2}, \dots, a_{i,d(v_i)} \rangle$ (see Figure 3).

In other words, the graph G' is completely defined by:

$$V' = V \cup A \cup B, \quad (15)$$

$$E' = E \cup E_1 \cup E_2, \quad (16)$$

where A and B were defined in equation (4), and

$$E_1 = \left(\bigcup_{v_i \in V} \{(v_i, a_{i,j}), (a_{i,j}, b_{i,j}) \mid a_{i,j} \in A_{v_i}, b_{i,j} \in B_{v_i}\} \right), \quad (17)$$

$$E_2 = \left(\bigcup_i \{(a_{i,k}, a_{i,k+1}) \mid 1 \leq k \leq d(v_i) - 1\} \right). \quad (18)$$

To complete the construction, let Q be the total number of components and let $K' = K + Q$.

Assume $S \subseteq V$ is a dominating set in G with $|S| \leq K$. Let

$$S' = S \cup A, \quad (19)$$

and let's prove that S' is a global powerful alliance in G' with size K' or less.

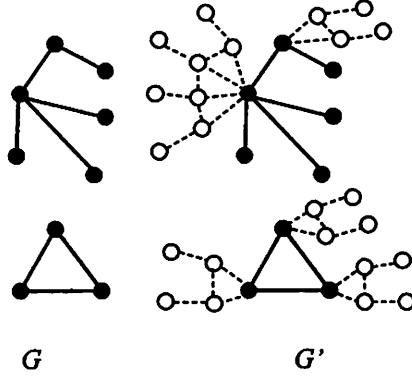


Figure 3. Construction of an instance of GPA from an instance of DS.

We only show that S' forms a powerful alliance in G' (the rest of the proof is the same as in the two preceding theorems). First, let v' be any vertex in S' and show that the *defensive alliance* property is satisfied for v' . If $v' \in S$, then by the definition of S' , it is guaranteed that $|N_{G'}[v'] \cap S'| > |N_{G'}[v'] \cap (V' - S')|$. Otherwise, if $v' \in A$, then $|N_{G'}[v'] \cap S'| = 3$ or $|N_{G'}[v'] \cap S'| = 4$, whereas $|N_{G'}[v'] \cap (V' - S')| = 1$. Next, we show that the *offensive alliance* property is satisfied for every vertex in $\partial(S')$. Observe that

$$\partial(S') = V' - S' = (V - S) \cup B. \quad (20)$$

Let $v' \in \partial(S')$. If $v' \in (V - S)$, then since S is a dominating set in G , there must exist a vertex $w' \in S$ such that $(v', w') \in E$. Hence, it follows that $|N_{G'}[v'] \cap S'| > |N_{G'}[v'] \cap (V' - S')|$. Finally, if $v' \in B$, then $|N_{G'}[v'] \cap S'| = |N_{G'}[v'] \cap S'| = 1$.

Conversely, let S' be a global powerful alliance in G' with K' or less vertices. First, it can be verified that every dominating set in G' , must contain at least one vertex from each component $C_{i,j}$. Thus, S' contains at least Q component vertices. Letting

$$S = S' \cap V, \quad (21)$$

yields $|S| \leq K$. Since, S' is a global powerful alliance (and hence a global offensive alliance), an argument similar to the one in proof of Theorem 2 establishes that S forms a dominating set in G . \square

3 Conclusion

We proved that finding optimal defensive, offensive, or powerful global alliances are NP-hard problems. An interesting open question with significant practical ramifications is to determine whether these and other similar alliance-mining problems are easier to solve in the class of dynamic random graph models of web-like networks [2].

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