Quasigroups and Approximate Symmetry

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Abstract

Groups provide the mathematical language for exact symmetry. Applications in biology and other fields are now raising the problem of developing a rigorous theory of approximate symmetry. In this paper, it is shown how approximate symmetry is determined by a quasigroup.

1 Introduction

Exact symmetry is modelled mathematically by group structures. Let P be a subgroup of a group Q. Then Q acts on the homogeneous space

$$P \setminus Q = \{Px \mid x \in Q\} \tag{1}$$

of cosets of P by permutations

$$q : P \setminus Q \to P \setminus Q; Px \mapsto Pxq \tag{2}$$

for elements q of Q. For example, the symmetry of a square is described by the action of the 8-element dihedral group

$$Q = \langle \rho, \tau \mid \rho^4 = \tau^2 = (\rho\tau)^2 = 1 \rangle$$

on the homogeneous space given as above by the 2-element subgroup $P = \langle \tau \mid \tau^2 = 1 \rangle$.

In the real world, symmetry is rarely exact. Usually, symmetries are approximate, like the approximate bilateral symmetry of a face or most

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animals' anatomy. Even the rotational symmetry of an engine is only approximate. Although the engine may appear to have returned to its initial state after one or two rotations (for 4-cycle engines), some inevitable wear takes place during the rotation. Unfortunately, there does not yet seem to be a good general mathematical theory of approximate symmetry capable of mirroring the success of group theory in dealing with exact symmetry. As Rosen [4, p. 127] states:

> Since the general theory of approximate symmetry is not very well developed[,] I do not think it worthwhile to go into many details.

Attempting to strike a more positive note, Rosen continues:

> It suffices to state that it is possible to define approximate symmetry groups for state spaces equipped with metrics, and it is possible to define a measure of goodness of approximation for each approximate symmetry group.

Here, he uses the term "metric" to denote what mathematicians usually call a pseudometric, i.e. he allows distinct points of the state space to have zero distance between them. However, Petitjean [3, p. 294] has pointed out the problems arising from the use of a pseudometric rather than a metric. Rosen gives no reference for his definitions of "approximate symmetry group" or "goodness of approximation," but the concept of syntopy introduced by Maruani and Mezey [1] may be taken as typical.

The various shortcomings of the currently available qualitative models motivate the initiation of a rigorous mathematical theory of approximate symmetry, understood as a property of complex systems that are characterized by the presence of various different parts and levels. Approximate symmetry is then defined as exact symmetry at one part or level of a complex system [8]. For example, the wear of a well lubricated engine takes place at a much finer spatial scale than the macroscopic scale governing the main functions of the engine. Thus the engine's approximate rotational symmetry is manifest as exact rotational symmetry at the macroscopic scale, a symmetry that does not hold on the microscopic scale. The bilateral symmetry of vertebrate anatomy is an exact symmetry of that part of the animal's complex system that concerns itself with forward locomotion (symmetry of an arrow $\longrightarrow$ in a two-dimensional plane), but is not reflected in the disposition of most internal organs. (The symmetry of the lungs is presumably a vestige of the symmetric location of a fish's gills.)

In this paper, quasigroups are used to furnish models for one kind of approximate symmetry, exact symmetry holding at the macroscopic level of a two-level hierarchical system. Quasigroups, as "non-associative groups," are briefly introduced in Section 2. The following section recalls the construction of quasigroup homogeneous spaces (cf. [5] [6] [7]). Section 4 provides a simple example to illustrate the construction, and to show how it may lead to an instance of non-trivial approximate symmetry. Section 5 then presents the general version of the model. The section includes a new concept in the theory of quasigroups, the core congruence of a subquasigroup, as a quasigroup analogue and generalization of the group-theoretical concept of the core of a subgroup. The final section gives a brief discussion of potential developments in the rigorous theory of approximate symmetry.

For algebraic definitions and notations used in the paper, readers are referred to [9]. In particular, mappings are generally placed in the natural position on the right of their arguments, either in line or as an index. These conventions help to minimize the number of brackets, which otherwise proliferate in the study of non-associative systems such as quasigroups.

2 Quasigroups

A *quasigroup* is a set Q equipped with a binary multiplication, denoted by $\cdot$ or mere juxtaposition, such that in the equation

$$x \cdot y = z ,$$

knowledge of any two of x, y, z specifies the third uniquely. (Combinatorially, this means that the body of the multiplication table of a finite, nonempty quasigroup is just a Latin square.) Equivalently, quasigroups may be construed as sets $(Q, \cdot, /, \backslash)$ equipped with three binary operations of multiplication, *right division* $/$ and *left division* $\backslash$, satisfying the identities:

$$\begin{array}{ll} \text{(IL)} & y\backslash(y \cdot x) = x; \\ \text{(IR)} & x = (x \cdot y)/y; \\ \text{(SL)} & y \cdot (y\backslash x) = x; \\ \text{(SR)} & x = (x/y) \cdot y. \end{array}$$

Groups are quasigroups, but general quasigroups are not required to have an associative multiplication. A subset P of a quasigroup Q is a *subquasigroup* of Q if it is closed under the three binary operations. More generally, the definition by satisfaction of the identities on the three binary operations means that quasigroups form a variety in the sense of universal algebra, and are thus susceptible to study by the concepts and methods of that subject [9]. In particular, an equivalence relation α on a quasigroup Q is a *congruence* if it is a subquasigroup of $Q \times Q$. Then the natural projection

$$\operatorname{nat} \alpha : Q \rightarrow Q^{\alpha}; q \mapsto q^{\alpha}$$

(with $q^\alpha = \{r \in Q \mid (q,r) \in \alpha\}$ for $q \in Q$) onto the quotient

$$Q^\alpha = \{q^\alpha \mid q \in Q\}$$

is a quasigroup homomorphism. For a non-empty quasigroup Q, the *group replica congruence* is the smallest congruence γ on Q such that the quotient Q^γ is associative (a group).

For each element q of a quasigroup Q, the *right multiplication*

$$R(q) : Q \to Q; x \mapsto x \cdot q$$

and *left multiplication*

$$L(q) : Q \to Q; x \mapsto q \cdot x$$

are elements of the group $Q!$ of bijections from the set Q to itself. For a subquasigroup P of a quasigroup Q, the *relative left multiplication group* of P in Q is the subgroup $\mathrm{LMlt}_Q(P)$ of $Q!$ generated by

$$L(P) = \{L(p) : Q \to Q \mid p \in P\}. \tag{3}$$

3 Quasigroup homogeneous spaces

The construction of a quasigroup homogeneous space for a finite quasigroup is analogous to the transitive permutation representation of a group on the homogeneous space of cosets of a subgroup. Let P be a subquasigroup of a finite quasigroup Q. Let $P \setminus Q$ be the set of orbits of the relative left multiplication group $\mathrm{LMlt}_Q(P)$ on the set Q. If Q is a group, and P is nonempty, then this notation is consistent with (1). Let A be the incidence matrix of the membership relation between the set Q and the set $P \setminus Q$ of subsets of Q. Let A^+ be the pseudoinverse or "Penrose inverse" of the matrix A [2]. This is the unique matrix A^+ satisfying the equations

$$\begin{aligned} AA^+A &= A\,, & (4)\\ A^+AA^+ &= A^+, & (5)\\ (A^+A)^* &= A^+A, & (6)\\ (AA^+)^* &= AA^+, & (7) \end{aligned}$$

in which the * denotes the conjugate transpose.

For each element q of Q, right multiplication in Q by q yields a permutation of Q. Let $R_Q(q)$ be the corresponding permutation matrix. Define a new matrix

$$R_{P\setminus Q}(q) = A^+ R_Q(q) A\,. \tag{8}$$

[In the group case, the matrix (8) is just the permutation matrix given by the permutation (2).] Then in the homogeneous space of the quasigroup Q, each quasigroup element q yields a Markov chain on the state space $P \setminus Q$ with transition matrix $R_{P\setminus Q}(q)$ given by (8).

4 An example

Consider the quasigroup Q whose multiplication table is

Q	1	2	3	4	5	6
1	1	3	2	5	6	4
2	3	2	1	6	4	5
3	2	1	3	4	5	6
4	4	5	6	1	2	3
5	5	6	4	2	3	1
6	6	4	5	3	1	2

Let P be the singleton subquasigroup $\{1\}$. Note that $\mathrm{LMlt}_Q P$ is the cyclic subgroup of $Q!$ generated by (23)(456). Thus

$$P\backslash Q = \{\{1\},\{2,3\},\{4,5,6\}\}, \tag{9}$$

yielding

$$A_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and } A_P^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} .$$

Now (8) gives

$$R_{P\backslash Q}(5) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} . \tag{10}$$

One may view this Markov chain action graphically according to Figure 1. Denote the elements of the state space $P\backslash Q$, the orbits of $\mathrm{LMlt}_Q P$ on Q, respectively as

$$a = \{1\}, \quad a' = \{2,3\}, \quad b = \{4,5,6\} .$$

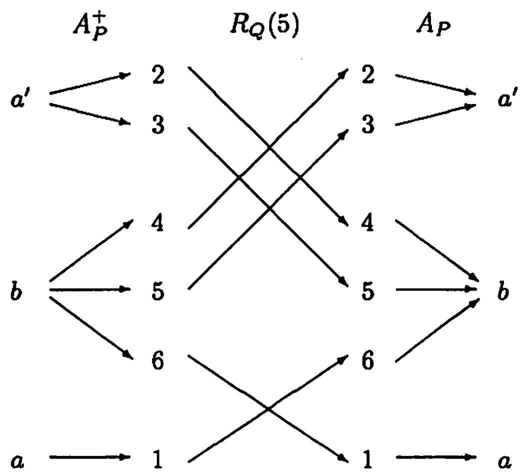

Figure 1: The Markov chain $R_{P\backslash Q}(5)$.

The incidence matrix A_P, giving the assignment of quasigroup elements to state space elements, is represented by the right-hand side of the figure.

The permutation $R_Q(5)$ of Q is represented in the center of the figure. The left-hand side represents the pseudoinverse A_P^+. In the Markov chain, each element of the state space on the left of the figure has a uniform chance of transitioning along each of the arrows leading from it. After that, its path through Q and back to the state space $P\backslash Q$ is uniquely specified, according to the matrix $R_{P\backslash Q}(5)$. For example, the element b has a two-thirds chance of transitioning to a', and a one-third chance of transitioning to a.

In order to study the action of the full quasigroup Q on $P\backslash Q$, define Markov matrices

$$\iota = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \tau = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}. \tag{11}$$

Note that $R_{P\backslash Q}(1) = \iota$, $R_{P\backslash Q}(2) = R_{P\backslash Q}(3) = \varepsilon$, and $R_{P\backslash Q}(4) = R_{P\backslash Q}(5)$ $= R_{P\backslash Q}(6) = \tau$. Moreover, the matrices (11) commute with each other. Consider the monoid generated by these matrices. Each element of the monoid may be expressed uniquely in the form $\varepsilon^l\tau^m$ for non-negative integers l and m. The action of these elements on the state space $\{a, a', b\}$ is then given by Figure 2, which displays the image of a under $\varepsilon^l\tau^m$. The symbol k stands for any positive integer. The information in the table is

$m \setminus l$	0	1	2	3	4	...
0	a	a'	$\frac{1}{2}a+\frac{1}{2}a'$	$\frac{1}{4}a+\frac{3}{4}a'$	$\frac{3}{8}a+\frac{5}{8}a'$	...
1	b	b	b	b	b	...
$2k$	$\frac{1}{3}a+\frac{2}{3}a'$	$\frac{1}{3}a+\frac{2}{3}a'$	$\frac{1}{3}a+\frac{2}{3}a'$	$\frac{1}{3}a+\frac{2}{3}a'$	$\frac{1}{3}a+\frac{2}{3}a'$	...
$2k+1$	b	b	b	b	b	...

Figure 2: Permutation action of Q on $\{a, a', b\}$

complete, since $b = a\tau$ and $a' = a\varepsilon$. In other words, $a'\varepsilon^l\tau^m = a\varepsilon^{l+1}\tau^m$ and $b\varepsilon^l\tau^m = a\varepsilon^l\tau^{m+1}$. Convex combinations of states are used to specify finite probability distributions. Thus $\frac{1}{3}a + \frac{2}{3}a'$ for example denotes the mixed state consisting of a one-third chance of state a and a two-thirds chance of state a'.

The quasigroup action may be interpreted as an approximate two-fold symmetry between the state b on the one hand, and the states a, a' on the other. If the distinction between a and a' is suppressed, then one obtains an exact two-fold symmetry between a and b, with ε acting as an identity element (just like ι), while τ acts as a transposition between a and b. Acknowledging the distinction between a and a', however, this symmetry is seen to be only approximate. For example, applying τ once to a gives b, but a repeated application of τ leads back to a only with probability one-third, and otherwise gives a'. Interpreting approximate symmetry as exact symmetry holding at one level of a hierarchical system, one may observe that in the present case, there is a hierarchy with just two levels: macroscopic and microscopic. The macrostates are $\{a, a'\}$ and $\{b\}$, the distinction between a and a' lying at the microscopic level. The approximate symmetry consists of exact two-fold symmetry at the macroscopic level.

5 The general model

In order to establish a general framework for approximate symmetry of the kind observed in the model of the previous section, it is necessary to consider some concepts in quasigroup theory. Let Q be a quasigroup with a congruence α. A subquasigroup Q_0 of Q is said to be *compatible* with α if it is the preimage of its image under the natural projection by α, i.e. if

$$Q_0 = (\operatorname{nat}\alpha)^{-1}(Q_0^\alpha).$$

In combinatorial terms, compatibility means that Q_0 is a union $\bigcup Q_0^\alpha$ of α-classes. (Compare the discussion of the Second Isomorphism Theorem in [9, IV, §1.2].) The *core* or *core congruence* of a subquasigroup Q_0 in Q is defined to be the largest congruence κ or $\kappa(Q_0)$ or $\kappa_Q(Q_0)$ on Q that is compatible with Q_0. This concept matches its group-theoretical analogue:

Proposition 5.1 *Let H be a subgroup of a group Q. Then the group-theoretical core $K_Q(H)$ of H in Q is the class of the identity element 1 of Q under the quasigroup-theoretical core $\kappa_Q(H)$ of H in Q.*

Proof Recall that $K_Q(H)$ is the intersection $\bigcap_{q\in Q} H^q$ of all the conjugates of H. As such, it is the largest normal subgroup N of Q contained in H. The map $\alpha \mapsto 1^\alpha$ provides an order-preserving isomorphism from the set of congruences compatible with H to the set of normal subgroups contained in H. Under this isomorphism, one has $\kappa_Q(H) \mapsto K_Q(H)$. □

The following definition serves to specify the kind of approximate symmetry under discussion. In the definition, an exact symmetry is described by a certain transitive permutation action, a faithful (or, in analysts' terminology, "effective") group homogeneous space.

Definition 5.2 *Let G be a group, and let (X, G) be a faithful homogeneous space for G. A system is said to exhibit* macroscopic approximate symmetry *of type (X, G) if it consists of two hierarchical levels, macroscopic and microscopic, with an exact symmetry of type (X, G) holding at the macroscopic level.*

Theorem 5.3 *Suppose that a non-empty finite quasigroup Q contains a subquasigroup Q_0 compatible with the group replica congruence of Q. Let κ be the core of Q_0 in Q. Then for a subquasigroup P of Q_0, the homogeneous space $P\backslash Q$ exhibits macroscopic approximate symmetry of type $(Q_0^\kappa\backslash Q^\kappa, Q^\kappa)$.*

Proof Since Q_0 is compatible with the group replica congruence of Q, the quotient Q^κ is a group. As a consequence of the isomorphism theorems, $K_{Q^\kappa}(Q_0^\kappa)$ is trivial, so the group homogeneous space $(Q_0^\kappa\backslash Q^\kappa, Q^\kappa)$ is faithful.

The microstates of the homogeneous space $P\backslash Q$ are its elements, the $\mathrm{LMlt}_Q(P)$-orbits on Q. The macrostates are the $\mathrm{LMlt}_Q(Q_0)$-orbits on Q. Since P is a subquasigroup of Q_0, it is immediate that each macrostate is a union of microstates, so that $P\backslash Q$ forms a two-level hierarchical system.

Suppose that $(x, y) \in \kappa$. Let e be an element of Q for which e^κ is the identity element of the group Q^κ. Then

$$y/x \in (y/x)^\kappa = y^\kappa / x^\kappa = e^\kappa .$$

Now e^κ is a subquasigroup of Q_0. Since $xL(y/x) = y$ by (SR), it follows that each $\mathrm{LMlt}_Q(Q_0)$-orbit is a union of κ-classes.

For x in Q, the map

$$\beta : Q_0 \backslash Q \to Q_0^\kappa \backslash Q^\kappa ; x\mathrm{LMlt}_Q(Q_0) \mapsto x^\kappa \mathrm{LMlt}_{Q^\kappa}(Q_0^\kappa)$$

bijects. Certainly it is well-defined, since $xL(q_1)^{\pm 1} \dots L(q_r)^{\pm 1} = y$ (with $x, y \in Q$ and $q_1, \dots, q_r \in Q_0$) implies $x^\kappa L(q_1^\kappa)^{\pm 1} \dots L(q_r^\kappa)^{\pm 1} = y^\kappa$. The map β is clearly surjective. For the injectivity, suppose

$$x^\kappa L(p_1^\kappa)^{\pm 1} \dots L(p_r^\kappa)^{\pm 1} = y^\kappa L(q_1^\kappa)^{\pm 1} \dots L(q_s^\kappa)^{\pm 1}$$

for $x, y \in Q$ and $p_1, \dots p_r, q_1, \dots, q_s \in Q_0$. Then

$$(xL(p_1)^{\pm 1} \dots L(p_r)^{\pm 1}, yL(q_1)^{\pm 1} \dots L(q_s)^{\pm 1}) \in \kappa ,$$

so x and y share the same $\mathrm{LMlt}_Q(Q_0)$-orbit.

Finally, for $x, y, q \in Q$ and $q_1, \dots, q_r \in Q_0$, the equation

$$xL(q_1)^{\pm 1} \dots L(q_r)^{\pm 1} R(q) = y$$

implies $x^\kappa L(q_1^\kappa)^{\pm 1} \dots L(q_r^\kappa)^{\pm 1} R(q^\kappa) = y^\kappa$. Thus the transition matrix of $R_{Q_0 \backslash Q}(q)$ on $Q_0 \backslash Q$ is the permutation matrix of $R_{Q_0^\kappa \backslash Q^\kappa}(q^\kappa)$ on $Q_0^\kappa \backslash Q^\kappa$. It follows that the macroscopic homogeneous space $(Q_0 \backslash Q, Q)$ has the required symmetry type $(Q_0^\kappa \backslash Q^\kappa, Q^\kappa)$. □

6 Discussion

Theorem 5.3 shows how a finite quasigroup may model macroscopic approximate symmetry. Examples such as that of the rotating engine, in which the symmetry group is infinite, are not covered by the theorem. A combinatorial attempt to extend the theorem to infinite quasigroups founders on the lack of an operator-theoretic analogue for the pseudoinverse of the incidence matrix of Q in $P \backslash Q$. However, one might well consider a measure-theoretical approach in this case.

The macroscopic approximate symmetry of Definition 5.2 is only one kind of approximate symmetry. For example, [8, §5.2] considers an approximately symmetric three-level hierachical system in which the exact symmetry holds at the mesoscopic level, but not at the macroscopic or microscopic levels. It would be of interest to develop combinatorial models for different kinds of approximate symmetry.

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