Some Coefficients of the Flow Polynomial of K_n

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Abstract

We determine some coefficients of the flow polynomial of the complete graph K_n.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The flow polynomial of G, $F(G,\lambda)$, is a polynomial in λ which gives the number of nowhere-zero λ-flows in G independent of the chosen orientation. Many properties of the flow polynomial, as well as more details on nowhere-zero flows can be found in [1] and [2]. The chromatic polynomial of the complete graph on n vertices is $P(K_n,\lambda) = \lambda\cdot(\lambda-1)\cdot(\lambda-2)\dots(\lambda-n+1)$. The flow polynomial of the complete graph on n vertices beyond very small values of n is not known and is difficult to compute. It is known that if G is a planar graph, then there is indeed a "λ-to-1" correspondence between the number of λ proper vertex colorings of G and the number of no-where zero λ flows of G^*, the planar dual of G. Since K_n is non-planar for $n \geq 5$, then we can no longer use $P(K_n,\lambda)$ and planar duality to compute $F(G,\lambda)$. The Contraction -Deletion Principle is an essential tool in computing flow polynomial of graphs by successively selecting a directed edge and reducing the problem to computing the flow polynomials of two smaller graphs, i.e., If e is any edge of G, then $F(G,\lambda) = F(G'',\lambda) - F(G',\lambda)$, where G' and G'' are obtained from G by deleting and contracting the edge e, respectively. Also, if G has no edges, then $F(G,\lambda) = 1$, if G has a bridge, then $F(G,\lambda) = 0$, and if G is a cycle, then $F(G,\lambda) = \lambda - 1$. in this note we make a few observations about some of the coefficients of $F(K_n,\lambda)$. For convenience, we use $\omega = 1 - \lambda$.

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2 The Ubiquitous Binomial Coefficients

The flow polynomial of the complete graphs K_n for $3 \leq n \leq 8$ are:

$$
\begin{aligned}
F(K_3,\omega) &= \omega \\
F(K_4,\omega) &= \omega^3+3\omega^2+2\omega \\
F(K_5,\omega) &= \omega^6+4\omega^5+10\omega^4+15\omega^3+15\omega^2+6\omega \\
F(K_6,\omega) &= \omega^{10}+5\omega^9+15\omega^8+35\omega^7+64\omega^6+96\omega^5+120\omega^4+ \\
&\quad 120\omega^3+80\omega^2+24\omega \\
F(K_7,\omega) &= \omega^{15}+6\omega^{14}+21\omega^{13}+56\omega^{12}+126\omega^{11}+245\omega^{10}+420\omega^9 \\
&\quad +645\omega^8+895\omega^7+1120\omega^6+1260\omega^5+1225\omega^4+945\omega^3+ \\
&\quad 490\omega^2+120\omega \\
F(K_8,\omega) &= \omega^{21}+7\omega^{20}+28\omega^{19}+84\omega^{18}+210\omega^{17}+462\omega^{16}+916\omega^{15} \\
&\quad +1660\omega^{14}+2779\omega^{13}+4333\omega^{12}+6328\omega^{11}+8680\omega^{10}+ \\
&\quad 11200\omega^9+13620\omega^8+15464\omega^7+16261\omega^6+15400\omega^5+ \\
&\quad 12495\omega^4+7980\omega^3+3444\omega^2+720\omega
\end{aligned}
$$

A careful study of the above coefficients reveals that many of them can be written as linear combinations of binomial coefficients as following:

$$
\begin{aligned}
F(K_3,\omega) &= \binom{1}{0} \\
F(K_4,\omega) &= \binom{2}{0}, \binom{3}{0}, 2! \\
F(K_5,\omega) &= \binom{3}{0}, \binom{4}{1}, \binom{5}{2}, \binom{6}{3} - \binom{5}{1}, \binom{7}{4} - 4\binom{5}{1} + 3! \\
F(K_6,\omega) &= \binom{4}{0}, \binom{5}{1}, \binom{6}{2}, \binom{7}{3}, \binom{8}{4} - \binom{6}{1}, \binom{9}{5} - 2\binom{6}{2}, \binom{10}{6} \\
&\quad -6\binom{6}{2}, \binom{11}{7} - \binom{6}{2}, \ldots, 4! \\
F(K_7,\omega) &= \binom{5}{0}, \binom{6}{1}, \binom{7}{2}, \binom{8}{3}, \binom{9}{4}, \binom{10}{5} - \binom{7}{1}, \binom{11}{6} - 2\binom{7}{2}, \\
&\quad \binom{12}{7} - 7\binom{7}{2}, \binom{13}{8} - 14\binom{7}{2}, \binom{14}{9} - 42\binom{7}{2}, \ldots, 5! \\
F(K_8,\omega) &= \binom{6}{0}, \binom{7}{1}, \binom{8}{2}, \binom{9}{3}, \binom{10}{4}, \binom{11}{5}, \binom{12}{6} - \binom{8}{1}, \\
&\quad \binom{13}{7} - 2\binom{8}{2}, \binom{14}{8} - 8\binom{8}{2}, \binom{15}{9} - 24\binom{8}{2},
\end{aligned}
$$

$$\binom{16}{10} - 30\binom{8}{3}, \binom{17}{11} - 66\binom{8}{3}, \ldots, 6!$$

We should remind the reader that $F(G,\omega)$ is a polynomial of degree $\nu = \nu(G)$, where $\nu(G)$, the cyclomatic number of G, is defined as $\nu(G) = |E(G)| - |V(G)| + \kappa(G)$ where $\kappa(G)$ denotes the number of components. Coefficient of ω^ν is $(-1)^\nu$ and all terms in $F(G,\omega)$ have the same sign. This is one of the reasons why dealing with $F(G,\omega)$ is easier than $F(G,\lambda)$.

3 Some Facts, Conjectures and Observations

Some of the facts we state here can be proved easily:

- The number of terms is $f(n) = \binom{n}{2} - n + 1 = \frac{1}{2}n^2 - \frac{3}{2}n + 1$.
- Largest Coefficient is achieved by the term ω^{n-2} where $n - 2 = \lceil \frac{d}{dx} f(n) \rceil$
- For even n, the alternating sum is zero.
- For odd n, the alternating sum is ± 1.
- Given $P(G,\lambda) = \sum_{i=0}^{n-1} a_i \lambda^{n-i}$, Read's Unimodal Conjecture states $1 \leq |a_1| \leq |a_2| \leq \ldots \leq |a_{i-1}| \leq |a_i| \geq |a_{i+1}| \geq \ldots \geq |a_n - 1| \geq |a_n| = 0$ for some $1 \leq i \leq n$. In addition, Hoggar's Strong Logarithmic Concavity Conjecture states that $a_i a_{i+2} < a_{i+1}^2$. The author is not aware of these conjectures being made about $F(G,\lambda)$ and we would like to state them here for the sake of completion.
- Given $F(K_n,\omega) = \sum_{i=0}^{\nu(G)-1} a_i \lambda^{\nu(G)-i}$, we may recursively see the following coefficients:

$$a_1 = \binom{n-2}{0}, a_2 = \binom{n-1}{1}, a_3 = \binom{n}{2}, a_4 = \binom{n+1}{3}, \ldots,$$

$$a_{n-3} = \binom{2n-6}{n-4}, a_{n-2} = \binom{2n-5}{n-3},$$

$$a_{n-1} = \binom{2n-4}{n-2} - \binom{n}{1},$$

$$a_n = \binom{2n-3}{n-1} - 2\binom{n}{2}$$

$$a_{n+1} = \binom{2n-2}{n} - n\binom{n}{2}$$

$$a_{n+2+k} = \binom{2n-1}{n+1} - f(k)\binom{n}{3} \quad for \quad k \geq 0,$$

$$\ldots,$$

$$a_{\nu(G)} = (n-2)!$$

Many coefficients are still remaining undetermined. However, we should add that it takes *Maple 10* close to 50 minutes to determine the flow polynomial of K_9. Our results state that $F(K_9, \omega)$ should be the following statement:

$$\begin{aligned} F(K_9,\omega) \quad = \quad & \binom{7}{0}, \binom{8}{1}, \binom{9}{2}, \binom{10}{3}, \binom{11}{4}, \binom{12}{5}, \binom{13}{6}, \binom{14}{7} - \binom{9}{1}, \\ & \binom{15}{8} - 2\binom{9}{2}, \binom{16}{9} - 9\binom{9}{2}, \binom{17}{10} - 30\binom{9}{2}, \\ & \binom{18}{10} - ?\binom{?}{?}, \ldots, 7! \end{aligned}$$

And in fact, our result agrees with the *Maple 10* computation.

References

[1] Hossein Shahmohamad, "On nowhere-zero flows, chromatic equivalence and flow equivalence of graphs", PhD Thesis, University of Pittsburgh, 2000.

[2] William H. Tutte, Graph Theory, *Addison-Wesley, Reading, Mass.*, 1984.