On Certain Series Expansions of the Sine Function Containing Embedded Catalan Numbers: A Complete Analytic Formulation

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Abstract

This article continues the study of a class of non-terminating expansions of $\sin(m\alpha)$ (even $m \geq 2$) which in each case possesses embedded Catalan numbers. A known series form of the sine function (said to be associated with Euler) is taken here as our basic representation, the coefficient of the general term being developed analytically in an interesting fashion and shown to be dependent on the Catalan sequence in the manner expected. The work, which has a historical backdrop to it, is discussed in the context of prior results by the author and others.

Introduction

Background

The earliest awareness of the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$, with general $(n+1)$th term

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

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has so far been traced to Antu Ming who found them occurring in some expansions of the sine function. In [1] the author discussed this historical point and, having established formally the infinite series forms (in odd powers of $\sin(\alpha)$) of $\sin(2\alpha)$ and $\sin(4\alpha)$, went on to develop those for $\sin(6\alpha), \sin(8\alpha)$ and $\sin(10\alpha)$ using a methodical algebraic process. The paper concluded with a predicted generalised structural format for $\sin(2p\alpha)$ (integer $p \geq 1$), convergent when $|\alpha| < \frac{\pi}{2}$. It was standardised to the following not long afterwards by Xinrong [2] in his work on the topic:[1]

$$\begin{aligned}\sin(2p\alpha) &= 2\left\{\sum_{n=1}^{p} \alpha_n^{(p)} \sin^{2n-1}(\alpha)\right. \\ &\qquad \left. + \sum_{n=1}^{\infty} \frac{h_p(c_{n-1},\ldots,c_{n+p-2})}{2^{2(n+p)-3}} \sin^{2(n+p)-1}(\alpha)\right\}. \qquad (2)\end{aligned}$$

The key feature here is a linearity of the function h_p in the p Catalan elements $c_{n-1},\ldots,c_{n+p-2}$ that was asserted by Larcombe in [1] (w.r.t. an equivalent function g_p, see p.45). Based on (2), the linear combination

$$h_p(c_{n-1},\ldots,c_{n+p-2}) = \beta_0^{(p)} c_{n-1} + \beta_1^{(p)} c_n + \cdots + \beta_{p-1}^{(p)} c_{n+p-2} \qquad (3)$$

has been shown by Xinrong to have a general coefficient

$$\beta_n^{(p)} = [x^n]\{H_p(x)\}, \qquad n = 0,\ldots,p-1, \qquad (4)$$

where, for $p \geq 1$ (note the sign factor omitted in [2, Theorem 1.2, p.158]),

$$H_p(x) = (-1)^p \frac{(2+\sqrt{4-x})^{2p} - (2-\sqrt{4-x})^{2p}}{8\sqrt{4-x}} \qquad (5)$$

is a degree $p-1$ polynomial in x; such polynomials (computed symbolically) are listed in [2], amongst which those for values $p = 1,\ldots,5$ validate results of the author given in [1]. We note, in passing, that $H_p(x)$—clearly an (ordinary) generating function for the finite p-sequence $\{\beta_0^{(p)},\ldots,\beta_{p-1}^{(p)}\}$ of coefficients—can be written in terms of the corresponding function

$$G(x) = \frac{1-\sqrt{1-4x}}{2x} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \qquad (6)$$

for the Catalan sequence as

$$H_p(x) = (-1)^p 4^{p-2} \frac{[2-I(x)]^{2p} - I^{2p}(x)}{1-I(x)}, \qquad (7)$$

[1] A preprint of Xinrong's 2004 article [2] was sent to this author privately in the spring of 2001, and some of its findings developed in [3].

with $I(x) = 1 - \sqrt{1-\frac{x}{4}} = \frac{x}{8}G(\frac{x}{16})$.

The potential to reduce the dependency of h_p as resting on fewer of the p aforesaid Catalan numbers arises from the relation (valid for $n \geq 0$, given $c_0 = 1$)

$$c_{n+1} = 2\frac{(2n+1)}{(n+2)}c_n \tag{8}$$

between neighbouring members of the Catalan sequence. By first identifying $\beta_n^{(p)}$ (4) as a binomial coefficient sum, the author arrived in [3, p.218] at a surprisingly compact (and, mathematically, entirely natural) representation of $h_p = h_p(c_{n-1})$ by employing hypergeometric function theory. We repeat here, for reference purposes later, the result stated therein:

Theorem 1 For $p, n \geq 1$,

$$h_p(c_{n-1}) = (-1)^p p(n+p)\frac{[2(n+2p-1)]!n!}{(n+2p-1)![2(n+p)]!}c_{n-1}.$$

Concerning the characteristics of h_p as initially proposed in (2), the theorem constituted a further step forward from the study of this class of expansions by Xinrong, and it was duly verified for values $p = 1, 2, 3$ in [3]. The univariate function $h_p(c_{n-1})$ evidently has a tighter form, and is consistent with the main result of this article in which we see that in an alternative expansion of $\sin(2p\alpha)$ the coefficients of *all* powers of $\sin(\alpha)$ are available from a new and different type of generic expression.

This Paper

Here, the analysis of [3] is extended by virtue of the fact that we find the general coefficient in the *complete* expansion

$$\sin(m\alpha) = \sum_{n=0}^{\infty} S_n^{(m)} \sin^{2n+1}(\alpha), \tag{9}$$

where $m \geq 2$ is even. As alluded to already, in seeking such a formulation the r.h.s. of (9) has the obvious advantage of being a *single* sum (that covers all powers of $\sin(\alpha)$) when compared to the split sum format of (2). The motivation for (9) stems from a series version of $\sin(m\alpha)$ associated with Leonhard Euler. Computational investigation of the latter gives rise to a result involving the Gamma function that paves the way for the development of a closed form for $S_n^{(m)}$ which, like h_p, is expressible as a (functional) multiple of c_{n-1}, and which confirms the asymptotic behaviour of h_p found in

[3]. The next section details all of this, and ends with a couple of corollaries to the work. A short summary concludes the paper.

Formulation of $S_n^{(m)}$

With reference to (9), we start by considering the equation

$$\begin{aligned}
&\sin(m\alpha) \\
&= m\sin(\alpha) - \left[\frac{m(m^2-1^2)}{3!}\right]\sin^3(\alpha) \\
&\quad + \left[\frac{m(m^2-1^2)(m^2-3^2)}{5!}\right]\sin^5(\alpha) \\
&\quad - \left[\frac{m(m^2-1^2)(m^2-3^2)(m^2-5^2)}{7!}\right]\sin^7(\alpha) + \cdots \\
&= m\left[\sin(\alpha) + \sum_{n=1}^{\infty}\left\{(-1)^n\frac{1}{(2n+1)!}\prod_{i=1}^{n}[m^2-(2i-1)^2]\right\}\sin^{2n+1}(\alpha)\right] \\
&= m\left[\sin(\alpha) + \sum_{n=1}^{\infty}\left\{\frac{1}{(2n+1)!}\prod_{i=1}^{n}[(2i-1)^2-m^2]\right\}\sin^{2n+1}(\alpha)\right] \\
&= \sum_{n=0}^{\infty} S_n^{(m)}\sin^{2n+1}(\alpha) \qquad (10)
\end{aligned}$$

where, defining $\prod_{i=1}^{0}[(2i-1)^2-m^2]=1$ (so that $S_0^{(m)}=m$, as it must),

$$S_n^{(m)} = \frac{m}{(2n+1)!}\prod_{i=1}^{n}[(2i-1)^2-m^2], \qquad n \geq 0. \qquad (11)$$

Given (11), the expansion (10) holds for all integer $m \geq 1$ (it indeed generates correctly those finite series for m odd), and is credited to Euler by Chinese scientific historian Luo.[2] We shall proceed to derive a closed form for $S_n^{(m)}$ from (11), under the assumption that $m \geq 2$ is even.

Remark 1 It is a straightforward matter to re-produce, based on Luo's representation

$$\sin(m\alpha) = m\sin(\alpha) + \sum_{n=1}^{\infty}\left\{\frac{A_n^{(m)}}{4^{n-1}}\right\}\sin^{2n+1}(\alpha), \qquad (12)$$

[2] J. Luo disseminated an appreciation of the work of Antu Ming in textbook form in 1998 (see [1] for details, and Remark 6 later for some recent news on Ming). His first publication relevant to the topic of this paper appeared a decade earlier [4].

the recursion for his coefficients (no closed form is, of course, given by Luo). Comparing (12) with the penultimate line in (10),

$$\frac{A_n^{(m)}}{4^{n-1}} = \frac{m}{(2n+1)!}\prod_{i=1}^{n} f(i;m) \tag{13}$$

where $f(i;m) = (2i-1)^2 - m^2$. It then follows that

$$\begin{aligned} \frac{A_{n+1}^{(m)}}{A_n^{(m)}} &= \frac{2}{(n+1)(2n+3)} f(n+1;m) \\ &= 2\frac{[2n-(m-1)][2n+(m+1)]}{(n+1)(2n+3)}, \qquad n \geq 1, \end{aligned} \tag{14}$$

with $A_1^{(m)} = -m(m-1)(m+1)/6$ an initial value to begin Luo's recursive procedure to calculate the coefficients in the sum of (12) for fixed m; it is evident from (10),(12) that

$$\begin{aligned} S_0^{(m)} &= m, \\ S_n^{(m)} &= \frac{A_n^{(m)}}{4^{n-1}}, \qquad n \geq 1. \end{aligned} \tag{15}$$

Regarding $S_n^{(m)}$, we now state and prove (partially) a result which underpins our formulation of it, having been obtained via one of the mainstream computer algebra systems currently available commercially.

<u>Lemma 1</u> For integer m (even) ≥ 2, $n \geq 1$,

$$\prod_{i=1}^{n}[(2i-1)^2 - m^2] = (-1)^{\frac{m}{2}}\frac{4^n}{\pi}\Gamma\left(n+\frac{1}{2}-\frac{m}{2}\right)\Gamma\left(n+\frac{1}{2}+\frac{m}{2}\right).$$

Two properties of the Gamma function to be used in the proof are first noted. These are

$$\Gamma\left(s+\frac{1}{2}\right) = \sqrt{\pi}\frac{1\cdot 3\cdot 5\cdot \;\cdots\; \cdot(2s-1)}{2^s}, \qquad s = 1,2,3,\ldots, \tag{16}$$

and

$$s\Gamma(s) = \Gamma(s+1), \qquad s \neq 0,-1,-2,\ldots \tag{17}$$

<u>Proof</u> An inductive argument suffices. When $m = 2$, $n = 1$, the l.h.s. of Lemma 1 is $1^2 - 2^2 = -3$, whilst (given $\Gamma(\frac{1}{2}) = \sqrt{\pi}$) the r.h.s. evaluates to

$-(4/\sqrt{\pi})\Gamma(\frac{5}{2}) = -3$ also (since $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$ by (16)). Now, let $m \geq 2$ be fixed, and assume Lemma 1 holds for some $n \geq 1$. Then

$$\begin{aligned}
&\prod_{i=1}^{n+1}[(2i-1)^2 - m^2] \\
&\quad = [(2n+1)^2 - m^2]\prod_{i=1}^{n}[(2i-1)^2 - m^2] \\
&\quad = 4\left(n + \frac{1}{2} - \frac{m}{2}\right)\left(n + \frac{1}{2} + \frac{m}{2}\right)\prod_{i=1}^{n}[(2i-1)^2 - m^2].
\end{aligned}$$

By hypothesis,

$$\begin{aligned}
&= 4\left(n + \frac{1}{2} - \frac{m}{2}\right)\left(n + \frac{1}{2} + \frac{m}{2}\right) \times \\
&\qquad (-1)^{\frac{m}{2}}\frac{4^n}{\pi}\Gamma\left(n + \frac{1}{2} - \frac{m}{2}\right)\Gamma\left(n + \frac{1}{2} + \frac{m}{2}\right) \\
&= (-1)^{\frac{m}{2}}\frac{4^{n+1}}{\pi}\Gamma\left(n + \frac{3}{2} - \frac{m}{2}\right)\Gamma\left(n + \frac{3}{2} + \frac{m}{2}\right)
\end{aligned}$$

as required using (17); this completes the inductive step on n. That on m is more involved algebraically but, being fairly straightforward nonetheless (it utilises (17) in a similar way to that just seen), it is omitted.□

Consider now setting $n = 0$ in Lemma 1. It reads

$$\begin{aligned}
\prod_{i=1}^{0}[(2i-1)^2 - m^2] &= (-1)^{\frac{m}{2}}\frac{1}{\pi}\Gamma\left(\frac{1}{2} - \frac{m}{2}\right)\Gamma\left(\frac{1}{2} + \frac{m}{2}\right) \\
&= (-1)^{\frac{m}{2}}\frac{1}{\pi}\cdot(-1)^{\frac{m}{2}}\pi \\
&= 1 \qquad (18)
\end{aligned}$$

as required, having used (16) and the counterpart equation

$$\Gamma\left(-s + \frac{1}{2}\right) = (-1)^s\sqrt{\pi}\frac{2^s}{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2s-1)} \qquad (19)$$

(for $s = 1, 2, 3, \ldots$) which together give

$$\Gamma\left(\frac{1}{2} - s\right)\Gamma\left(\frac{1}{2} + s\right) = (-1)^s\pi, \qquad s \geq 0, \qquad (20)$$

a special case of Euler's Reflection Formula (see, *e.g.*, [5, (1.2.1), p.9]). In view of (18) and Lemma 1, equation (11) can be modified accordingly to the following:

__Lemma 2__ For integer m (even) ≥ 2, $n \geq 0$,

$$S_n^{(m)} = (-1)^{\frac{m}{2}} \frac{m}{(2n+1)!} \frac{4^n}{\pi} \Gamma\left(n + \frac{1}{2} - \frac{m}{2}\right) \Gamma\left(n + \frac{1}{2} + \frac{m}{2}\right).$$

__Remark 2__ For fixed m the coefficients $S_n^{(m)}$ are those of the Taylor series of the function $f(\alpha) = \sin[m\sin^{-1}(\alpha)]$. We qualify this by noting that it can be shown without too much difficulty that in an appropriate domain

$$f(\alpha) = S_0^{*(m)}\alpha + S_1^{*(m)}\alpha^3 + S_2^{*(m)}\alpha^5 + \cdots = \sum_{n=0}^{\infty} S_n^{*(m)}\alpha^{2n+1}, \qquad (21)$$

say, whence

$$\begin{aligned} \sin(m\alpha) &= \sin(m\sin^{-1}[\sin(\alpha)]) \\ &= f(\sin(\alpha)) \\ &= \sum_{n=0}^{\infty} S_n^{*(m)} \sin^{2n+1}(\alpha), \end{aligned} \qquad (22)$$

so that $S_n^{*(m)} = S_n^{(m)}$ $(n \geq 0)$ by (10).

At this point we check our analysis thus far with a couple of results established in [1], noting that $\Gamma(s+\frac{1}{2})$, as defined by (16), is easily re-written as

$$\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi}\frac{(2s)!}{4^s s!}, \qquad s \geq 0, \qquad (23)$$

whereupon

$$\Gamma^2\left(s + \frac{1}{2}\right) = \pi\frac{(s+1)(2s)!}{16^s}c_s, \qquad s \geq 0, \qquad (24)$$

in terms of the Catalan number c_s (1). First, choosing $m = 2$ Lemma 2 gives

$$S_n^{(2)} = -\frac{2^{2n+1}}{(2n+1)!\pi}\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(n + \frac{3}{2}\right), \qquad n \geq 0. \qquad (25)$$

By (17), $\Gamma(n - \frac{1}{2}) = \Gamma(n + \frac{1}{2})/(n - \frac{1}{2})$ and $\Gamma(n + \frac{3}{2}) = (n + \frac{1}{2})\Gamma(n + \frac{1}{2})$, so that

$$\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(n + \frac{3}{2}\right) = \frac{(2n+1)}{(2n-1)}\Gamma^2\left(n + \frac{1}{2}\right), \qquad n \geq 0, \qquad (26)$$

which in conjunction with (24) means that in turn (25) becomes

$$S_n^{(2)} = -\frac{(n+1)}{2^{2n-1}(2n-1)}c_n, \qquad n \geq 0; \tag{27}$$

this gives (correctly) $S_0^{(2)} = 2$, and further, by (15), that for $n \geq 1$,

$$A_n^{(2)} = 4^{n-1}S_n^{(2)} = -\frac{(n+1)}{2(2n-1)}c_n = -c_{n-1} \tag{28}$$

using (8)—this agrees with the Appendix of [1]. If m is now set to 4 (for our second example here), then according to Lemma 2

$$\begin{aligned} S_n^{(4)} &= \frac{4^{n+1}}{(2n+1)!\pi}\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{5}{2}\right) \\ &= \frac{(n+1)(2n+3)}{4^{n-1}(2n-1)(2n-3)}c_n, \qquad n \geq 0, \end{aligned} \tag{29}$$

on deriving the relations

$$\begin{aligned} \Gamma\left(n-\frac{3}{2}\right) &= \frac{\Gamma\left(n-\frac{1}{2}\right)}{\left(n-\frac{3}{2}\right)} \\ &= \frac{\Gamma\left(n+\frac{1}{2}\right)}{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}, \\ \Gamma\left(n+\frac{5}{2}\right) &= \left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right) \\ &= \left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right) \end{aligned} \tag{30}$$

for $n \geq 0$ (by (17)) and again employing (24) as appropriate. We recover $S_0^{(4)} = 4$ trivially from (29), whilst using (8),(15) it yields

$$A_n^{(4)} = 4^{n-1}S_n^{(4)} = \frac{2(2n+3)}{(2n-3)}c_{n-1}, \qquad n \geq 1, \tag{31}$$

concurring once more with the Appendix of [1].

In summary, it is seen from (28),(31) that $S_n^{(2)} = -4^{1-n}c_{n-1}$ and $S_n^{(4)} = 2^{3-2n}[(2n+3)/(2n-3)]c_{n-1}$ for $n \geq 1$. The value $m = 6$, when verified, leads to $S_n^{(6)} = -3\cdot 4^{1-n}[(2n+3)(2n+5)/(2n-3)(2n-5)]c_{n-1}$ ($n \geq 1$), and other cases of m, treated in the same way, allow a final form for $S_n^{(m)}$ to be concluded by appealing to the result

$$\Gamma\left(n+\frac{1}{2}-\frac{m}{2}\right)\Gamma\left(n+\frac{1}{2}+\frac{m}{2}\right) = \Gamma^2\left(n+\frac{1}{2}\right)\prod_{i=1}^{\frac{1}{2}m}\frac{[2n+(2i-1)]}{[2n-(2i-1)]} \tag{32}$$

for $n \geq 0$ (the proof of which, by induction, is left as a reader exercise). From (8),(24) and (32), Lemma 2 may be re-cast as

$$\begin{aligned} S_0^{(m)} &= m, \\ S_n^{(m)} &= (-1)^{\frac{m}{2}} 2^{1-2n} m Q(n;m) c_{n-1}, \qquad n \geq 1, \end{aligned} \tag{33}$$

where

$$Q(n;m) = \begin{cases} 1 & m = 2 \\ \frac{(2n+3)(2n+5)\cdots[2n+(m-1)]}{(2n-3)(2n-5)\cdots[2n-(m-1)]} & m = 4,6,8,\ldots \end{cases} \tag{34}$$

Moreover, with $Q(n;m)$ as defined, we now introduce the Catalan number $c_{-1} = -\frac{1}{2}$ as an addition to the Catalan sequence (1). Then, since writing $Q(0;m) = (-1)^{\frac{m}{2}-1}\ \forall m \geq 2$ is consistent with (34) at $n = 0$, equation (33) can itself be re-stated in a slightly neater form to constitute our main result of the paper.

Theorem 2 For integer m (even) ≥ 2, $n \geq 0$,

$$S_n^{(m)} = (-1)^{\frac{m}{2}} 2^{1-2n} m Q(n;m) c_{n-1}.$$

Remark 3 We can easily relate the coefficient function h_p to the corresponding coefficient in (9). Writing $m = 2p$ ($p = 1,2,3,\ldots$), the expansion (9) reads

$$\begin{aligned} \sin(2p\alpha) &= \sum_{n=0}^{\infty} S_n^{(2p)} \sin^{2n+1}(\alpha) \\ &= \sum_{n=0}^{p-1} S_n^{(2p)} \sin^{2n+1}(\alpha) + \sum_{n=p}^{\infty} S_n^{(2p)} \sin^{2n+1}(\alpha), \end{aligned} \tag{35}$$

comparison of which with (2) gives by inspection that

$$\begin{aligned} 2\sum_{n=1}^{\infty} \frac{h_p}{2^{2(n+p)-3}} \sin^{2(n+p)-1}(\alpha) &= \sum_{n=p}^{\infty} S_n^{(2p)} \sin^{2n+1}(\alpha) \\ &= \sum_{n=1}^{\infty} S_{n+p-1}^{(2p)} \sin^{2(n+p)-1}(\alpha), \end{aligned} \tag{36}$$

and so

$$h_p = 4^{n+p-2} S_{n+p-1}^{(2p)}, \qquad p,n \geq 1. \tag{37}$$

By way of a sample check of (37), consider the case $p = 2$ for which

$$h_2 = 4^n S_{n+1}^{(4)} = 2\frac{(2n+5)}{(2n-1)} c_n \tag{38}$$

directly from (31). Expressing this in terms of c_{n-1} as

$$h_2(c_{n-1}) = 4\frac{(2n+5)}{(n+1)}c_{n-1} \tag{39}$$

using (8), it is found to match up with $h_2(c_{n-1})$ as given by Theorem 1.

We finish with two corollaries to the work presented.

Corollary 1 For finite (even) m,

$$\lim_{n\to\infty}\left\{\frac{S_{n+1}^{(m)}}{S_n^{(m)}}\right\} = 1.$$

Proof Equations (14),(15) give a coefficient ratio (this is also available via Theorem 2)

$$\frac{S_{n+1}^{(m)}}{S_n^{(m)}} = \frac{1}{2}\frac{[2n-(m-1)][2n+(m+1)]}{(n+1)(2n+3)},$$

from which the result is immediate.□

Corollary 2 For finite (even) m,

$$\lim_{n\to\infty}\left\{S_n^{(m)}\right\} = 0.$$

Proof By Theorem 2,

$$\begin{aligned}\lim_{n\to\infty}\left\{S_n^{(m)}\right\} &= (-1)^{\frac{m}{2}}2m\lim_{n\to\infty}\left\{2^{-2n}Q(n;m)c_{n-1}\right\}\\ &= (-1)^{\frac{m}{2}}2m\lim_{n\to\infty}\left\{2^{-2n}c_{n-1}\right\}\\ &= (-1)^{\frac{m}{2}}\frac{m}{2}\lim_{n\to\infty}\left\{2^{-2n}c_n\right\},\end{aligned}$$

using a simple change of limit variable and noting that $Q(n;m) \to 1$ as $n \to \infty$. Stirling's approximation for large $n!$ yields that for large n

$$c_n \sim \frac{1}{\sqrt{\pi}}\frac{4^n}{n\sqrt{n}},$$

hence

$$\lim_{n\to\infty}\left\{S_n^{(m)}\right\} = (-1)^{\frac{m}{2}}\frac{m}{2\sqrt{\pi}}\lim_{n\to\infty}\left\{\frac{1}{n\sqrt{n}}\right\} = 0.\square$$

Remark 4 It seems appropriate to show that the asymptotic form of $S_n^{(m)}$ is in line with that of h_p found in [3]. Theorem 2 gives that

$$S_{n+p-1}^{(2p)} = (-1)^p 4^{2-(n+p)} p Q(n+p-1; 2p) c_{n+p-2}, \tag{40}$$

which for large $n \gg p \geq 1$

$$\sim (-1)^p 4^{2-(n+p)} p \cdot 1 \cdot \frac{1}{\sqrt{\pi}} \frac{4^{n+p-2}}{n\sqrt{n}} = \frac{(-1)^p p}{\sqrt{\pi}} \frac{1}{n\sqrt{n}}; \tag{41}$$

as anticipated, h_p thus behaves asymptotically as

$$\frac{(-1)^p 4^{n+p-2} p}{\sqrt{\pi}} \frac{1}{n\sqrt{n}} \tag{42}$$

by (37), recovering the result derived in [3, (37), p.219].

Remark 5 For completeness we remark that another article [6], to which the interested reader is directed, has looked briefly at the role of hypergeometric functions in generating such series expansions of the sine function as considered here. It can in fact be used to provide a more direct route to Lemmas 1,2, as shown in the Appendix.

Summary

This paper, with reference to previous work by the author and others, has drawn on a particular result associated with Euler to develop a new representation of certain non-terminating expansions of the sine function which contain embedded terms of the Catalan sequence. The analysis has been validated as appropriate.

Appendix

Here, as alluded to in Remark 5, we give a different derivation of Lemma 2 (Lemma 1 is also easily recovered in the process).

Equation (A1) of [6, p.73] gives Euler's expansion of $\sin(m\alpha)$ (10),(11) as

$$\sin(m\alpha) = m\sin(\alpha)\, {}_2F_1\left(\begin{matrix} \frac{1}{2}-\frac{m}{2}, \frac{1}{2}+\frac{m}{2} \\ \frac{3}{2} \end{matrix} \middle| \sin^2(\alpha)\right) \tag{A1}$$

in hypergeometric form, so that, by comparison with (9),

$$S_n^{(m)} = m \frac{(\frac{1}{2}-\frac{m}{2})_n (\frac{1}{2}+\frac{m}{2})_n}{n! (\frac{3}{2})_n}, \qquad n \geq 0. \tag{A2}$$

Since, for $n \geq 0$, $(a)_n = \Gamma(a+n)/\Gamma(a)$,

$$\left(\frac{1}{2}-\frac{m}{2}\right)_n\left(\frac{1}{2}+\frac{m}{2}\right)_n = \frac{\Gamma(n+\frac{1}{2}-\frac{m}{2})\Gamma(n+\frac{1}{2}+\frac{m}{2})}{\Gamma(\frac{1}{2}-\frac{m}{2})\Gamma(\frac{1}{2}+\frac{m}{2})} = \frac{\Gamma(n+\frac{1}{2}-\frac{m}{2})\Gamma(n+\frac{1}{2}+\frac{m}{2})}{(-1)^{\frac{m}{2}}\pi} \quad \text{(A3)}$$

by (20) for even $m \geq 2$. Substituting (A3) into (A2) and noting that $n!(\frac{3}{2})_n = 4^{-n}(2n+1)!$ $(n \geq 0)$ yields Lemma 2.

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Remark 6 Antu Ming (c.1692-1763) studied mathematics under the personal supervision of emperor Kangxi who had established a royal college by decree in 1670. As well as being an exponent of the idea that geometric figures and numbers could be transformed into each other, contributing to the initial idea of a limit (in a geometrical context) and founding the theoretical notion of an inverse function, Antu Ming, as we have seen, worked with series representations of trigonometric functions for which he in fact found application in astronomy. Such was his influence in this particular field (he became a youth royal astronomer, and later a top-ranking national scholar and head of the Imperial Observatory, participating in the compilation and edition of three very important astronomical works) that, Chinese astronomers having found a new Minor Planet (No. 28242) in 1999, the Minor Planet Centre of the International Astronomical Union decreed on May 26th, 2002, it be named as "Antu Ming's Star". The year 2002 was designated the 310th anniversary of Ming's birth, and in August a so called Naming Meeting for the star and Nadamu[3] was held accordingly in the Inner Mongolian village of Chagannor (which lies within the local county of Zheng Bai Qi[4]) at which a conference was held to consider and reflect upon his scientific contribution (this also includes his work, as part of a team, on the early topographical mapping of China). More than 500 delegates and 20,000 local residents took part in the event, where the decisions to re-name the village after him and to build the Antu Ming Museum of Science and Technology were announced in recognition of his achievements and lasting legacy in China.[5] Further details about this fine mathematician and astronomer of the Qing Dynasty are available from the respected Chinese historian Professor J.J. Luo (he can be reached by post at the Institute for the History of Science, Inner Mongolia Normal University, Huhhot, Inner

[3] A "Nadamu" has evolved, from ancient times, to become a comprehensive modern day celebration activity integrated with ethnic sports, culture and art, trading and fairs, *etc.*

[4] Zheng Bai Qi (6,229 km^2) is part of the city of Xilingolemeng (202,580 km^2), which in turn constitutes part of the province of Inner Mongolia (the latter covering an area of approximately 1.2×10^6 km^2). To clarify, in China "Village" ("Town") $\subset$ "County" ("Qi") $\subset$ "City" ("Meng") $\subset$ "Province".

[5] The so called "ancestral place"—where one's ancestors/parents were born or lived—has long been highly cherished in Chinese custom. The authorised record and filing systems in China are still using this identifier much more commonly than a "birthplace". Although in the literature there are a lot of cases relating to the use of birthplace, the Chinese are still rather respectfully retaining their ancestral places in formal personal documentation. Since the 1980s some local authorities in China have adopted a standard requiring that details about a person must contain both his or her ancestral place and birthplace. This came into force to facilitate data compatibility for those who are to travel or live abroad, so as to keep in line with some Western countries' filing formats which generally require birthplace.

It was once the convention that Chinese scholars named themselves according to the dwelling place of the very first ancestor(s) they could trace back in family lineage, full names being formed by the family name followed by that of the ancestral place. In the official volume which details the history of the Qing Dynasty, the entry for Antu Ming gives his ancestral place but there is no record of his actual birthplace. It would be almost impossible to find this since the Mongolians were, historically, a nomadic people. Chagannor Town is the place where the local governmental council of Zheng Bai Qi is situated (it houses the central authority of the county). Re-naming it after him was not, as some Western academic researchers wrongly believe, due to the fact that it is his exact place of birth (for this is unknown, as just stated), but rather that Zheng Bai Qi is truly the ancestral place of Antu Ming, a notion well understood, even in today's times, by the Chinese in terms of the tradition and culture of their country.