MATCHINGS DEFINED BY LOCAL CONDITIONS

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ABSTRACT. A graph has the neighbour-closed-co-neighbour, or ncc property, if for each of its vertices x, the subgraph induced by the neighbour set of x is isomorphic to the subgraph induced by the closed non-neighbour set of x. Graphs with the ncc property were characterized in [1] by the existence of a locally C_4 perfect matching M: every two edges of M induce a subgraph isomorphic to C_4. In the present article, we investigate variants of locally C_4 perfect matchings. We consider the cases where pairs of distinct edges of the matching induce isomorphism types including P_4, the paw, or the diamond. We give several characterizations of graphs with such matchings. In addition, we supply characterizations of graphs with matchings whose edges satisfy a prescribed parity condition.

1. INTRODUCTION

Matchings have been extensively studied in graph theory, and play an important role in combinatorial optimization; see for example, [7, 8]. A *disjoint neighbour perfect* or *dnp matching* M is a perfect matching with the property that no edge of M is in a triangle. For example, every perfect matching in a bipartite graph is dnp, and there is a unique dnp matching in the Cartesian product of an n-vertex clique with K_2, written $K_n \square K_2$.

We only consider graphs which are finite, undirected, and simple. We use the notation $G \restriction S$ for the subgraph of G induced by a set of vertices S, and the notation $G \cong H$ for isomorphic graphs. If x is a vertex of G, then define $N(x)$ to be the set of vertices of G joined to x. Define $N^c[x]$ to be the set $V(G) \setminus N(x)$. R. Nowakowski recently proposed the following vertex partition property as an analogue of similar properties for infinite graphs (such as the infinite random graph): a graph G has the *neighbour-closed-co-neighbour* or *ncc* property, if for all $x \in V(G)$, we have that $G \restriction N(x) \cong G \restriction N^c[x]$. There are many examples of such graphs, such as the bipartite cliques $K_{n,n}$ and the graphs $K_n \square K_2$. There are, however,

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many ncc graphs that are not one of these types. The class of ncc graphs were completely characterized in [1] using dnp matchings.

Theorem 1. *A graph G is ncc if and only if there is a positive integer n so that G has $2n$ vertices, G is n-regular, and G has a dnp matching.*

Theorem 1 implies the following.

Theorem 2. *A graph G is ncc if and only if G has a perfect matching M so that every pair of distinct edges of M induce a subgraph isomorphic to C_4.*

A dnp matching in an ncc graph acts "locally" as an isomorphism. This is made precise in the following theorem, which was proved as a claim in the converse of Theorem 2.1 from [1].

Theorem 3. *Let G be an ncc graph with a dnp matching $M = \{a_i b_i : 1 \leq i \leq n\}$. Then the mapping*

$$f : G \restriction \{a_i : 1 \leq i \leq n\} \to G \restriction \{b_i : 1 \leq i \leq n\}$$

defined by $f(a_i) = b_i$ is an isomorphism.

Following [1], we name the mapping f of the theorem an *M-isomorphism*. The conclusion of this theorem holds regardless of what "orientation" the matching is given. Hence, for each edge $xy \in M$, there are two choices for the "a" vertex and two for the "b", giving rise to 2^n distinct M-isomorphisms. In this way, we may view a matching as a mapping (which may not necessarily be an isomorphism), which we refer to as an *M-morphism*. This view leads to a new characterization of ncc graphs.

Theorem 4. *A graph G is ncc if and only if G has n^2 edges, has a perfect matching M so that every M-morphism is an isomorphism, and no two distinct edges of M induce a subgraph isomorphic to K_4.*

Before we prove Theorem 4, we need some notation. Let P_n denote the path with n edges. The graph $2K_2$ consists of two disjoint copies of K_2. The *paw* is K_3 plus one endvertex, and the *diamond* is K_4 minus an edge. See Figure 1. For more on these graphs, the reader is directed to [2].

Proof. The necessity follows by Theorems 1, 2, and 3. For sufficiency, fix distinct edges $e = ab$ and $e' = a'b'$ of M. Up to isomorphism, the graph H induced on the vertices of e and e' is one of $2K_2$, C_4, P_4, the paw, or the diamond. Suppose first that H is the paw, say with edges $ab, aa', ba', a'b'$. But then aa' is an edge, with bb' a non-edge, which violates that every M-morphism is an isomorphism. A similar argument excludes P_4 and the diamond. By Theorem 2, we need only exclude $2K_2$. If $H \cong 2K_2$, then each pair of distinct edges of M distinct from e, e' is joined by at most two

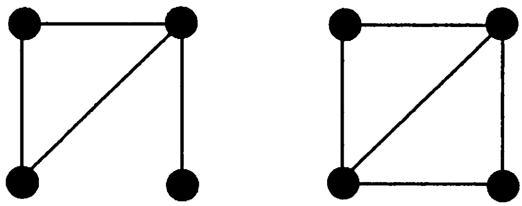

FIGURE 1. The paw and the diamond.

edges (since we have excluded all possibilities for H except $2K_2$ and C_4). But then

$$|E(G)| \leq n + 2\left(\binom{n}{2} - 1\right) < n^2,$$

which contradicts hypothesis. □

Let G have a perfect matching M. We say that M is *locally H* if each pair of distinct edges of M induce a graph isomorphic to H. Hence, a matching may be locally $2K_2$, C_4, P_4, the paw, the diamond, or K_4, with no other possibilities. A graph with a locally $2K_2$ perfect matching consists of n disjoint copies of K_2. Such matchings have been well-studied, and are sometimes called *induced* or *strong*; see [3]. A graph with a locally K_4 perfect matching is a clique. With this notation, we may restate Theorem 2 as follows.

Theorem 5. *A graph is ncc if and only if it has a locally C_4 perfect matching.*

From Theorem 5 and the above discussion, the remaining unexamined choices for H are P_4, the paw or the diamond. In each case, graphs with locally H perfect matchings give rise to an interesting class of graphs. For these graph classes, we prove structural characterizations similar to Theorem 4 in Theorems 6 and 8.

Graphs with locally H perfect matchings have diameter 2 or 3. In Section 3, we present a generalization of locally H perfect matchings to graphs with arbitrary diameter. This gives rise to *parity disjoint* perfect matchings, which are defined via certain distance conditions on the edges of the matching. We characterize such matchings in Theorem 10, and give a polynomial time recognition algorithm for them in Corollary 2.

2. Characterizing graphs with locally H perfect matchings

We now characterize graphs with locally H perfect matchings in a fashion similar to Theorem 4. However, we will use M-morphisms that are not necessarily isomorphisms.

Let $f : V(G) \to V(H)$ be a vertex mapping. We will abuse notation and write $f : G \to H$. The mapping f is a *homomorphism* if $xy \in E(G)$ implies that $f(x)f(y) \in E(H)$; in other words, it sends edges to edges. See the book [6] for more on homomorphisms. The map f is a *cohomomorphism* if $xy \in E(G)$ implies that $f(x)f(y) \notin E(H)$. Cohomomorphisms were first studied in [5]. An *anti-homomorphism* sends edges to non-edges, while an *anti-cohomomorphism* sends non-edges to edges. The mapping f is an *anti-isomorphism* if it is bijective and is both an anti-homomorphism and an anti-cohomomorphism.

Theorem 6. *Let G be a graph with $2n$ vertices, where n is a positive integer.*

(1) *The graph G has a locally P_4 perfect matching if and only if there is a perfect matching M of G so that every M-morphism is an anti-homomorphism, there are $\frac{n^2+n}{2}$ edges in G, and no two edges of M induce a subgraph isomorphic to $2K_2$.*

(2) *The graph G has a locally paw perfect matching M if and only there is a perfect matching M of G so that every M-morphism is an anti-isomorphism.*

(3) *The graph G has a locally diamond perfect matching M if and only if there is a perfect matching M of G so that every M-morphism is an anti-cohomomorphism, there are $\frac{3n^2-n}{2}$ edges in G, and no two edges of M induce a subgraph isomorphic to K_4.*

Proof. (1) For the forward direction, let M be a locally P_4 perfect matching with $M = \{a_ib_i : 1 \le i \le n\}$. Fix an M-morphism

$$f : G \restriction \{a_i : 1 \le i \le n\} \to G \restriction \{b_i : 1 \le i \le n\}$$

defined by $f(a_i) = b_i$, for $1 \le i \le n$. Since each pair of distinct edges a_ib_i and a_jb_j of M induce a P_4, if say a_ia_j is an edge, then b_ib_j is a non-edge. Hence, by symmetry, f is an anti-homomorphism. As each pair of edges of M are joined by exactly one edge, there are $n + \binom{n}{2} = \frac{n^2+n}{2}$ edges in G. As M is locally P_4 perfect, no two edges of M induce a subgraph isomorphic to $2K_2$.

For the reverse direction, fix distinct edges a_ib_i and a_jb_j of M. Without loss of generality, say $i = 1$ and $j = 2$. By hypothesis, the subgraph H induced by $\{a_1, a_2, b_1, b_2\}$ cannot be isomorphic to $2K_2$. We must therefore exclude the cases when H is C_4, a paw, diamond, or K_4. Suppose for a contradiction that H is C_4. As M is an anti-homomorphism, a_1 is not

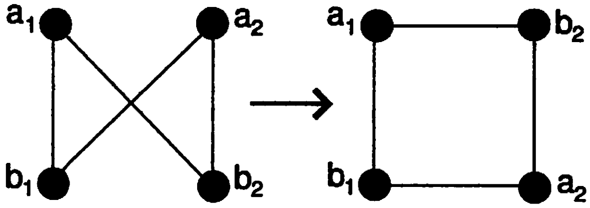

FIGURE 2. Excluding C_4 in the proof of (1).

joined to a_2 and b_1 is not joined to b_2; hence, a_1b_2 and a_2b_1 are edges. Define the M-morphism

$$f' : G \restriction \{a_1, b_2, a_3, \ldots, a_n\} \to G \restriction \{b_1, a_2, b_3, \ldots, b_n\}$$

by $f'(a_i) = \begin{cases} b_i & \text{if } i \neq 2; \\ a_2 & \text{else.} \end{cases}$ See Figure 2. The map f' fails to be anti-homomorphism, as $a_1b_2 \in E(G \restriction \{a_1, b_2, a_3, \ldots, a_n\})$ but $f'(a_1)f'(b_2) \in E(G \restriction \{b_1, a_2, b_3, \ldots, b_n\})$. Hence, H is not C_4. A similar argument excludes the diamond and K_4.

We have shown that each H is either P_4 or a paw. Suppose for a contradiction that some pair of distinct edges of M induces a paw. Let r be the number of pairs of edges of M with exactly 1 edge between them, and let s be the number of pairs of edges with exactly 2 edges between them. Then $r \geq 0$, $s \geq 1$, and $r + s = \binom{n}{2}$. Further,

$$\begin{aligned} |E(G)| &= n + r + 2s \\ &> n + \binom{n}{2} = \frac{n^2 + n}{2}, \end{aligned}$$

which contradicts hypothesis.

(2) For the forward direction, let M be a locally paw perfect matching, and fix an M-morphism $f : G \restriction \{a_i : 1 \leq i \leq n\} \to G \restriction \{b_i : 1 \leq i \leq n\}$ defined by $f(a_i) = b_i$ for $1 \leq i \leq n$. Since each pair of edges a_ib_i and a_jb_j of M induce a paw, if say a_ia_j is an edge, then b_ib_j is a non-edge. Hence, by symmetry, f is an anti-homomorphism. If a_ia_j is a non-edge, then a_ia_j is an edge; by symmetry, f is an anti-cohomomorphism, and thus, f is an anti-isomorphism.

For the reverse direction, fix distinct edges a_ib_i and a_jb_j of M. By hypothesis and arguments similar to those given in the proof of (1), the subgraph H induced by $\{a_i, a_j, b_i, b_j\}$ cannot be isomorphic to $2K_2$, P_4, C_4, the diamond, or K_4. Hence, M is locally paw.

(3) For the forward direction, let M be a locally diamond perfect matching, and fix an M-morphism $f : G \restriction \{a_i : 1 \leq i \leq n\} \rightarrow G \restriction \{b_i : 1 \leq i \leq n\}$ defined by $f(a_i) = b_i$. Since each pair of edges a_ib_i and a_jb_j of M induce a diamond, if say a_ia_j is a non-edge, then a_ia_j is an edge; by symmetry f is an anti-cohomomorphism. As each pair of edges of M are joined by exactly three edges, there are $n+3\binom{n}{2} = \frac{3n^2-n}{2}$ edges in G. As M is locally diamond perfect, no two edges of M induce a subgraph isomorphic to K_4.

For the reverse direction, fix a_ib_i and a_jb_j edges of M. By hypothesis and arguments similar to those of (1), the subgraph H induced by $\{a_i, a_j, b_i, b_j\}$ cannot be isomorphic to $2K_2, P_4, C_4$, or K_4. We must exclude the paw. Suppose for a contradiction that H is isomorphic to a paw. Hence, between each pair of edges in M there are either 2 or 3 edges. As in (1) there are integers $r \geq 1$ and $s \geq 0$ so that $r + s = \binom{n}{2}$, and

$$\begin{aligned} |E(G)| &= n + 2r + 3s \\ &< n + 3\binom{n}{2} = \frac{3n^2 - n}{2}, \end{aligned}$$

which is a contradiction. □

Planarity is a strong restriction on graphs with a locally H perfect matching, as witnessed by the following theorem.

Corollary 1. *There are only finitely many non-isomorphic planar graphs which have a locally H matching, where H is one of C_4, P_4, the paw, or the diamond.*

Proof. Fix H as in the statement of the corollary. A graph G with $2n$ vertices and a locally H perfect matching is *dense*, in the sense that $|E(G)| \in O(n^2)$. This fact, Theorem 6, and the well known property that if G is planar then $|E(G)| \leq 3|V(G)| + 6$ complete the proof. □

We now turn to another structural characterization of graphs with a locally H perfect matching. Suppose that G is a graph with perfect matching M, and let ab, $a'b'$ be distinct edges of M. Define an *interchange (with respect to M)* by interchanging the edges and non-edges of $G \restriction \{a, a', b, b\}$, leaving the edges ab and $a'b'$ unchanged, so that the isomorphism type of the subgraph induced by $\{a, a', b, b\}$ is unchanged. We write $G \sim_M G'$ if G' results from G by one C_4-interchange with respect to M. We write $G \sim_M^* G'$ if there is an integer $n \geq 0$, and graphs $G_0 = G, G_1, \ldots, G_n = G'$ so that for all $0 \leq i \leq n-1$, $G_i \sim_M^* G_{i+1}$. See Figure 3.

If G and H are graphs, then we write the *Cartesian product* of G and H as $G \square H$. The following theorem was proved in [1].

Theorem 7. *A graph G is ncc if and only if G has a perfect matching M so that $G \sim_M^* (K_n \square K_2)$.*

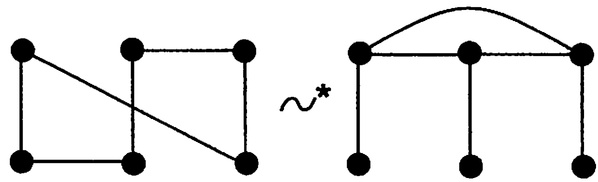

FIGURE 3. A sequence of interchanges in a graph with a locally P_4 perfect matching.

Define the graph K_n' by adding an endvertex joined to each vertex of K_n. Let the vertices of K_n be labelled $\{x_j : 1 \leq j \leq n\}$. Define the graph K_n'' by adding a set of n independent vertices y_i, so that for each $1 \leq i \leq n$, y_i is joined to all x_j with $j \geq i$. We use the notation $\overline{K_n}$ for the complement of K_n. Define the graph K_n''' by adding all edges between K_n and $\overline{K_n}$. The proof of the following theorem, which extends Theorem 7 to locally H matchings, follows from the definitions.

Theorem 8. *Let G be a graph.*

(1) *The graph G has a locally P_4 perfect matching M if and only if it has a matching M so that $G \sim_M^* K_n'$.*
(2) *The graph G has a locally paw perfect matching M if and only if it has a matching M so that $G \sim_M^* K_n''$.*
(3) *The graph G has a locally diamond perfect matching M if and only if it has a matching M so that $G \sim_M^* K_n'''$.*

Locally H graphs, where H is one of P_4, the paw, or the diamond are in a certain sense *universal*. We make this precise in the following theorem.

Theorem 9. *Let G be a fixed graph, and suppose that H is isomorphic to one of P_4, the paw, or the diamond. Then G is isomorphic to the induced subgraph of a graph G' with a locally H perfect matching, so that $|V(G')| \leq 2|V(G)|$.*

Proof. We give the construction for $H \cong P_4$, since the cases of the paw and diamond are handled analogously. Let $V(G) = \{x_1, \ldots, x_n\}$. To form G'', add to G vertices $\{y_1, \ldots, y_n\}$ so that for all i, y_i is only joined to x_i. Form G' as follows: if x_i is not joined to x_j in G'', then add an edge between y_i and y_j; add no other edges. It is straightforward to check that $\{x_i y_i : 1 \leq i \leq n\}$ is a locally P_4 perfect matching in G'. □

We do not know if the problems of recognizing a locally H perfect matching, where H is P_4, the paw, or the diamond, are polynomial time.

3. Parity disjoint matchings and pairings

All graphs in this section are connected. It is not hard to see that a graph with a locally H perfect matching, where H is connected, has diameter 2 or 3. In this section, we consider a variation of locally H perfect matchings to include graphs of arbitrary diameter. We denote by $d_G(u, v)$ the distance between u and v; we may drop the subscript G if it is clear from context.

A *pair* in a graph is an unordered set of two distinct vertices. A *parity disjoint* or *pd* pair is a pair $\{a, b\}$ of vertices with the property that for all vertices x

$$d(a, x) \equiv d(b, x) + 1 \pmod 2.$$

In other words, a pair is pd if every vertex of even (odd) distance to a is odd (even) distance to b. A *pd edge* is a pd pair that is an edge. For instance, an ncc graph G is diameter 2, so by Theorem 5 each edge in a dnp matching of G is pd. All edges in a bipartite graph is pd.

A *pairing* P is a set of pairwise disjoint pairs. In particular, a pairing is a matching if each pair forms an edge of the graph. A *pd pairing* is a pairing P so that

(1) for all $x \in V(G)$, there is a unique pair $p \in P$ so that $x \in p$;
(2) for each pair $\{a, b\} \in P$, $d(a, b)$ is odd;
(3) each pair in P is pd.

A *pd matching* is a pd pairing P where each pair in P is an edge. For example, an ncc graph or a balanced bipartite graph (that is, a bipartite graph whose vertex classes have the same cardinality) have dnp pairings.

Before we give a characterization of graphs with pd matchings and pairings, we need a few definitions. Define the graph G^{+odd} by joining all pairs of non-joined vertices of G that are an odd distance apart. See Figure 4 for an example of G^{+odd}.

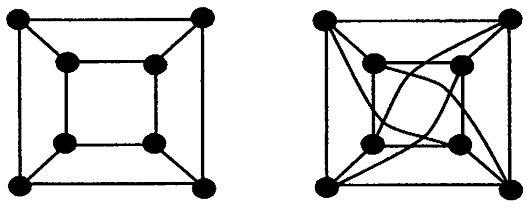

FIGURE 4. A graph G and G^{+odd}.

Let $f : G \to H$ be a vertex mapping. We say that f *preserves parity* if for all $x, y \in V(G)$,

$$d_G(x, y) \equiv d_H(f(x), f(y)) \pmod 2.$$

Define $e(x)$ to be the set of vertices of even distance to x in G (including x); the set $o(x)$ is defined analogously. A perfect matching M of G is *co-dnp* if for each edge $ab \in M$, there is no $x \in V(G)$ that is non-joined to both a and b.

Theorem 10. *Let G be a graph with $2n$ vertices.*

(1) *A graph G has a pd pairing if and only if G^{+odd} is ncc.*
(2) *A graph G has a pd matching if and only there is a perfect matching M of G so that every M-morphism preserves parity, and for all $x \in V(G)$, $|e(x)| = n$.*

Proof. (1) For the forward direction, assume that G has a pd pairing $P = \{\{a_i, b_i\} : 1 \leq i \leq n\}$. Since $d(a_i, b_i)$ is odd by hypothesis, a_ib_i is an edge of G^{+odd}. Hence, $M = \{a_ib_i : 1 \leq i \leq n\}$ is a perfect matching in G^{+odd}. By Theorem 5 we need only check that any two distinct edges of M induce C_4. Suppose that a_i and b_i have either a common neighbour or common non-neighbour z. In either case, $d_G(a_i, z) \equiv d_G(b_i, z) \pmod 2$, which is a contradiction. The result follows since a matching which is dnp and co-dnp is locally C_4.

For the reverse direction, suppose that G^{+odd} is ncc. Let $M = \{a_ib_i : 1 \leq i \leq n\}$ be a locally C_4 matching in G^{+odd}, and so $P = \{\{a_i, b_i\} : 1 \leq i \leq n\}$ is a pairing in G (some of the edges a_ib_i of G^{+odd} may not be present in G). If $z \in V(G)$ has the property that $d_G(z, a_i)$ and $d_G(z, b_i)$ have the same parity, then this would contradict that a_i and b_i has no common neighbour nor non-neighbour in G^{+odd}.

(2) For the forward direction, let G have a pd matching $M = \{a_ib_i : 1 \leq i \leq n\}$. We prove that the M-morphism f mapping a_i to b_i preserves parity. Now,

$$d(a_i, a_j) \equiv d(a_j, b_i) + 1 \equiv d(b_i, b_j) + 2 \equiv d(b_i, b_j) \pmod 2.$$

As f was arbitrary, every M-morphism preserves parity.

For all i and j, each edge a_jb_j of M has exactly one of a_j or b_j in $e(a_i)$. The same holds for $e(b_i)$. Hence, for all vertices x of G, we have that $|e(x)| = n$.

For the reverse direction, fix $M = \{a_ib_i : 1 \leq i \leq n\}$ a matching of G with the prescribed property. Consider the edge a_1b_1. Since $|e(a_1)| = n$, by relabelling if necessary, we may assume that $e(a_1) = \{a_1, \ldots, a_n\}$ and $o(a_1) = \{b_1, \ldots, b_n\}$. As every M-morphism preserves parity and since $|e(b_1)| = n$, we have that $e(b_1) = \{b_1, \ldots, b_n\}$ and $o(b_1) = \{a_1, \ldots, a_n\}$ Hence, $o(a_1) = e(b_1)$ and $e(a_1) = o(b_1)$. In particular, $a_1b_1 \in M$ is a pd edge.

Define a pair of distinct vertices x, y of G to be *even twins* if $e(x) = e(y)$. Since every M-morphism preserves parity, every even twin of a_1 among the a_i is mapped by M to an even twin of b_1 among the b_i. Further, there are the same number of even twins of a_1 among the a_i as even twins of b_1 among the b_i. Therefore, each even twin of a_1 is matched by M to an even twin of b_1. List the even twins of a_1 and b_1 as $u_1 = a_1, u_2, \ldots, u_k$ and $v_1 = b_1, v_2, \ldots, v_k$, respectively, so u_i is matched to v_i by M. Let $M_{u_1} = \{u_i v_i : 1 \le i \le k_{u_1}\}$, and let $M_{u_1} = M_{u_i} = M_{v_i}$ for all $1 \le i \le k_{u_1}$. Note that each edge $u_i v_i$ is a pd edge.

Define

$$M = \bigcup_{z \in V(G)} M_z.$$

We now prove that M is a pd matching. To see that M is a matching, suppose to the contrary that there are two edges uv and uv' in M. But then v and v' are in the set $\{v_i : 1 \le i \le k_u\}$. But u is matched by M with a unique element of $\{v_i : 1 \le i \le k_u\}$, which is a contradiction. The matching M is pd by construction. □

Theorem 10 (1) implies that if G has a pd matching, then G^{+odd} is ncc, but the converse is false. Consider the graph G formed from $K_3 \square K_2$ by deleting one edge in its unique dnp matching. The graph $G^{+odd} \cong G$ is ncc, but G has no pd matching.

We now demonstrate how to recognize graphs with pd matchings and pairings in polynomial time. To form the graph G^{-odd}, delete all edges ab with the property that there is a vertex x such that $d(a, x) \equiv d(b, x) \pmod 2$. The graph G^{-odd} may be constructed from G in polynomial time; the same is true with G^{+odd}. The graph G has a pd matching (pairing) if and only if G^{-odd} (G^{+odd}) has a perfect matching (is ncc). This gives rise to the following corollary of Theorem 10.

Corollary 2. *There is a polynomial-time algorithm to determine whether a graph has a pd matching (pairing).*

We conclude with a discussion of operations preserving pd matchings. If G and H are graphs (whose vertex set may intersect non-trivially), then we write $G \cup H$ for the graph with vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H)$.

Corollary 3. (1) *If G has a pd matching and H is any graph, then $G \square H$ has a pd matching.*

(2) *If G is any graph, then the graph G' formed by joining an endvertex to each vertex of G has a pd matching.*

(3) *If G has a pd matching, then the graph G'' formed by joining a path of length two to a fixed vertex has a pd matching.*

(4) *Let G and H have pd matchings M and M', respectively. If $V(G) \cap V(H) = \{a, b\}$, where ab is pd edge in M and M', then $M \cup M'$ is a pd matching of $G \cup H$.*

Proof. (1) Let $M = \{a_i b_i : 1 \leq i \leq n\}$ be a pd matching of G. Define

$$M_\Box = \{(a_i, x)(b_i, x) : 1 \leq i \leq n, x \in V(H)\}.$$

It is straightforward to verify that $M_\Box$ is a perfect matching of $G \Box H$. Now fix $i \in \{1, \dots n\}$, and $(u, v) \in V(G \Box H)$. Then working (mod 2) we have that

$$\begin{aligned} d_{G\Box H}((a_i, x), (u, v)) &= d_G(a_i, u) + d_H(x, v) \\ &\equiv d_G(b_i, u) + 1 + d_H(x, v) \\ &= d_{G\Box H}((b_i, x), (u, v)) + 1, \end{aligned}$$

where the first and second equality follows by properties of distance in $G \Box H$, and the congruence follows since M is a pd matching. As i and (u, v) were arbitrary, we have that $M_\Box$ is a pd matching of $G \Box H$.

(2) Let $a \in V(G') \setminus V(G)$ be an endvertex of G' joined to b. If z is vertex of G', then $d(a, z) = d(b, z) + 1$, so ab is a pd edge. Hence, $M = \{a_i b_i : 1 \leq i \leq n, b_i \in V(G), a_i \in V(G') \setminus V(G)$ is an endvertex joined to $b_i\}$ is a pd matching of G'. The proof of (3) is similar to the one given for (2), and so is omitted.

For (4), let $M = \{a_i b_i : 1 \leq i \leq m\}$ and $M' = \{a_i' b_i' : 1 \leq i \leq n\}$. Without loss of generality, let $a = a_m = a_1{}'$ and $b = b_m = b_1{}'$. To see that $M \cup M'$ is a pd matching of $G \cup H$, we show that $a_1 b_1$ is a pd edge in $G \cup H$ (the other cases are similar). Fix $z \in V(G) \cup V(H)$. If z is in $V(G)$, then a shortest path from z to a_1 or b_1 must have all of its vertices in G. Since $a_1 b_1$ is a pd edge in G, the distances in $G \cup H$ from z to a_1 and to b_1 are of opposite parities.

Now let $z \in V(H) \setminus V(G)$. Any shortest path connecting z to a_1 or b_1 must go through one of a or b.

Case 1: The shortest paths P from z to a_1 and Q from z to b_1 both traverse through a. (The case when P and Q traverse through b is similar and so is omitted.)

Hence, if x is a_1 or b_1 then

$$d_{G\cup H}(x, z) = d_G(x, a) + d_H(a, z). \tag{3.1}$$

Let P' be the subpath of P in G from a_1 to a and Q' the subpath of Q in H from b_1 to a. See Figure 5.

The *parity* of a path is even (odd) if its number of edges is even (odd). Then P' and Q' have opposite parities since ab is pd in G. It follows by (3.1) that P and Q have opposite parities in $G \cup H$.

Case 2: The path P traverses through a and Q through b. (The case when P goes through b and Q through a is analogous and so is omitted.)

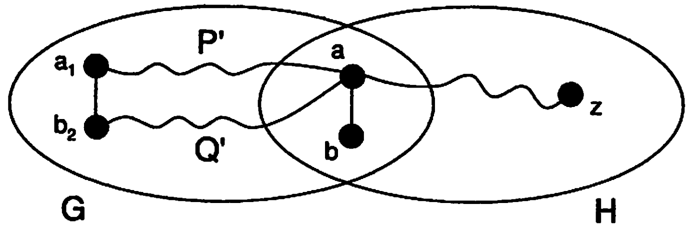

FIGURE 5. Case 1.

Then

$$d_{G\cup H}(a_1, z) = d_G(a_1, a) + d_H(a, z), \quad (3.2)$$

$$d_{G\cup H}(b_1, z) = d_G(b_1, b) + d_H(b, z). \quad (3.3)$$

Let P' be the subpath of P in G from a_1 to a and Q' the subpath of Q in G from b_1 to b. Let P'' be the subpath of P in H from a to z, and let Q'' be the subpath of Q in H from b to z. See Figure 6.

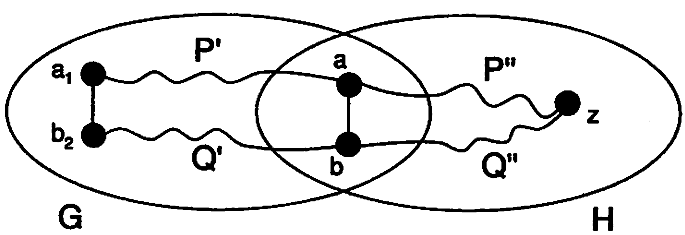

FIGURE 6. Case 2.

Then P'' and Q'' have opposite parities, since ab is pd in H. As ab and a_1b_1 are pd in G, we have that P' and Q' have the same parities. Hence, by (3.2) and (3.3) P and Q have opposite parities in $G \cup H$. □

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