Investigations on the Existence of Some Balanced Arrays with Two Symbols

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Abstract

In this paper we obtain some necessary conditions for the existence of balanced arrays (B-arrays) with two symbols and having strength seven. We then describe how these conditions involving the parameters of the array can be used to obtain an upper bound on the constraints of such arrays, and give some illustrative examples to this effect.

1 Introduction and Preliminaries

For ease of reference, we recall here the definition of a balanced array of strength t and having two symbols.

Definition 1.1. A balanced array (B-array) T with m rows(constraints), N columns (runs, treatment combinations), of strength t ($t \leq m$), and with two symbols(say, 0 and 1) is a matrix of size ($m \times N$) such that in every ($t \times N$) sub-matrix T^* of T, every t rowed column vector $\underline{\alpha}$ of weight i ($0 \leq i \leq t$; the weight of $\underline{\alpha}$ means the number of non-zero elements in it) appears with the same frequency (say) μ_i. The vector $\underline{\mu}' = (\mu_0, \mu_1, \cdots, \mu_t)$ is called the index set of T.

Remark: It is quite obvious that $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$.

We can easily extend the above definition to B-arrays with s symbols. In this paper, we restrict ourselves to arrays with $t = 7$.

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Definition 1.2. An orthogonal array (O array) is a B-array for which $\mu_i = \mu$ for each i. Thus, N here equals $\mu 2^t$.

Thus, O-arrays form a subset of B-arrays. These arrays have been extensively used to construct symmetrical as well as asymmetrical fractional factorial designs. B-arrays with different values of t give rise to factorial designs of different resolutions. For example a B array with t=7 will give us a balanced factorial design of resolution VIII. A design of resolution VIII will allow us to estimate all the effects up to and including three factor interactions in the presence of four factor interactions under the assumption that higher order interactions are negligible. B-arrays, a generalization of O-arrays, are also related to other combinatorial structures such as balanced incomplete block (BIB) designs, rectangular designs, group divisible designs, nested BIB designs, etc.. Thus the existence and construction of such arrays is very important from the point of view of applications as well as to study the combinatorial entities. To gain further insight into the importance of B-arrays to combinatorics and to statistical design of experiments, the interested readers may consult the list of references (by no means an exhaustive list) at the end of the paper, and also further references listed therein.
The problem of constructing a B-array for a given m($m \geq 8$) and an arbitrary index set $\underline{\mu}' = (\mu_0, \mu_1, \cdots, \mu_7)$ is clearly a nontrivial problem. To find the maximum value of m for a given $\underline{\mu}'$ is an important problem both in design theory and in combinatorics. Such problems for B-arrays and O-arrays have been discussed, among others, by Chopra and/or Dios [6,7], Hedayat et.al [8], Rafter and Seiden [13], Rao [14,15], Saha et.al [17], Seidan and Zemach [18], Yamammoto et.al [21], etc..
In this paper we derive some inequalities involving the parameters m and $\underline{\mu}'$ for B-arrays with t=7. For a B-array to exist, these inequalities must be satisfied by the given values of m and $\underline{\mu}'$. If we obtain a contradiction in at least one such condition, then that B-array will not exist. On the other hand, the B-array may or may not exist if all the inequalities are satisfied. For a given $\underline{\mu}'$, we indicate the use of these inequalities to obtain an upper bound on the number of constraints m.

2 Main Results with Discussion

The following results can be easily derived..

Lemma 2.1. *A B-array with $m = t = 7$ and an arbitrary index set $\underline{\mu}'$ always exists.*

Lemma 2.2. *A B-array T with $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_7)$ is also of strength k $(0 \le k \le 7)$.*

Note: Considered as an array of strength k, let $A(j,k)$ denote the j-th element $(j = 0, 1, 2, \ldots, k)$ of T. Clearly, $A(j,k) = \sum_{i=0}^{7-k} \binom{7-k}{i} \mu_{i+j}$. It is quite clear that $A(j,k)$ are merely linear functions of the μ_i's, and can be easily calculated once we know $\mu_0, \mu_1, \ldots$, etc. For example, $A(2,4) = \sum_{i=0}^{3} \binom{3}{i} \mu_{i+2} = \mu_2 + 3\mu_3 + 3\mu_4 + \mu_5$.

Lemma 2.3. *Consider a B-array T with index set $\underline{\mu}'$ and m rows. Let $x_j (0 \le j \le m)$ be the number of columns of weight j in T, Then the following results are true:*

$$\begin{aligned} \sum x_j &= N \\ \sum j^k x_j &= m_k A(k,k) + \sum_{l=1}^{k-1} a(l, k-1) M_l \end{aligned} \tag{2.1}$$

where $m_k = m(m-1)\ldots(m-k+1)$, and $k = 1, 2, \ldots, 7$.

Remark: Results in (2.1) express the moments of the column weights in terms of the parameters of the array T. In fact, the R.H.S. of each is merely a polynomial function in m with coefficient function of the μ_i's. To derive (2.1), one has merely to count the number of vectors of weight k in two ways (through rows and columns) by considering T as an array of strength k $(k = 1, 2, \ldots, 7)$.

Note: For ease in computation we next provide the values of various coefficients in (2.1)

We obtain the values of $A(k,k)$, $k = 1, 2, \ldots, 7$, by using $\sum_{i=k}^{7} \binom{7-k}{i-k} \mu_i$, and of $a(l, k-1)$ with $l = 1, 2, \ldots, k-1$, and $k = 1, 2, \ldots, 7$ are given by, 0; 1; 3, -2; 6, -11, 6; 10, -35 , 50, -24; 15, -85, 225, -274, 120; and 21, -175, 735, -1624, 1764, -720.

Theorem 2.1. *Consider a B-array T with m rows and index set $\underline{\mu}'$. For T to exist, the following results must be satisfied:-*

$$\begin{aligned} (a) \quad M_6^2 &\le M_7 M_5 \\ (b) \quad M_5^2 &\le M_7 M_3 \end{aligned} \tag{2.2}$$

Proof: We make use of the Cauchy's inequality which is

$$\left(\sum a_k b_k\right)^2 \le \sum a_k^2 \sum b_k^2 \quad .$$

Where $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are sequences of reals. We set $a_k = j^{\frac{7}{2}}\sqrt{x_j}$, and $b_k = j^{\frac{5}{2}}\sqrt{x_j}$ in the Cauchy's inequality to obtain

$$\begin{aligned}(\sum_{j=0}^{m} j^6 x_j)^2 &\leq \sum_{j=0}^{m} j^7 x_j \sum_{j=0}^{m} j^5 x_j \\ \text{i.e., } M_6^2 &\leq M_7 M_5 \text{ which is (a) above}\end{aligned}$$

To obtain (b), we use $a_k = j^{\frac{7}{2}}\sqrt{x_j}$, and $b_k = j^{\frac{3}{2}}\sqrt{x_j}$.

Theorem 2.2. *For a B-array T with m rows and index set $\underline{\mu}'$. For T to exist, the following results must be satisfied:-*

$$\begin{aligned}&(a) \quad M_4^4 \leq M_2^2 M_5 M_7 \\ &(b) \quad M_4^4 \leq M_7 M_3^3 \qquad (2.3)\end{aligned}$$

Proof: We use the following classical inequality to obtain (2.3)

$$(\sum(a_k b_k c_k))^4 \leq \sum a_k^4 \sum b_k^4 (\sum c_k^2)^2$$

Set $a_k = j^{\frac{7}{4}}\sqrt[4]{x_j}$ and $b_k = j^{\frac{5}{4}}\sqrt[4]{x_j}$ and $c_k = j\sqrt{x_j}$ for (a);

and set $a_k = j^{\frac{7}{4}}\sqrt[4]{x_j}$ and $b_k = j^{\frac{3}{4}}\sqrt[4]{x_j}$ and $c_k = j^{\frac{3}{2}}\sqrt{x_j}$ to obtain (b).

Theorem 2.3. *For an m-rowed B-array T with index set $\underline{\mu}'$ to exist, we must have the following:*

$$\begin{aligned}&(a) \quad M_5^3 \leq M_1 M_7^2 \\ &(b) \quad M_6^4 \leq M_3 M_7^3 \qquad (2.4)\end{aligned}$$

Proof: To derive (2.4), we use the Hölder Inequality:

For $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have $\sum a_k^{\frac{1}{p}} b_k^{\frac{1}{q}} \leq (\sum a_k)^{\frac{1}{p}} (\sum b_k)^{\frac{1}{q}}$

We pick $p = 3$; thus $\frac{1}{q} = \frac{2}{3}$ and the above is

$$\begin{aligned}\sum a_k^{\frac{1}{3}} b_k^{\frac{2}{3}} &\leq (\sum a_k)^{\frac{1}{3}} (\sum b_k)^{\frac{2}{3}} \\ \text{which gives } (\sum a_k^{\frac{1}{3}} b_k^{\frac{2}{3}})^3 &\leq \sum a_k (\sum b_k)^2\end{aligned}$$

To obtain (a) we set $a_k = j x_j$, $b_k = j^7 x_j$, we get

$$\begin{aligned}(\sum j^5 x_j)^3 &\leq \sum j x_j (\sum j^7 x_j)^2 \\ M_5^3 &\leq M_1 M_7^2.\end{aligned}$$

To obtain (b) above, we set $p = 4$, $q = \frac{4}{3}$

$$\begin{aligned} \sum a_k^{\frac{1}{4}} b_k^{\frac{3}{4}} &\le (\sum a_k)^{\frac{1}{4}} (\sum b_k)^{\frac{3}{4}} \\ \text{i.e., } (\sum a_k^{\frac{1}{4}} \sum b_k^{\frac{3}{4}})^4 &\le \sum a_k (\sum b_k)^3 \end{aligned}$$

Set $a_k = j^3 x_j$, $b_k = j^7 x_j$, and we obtain (b)

Theorem 2.4. *For a B-array T with m rows and index set $\underline{\mu}'$. For T to exist, the following results must be satisfied:-*

$$\begin{aligned} & (M_1 + 3M_2 + 6M_3 + 7M_4 + 6M_5 + 3M_6 + M_7)^{\frac{1}{3}} \\ \le \; & M_1^{\frac{1}{3}} + M_4^{\frac{1}{3}} + M_7^{\frac{1}{3}} \end{aligned} \tag{2.3.1}$$

Proof: Here we use Minkowski's inequality:

$$(\sum (a_k + b_k + c_k)^p)^{\frac{1}{p}} \le (\sum a_k^p)^{\frac{1}{p}} + (\sum b_k^p)^{\frac{1}{p}} + (\sum c_k^p)^{\frac{1}{p}}, \quad \text{where } p > 1.$$

We pick here $p = 3$, and set $a_k = j^{\frac{1}{3}} x_j^{\frac{1}{3}}$, $b_k = j^{\frac{4}{3}} x_j^{\frac{1}{3}}$, and $c_k = j^{\frac{7}{3}} x_j^{\frac{1}{3}}$, and then

$$(\sum j(1 + j + j^2)^3 x_j)^{\frac{1}{3}} \le (\sum j x_j)^{\frac{1}{3}} + (\sum j^4 x_j)^{\frac{1}{3}} + (\sum j^7 x_j)^{\frac{1}{3}}$$

After some simplification, we get the desired result.

A computer program was prepared involving m, $\underline{\mu}'$, and $t = 7$. If we are given m and $\underline{\mu}'$, we substitute these values in (2.2)-(2.4). If any one condition is contradicted, then T does not exists for that m and $\underline{\mu}'$. We must caution that these are merely necessary conditions, and even if all are satisfied that does not mean that T will exist. These conditions can also be used to obtain the $max(m)$ for a given $\underline{\mu}'$. For the sake of illustration, we provide below some values of $\underline{\mu}'$ and list the $max(m)$ for each by using the conditions (2.2)-(2.4)

Example1. Select $M_6^2 \le M_7 M_5$

a) Consider the B-array $(0, 1, 1, 0, 1, 1, 0, 0)$ This array contradicts the above inequality for $m = 9$. $(2.291863E + 12$ should be less than $2.270575E + 12)$. Hence, we require $m \le 8$.

b) Consider $(0, 1, 2, 0, 0, 2, 1, 0)$. This array contradicts the above inequality for $m = 9$. $(1.241283E + 13$ should be less than $1.22991E + 13)$.
Hence, we require $m \le 8$.

c) Consider $(1,1,0,0,0,0,1,1)$. Again we contradict the above inequality for $m = 9$. ($3.341087E + 12$ should be less than $3.323769E + 12$). Hence we require $m \leq 8$.

Example2. Select $M_5^2 \leq M_7 M_3$

a) Consider $(1,0,1,0,1,0,1,0)$. We contradict the above inequality for $m = 19$. ($6.017366E + 13$ should be less than $6.007429E + 13$). Hence, we require $m \leq 18$.

b) Consider $(1,1,1,1,0,0,0,0)$. We contradict the above inequality for $m = 9$. ($3.716413E + 08$ should be less than $3.578778E + 08$). Hence, $m \leq 8$.

References

[1] E. F. Beckenback and R Bellman, *Inequalities*, Springer Verlag, New York (1961).

[2] R. C. Bose, *On the Construction of Balanced Incomplete Block Designs*, Ann. Eugenics **9** (1939), 358–398.

[3] R. C. Bose, *On Some Connections Between the Design of Experiments and Information Theory*, Bull. Internat. Statist. Inst. **38** (1961), 257–271.

[4] T.M. Chakravarti, *Fractional Replication in Symmetrical Factorial Designs and Partially Balanced Arrays*, Sankhya **17** (1956), 143–164.

[5] C.S. Cheng, *Optimality of Some Weighing and 2^m Fractional Designs*, Ann. Statist. **8** (1980), 436–444.

[6] D. V. Chopra, *On Balanced Arrays with Two Symbols*, Ars Combinatoria **20A** (1985), 59–63.

[7] R. Dios and D. V. Chopra., *A Note on Balanced Arrays of Strength Eight*, J. Combin. Math. Combin. Comput. **41** (2002), 133–138.

[8] A. Hedayat, N. J. Sloane, and J. Stufken. *Orthogonal Arrays (theory and applications)*, Springer Verlag, New York (1999).

[9] S. K. Houghton, L. Thiel, J. Janssen and C. W. Lam, *There Is No (46,6,1) block design*, J. Combin. Designs **9**(2001), 60–71.

[10] J. P. C. Kleijnen and Ozge Pala, *Maximizing the Simulation Output: a Competition*, Simulation **73** (1999), 168–173.

[11] J. Q. Longyear, *Arrays of Strength s on Two Symbols*, J. Statist. Plann. Inf. **10**(1984), 227-239.

[12] D. S. Mitrinovicć, *Analytic Inequalities* , Springer Verlag, New York (1970).

[13] J.A. Rafter , E. Seiden, *Contributions to the Theory and Construction of Balanced Arrays*, Ann. Statistics **2**(1974), 1256–1273

[14] C. R. Rao, *Hypercube of Strength 'd' Leading to Confounded Designs in Factorial Experiments*, Bull. Calcutta Math. Soc. **38**(1946), 67–78.

[15] C. R. Rao, *Factorial Experiments Derivable from Combinatorial Arrangements of Arrays*, J. Roy. Statist. Soc. Suppl. **9**(1947), 128–139.

[16] C. R Rao, *Some Combinatorial Problems of Arrays and Applications to Design of Experiments*, A Survey of Combinatorial Theory (edited by J. N. Srivastava et. al), North-Holland Publishing Co. (1973), 349–359.

[17] G. M. Saha, R. Mukherjee and S. Kageyama, *Bounds on the Number of Contraints for Balanced Arrays of Strength t*, J. Statist. Plann. Inf. **18**(1988), 255–265.

[18] E. Seiden and R. Zemach, *On Orthogonal Arrays*, Ann. Math. Statist. **27**(1966), 1355–1370.

[19] K. Sinha, V. Dhar, G. Saha and S. Kageyama, *Balanced Arrays of Strength Two from Block Design*, J. Combin. Designs. **10**(2002), 303–312.

[20] W. D. Wallis, *Combinatorial Designs*, Marcel Dekker Inc., New York(1988).

[21] S. Yamamoto, Y. Fujii and M. Mitsuoka, *Three-Symbol Orthogonal Arrays of Strength Two and Index Two Having Maximal Constraints: Computational Study*, J. Comb. Inf. and Syst. Sci. **18**(1993), 209–215.