Classification of Optimal Binary Self-Orthogonal Codes

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Abstract

In this paper, we complete the classification of optimal binary linear self-orthogonal codes up to length 25. Optimal self-orthogonal codes are also classified for parameters up to length 40 and dimension 10. The results were obtained via two independent computer searches.

Keywords–Optimal binary linear codes, self-dual codes, self-orthogonal codes.

1 Introduction

A binary linear $[n, k]$ code C is a k-dimensional vector subspace of $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the finite field of two elements. In the sequel, all codes are assumed to be binary. The rate of a linear $[n, k]$ code C is defined as k/n. The elements of C are called codewords. The weight $wt(x)$ of a codeword x is the number of its non-zero coordinates. The minimum weight of C is the smallest weight among all non-zero codewords of C. An $[n, k, d]$ code is an $[n, k]$ code with minimum weight d. Two codes C and C' are equivalent if one can be obtained from the other by permuting the coordinates. The automorphism group of C is the set of permutations of the coordinates which preserve C. Every linear code is equivalent to a code with a systematic generator matrix of the form

$$G = [\, I \mid A \,].$$

The weight enumerator of C is $W_C(y) = \sum_{i=0}^{n} B_i y^i$, where B_i is the number of codewords of weight i in C. A code is called *even* if the weights of all codewords are even and *odd* otherwise. If all codewords in an even code have weights divisible by 4, we say the code is *doubly-even*, otherwise it is *singly-even*.

The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in \mathbb{F}_2^n : x \cdot y = 0 \text{ for all } y \in C\}$ where $x \cdot y$ denotes the standard inner product of x and y. A code C

is *self-dual* (SD) if $C = C^{\perp}$ and *self-orthogonal* (SO) if $C \subseteq C^{\perp}$. All self-orthogonal codes are even. All doubly-even codes are self-orthogonal. As the dual code of an $[n, k]$ linear code has dimension $n - k$, self-orthogonal codes exist only when $n \geq 2k$. For an introduction to coding theory in general and self-dual and self-orthogonal codes in particular, see [15] and [22].

Very little has previously been known about the minimum distance and number of self-orthogonal codes, since most of the research effort has been expended on the special case of self-dual codes. Self-dual codes have been classified up to length 34 in [1, 2, 8, 9, 20, 21]. Recently the optimal rate 1/2 linear codes have been classified for lengths up to 28 [12]. The Self-orthogonal $[n, k]$ codes for $n = 2k + 1$, $k \leq 9$ have been classified by Pless in [20] and for $11 \leq n - k \leq 16$, $4 \leq k \leq 10$ by the second author [5]. To the best of our knowledge, there have been no systematic investigations of SO codes (that are not self-dual) for larger lengths.

A linear $[n, k]$ code C is *optimal* if C has the highest minimum weight among all linear $[n, k]$ codes (see [6, 7] for known bounds on the highest minimum weight). For some values of n and k no optimal linear $[n, k]$ code is self-orthogonal. So we say that a self-orthogonal $[n, k]$ code with the highest minimum weight among all self-orthogonal $[n, k]$ codes is an *optimal SO code*. This definition is analogous to the definition of an *optimal SD code* in [22, Section 12].

In this paper, we complete the classification of optimal self-orthogonal codes up to length 25. Optimal SO codes are also classified for parameters up to length 40 and dimension 10 where the number of optimal codes is not too large. This appears to be the current computational limit given the increase in the size and number of optimal codes as n increases. Two different methods are used in the classification, so (in nearly all cases) the results have been obtained by two independent algorithms. (Note that mass formulae cannot be used to check the results since only optimal SD codes are classified.) The classified codes are listed and discussed whenever the number of codes with no zero columns is relatively small (at most 10–25). Generator matrices of the classified codes, and information about the dual distance and the orders of their automorphism groups, can be obtained electronically from the first author.

2 Classification of Self-Orthogonal Linear Codes

Determining the equivalence of codes plays a central role in any classification algorithm. Not only must one make sure that all resulting codes are inequivalent, but determining equivalence of partial codes is also important for efficiency reasons. In one approach, the authors utilize an algorithm for determining code equivalence that is based on the ideas of [3] and [16]. The second approach depends on the graph isomorphism program *nauty* [16, 17] for determining code equivalence; see [19] for further details. The approaches differ in the ways the codes are built up via smaller codes. Analogous methods have been used in [4] to classify ternary and quaternary self-orthogonal codes.

The first approach utilizes results on the parameters of residuals of codes. Let $\mathbf{G}$ be a generator matrix of a linear $[n, k, d]$ code C. Then the *residual code* $\mathrm{Res}(C, \mathbf{c})$ of C with respect to a codeword $\mathbf{c}$ is the code generated by the restriction of $\mathbf{G}$ to the columns where $\mathbf{c}$ has a zero entry. In addition to constructing $[n, k, d]$ codes from their $[n-w, k-1, d']$ residual codes, one may also start from $[n-i, k, d']$ codes. On the bottom of this hierarchy of extensions is the trivial $[k, k, 1]$ code.

In the second approach, $[n, k, d]$ codes are constructed by extending $[n-i, k-i, d]$ or $[n-i, k-i+1, d]$ codes. The following result shows when the latter type of codes can be used [15, p. 592].

Lemma 1 *Let C be an $[n, k, d]$ code. If there exists a codeword $\mathbf{c} \in C^{\perp}$ with $wt(\mathbf{c}) = i$, then there is an $[n-i, k-i+1, d]$ code.*

The subcodes through which the codes are constructed must also be self-orthogonal. For the approach via residual codes, on the other hand, such a restriction does not apply.

If $i = 1$ in the second approach, we get the method used in [19], where $[n, k, d]$ codes are obtained from $[n-1, k-1, d]$ codes by adding a new column in all possible ways to the parity check matrix, checking the minimum distance and orthogonality of the new code, and finally removing copies of equivalent codes. We shall now see how the equivalence test can be enhanced for this particular variant.

To speed up the algorithm and reduce the need for extensive tables of intermediate codes, a classification technique developed by McKay [18] was implemented. Essentially, the idea is that a code can be obtained from several subcodes, only one of which is identified as the "parent" of the new code. Then a new code is rejected unless it was obtained from its parent. Note that identifying a certain subcode means identifying a coordinate, and with the encoding used in [19] the output of *nauty* can be used to get a canonical labelling of the coordinates.

Shortening an $[n, k, d]$ linear code by deleting one coordinate and keeping the codewords with a 0 in the given coordinate gives an $[n-1, k', d]$ code with $k' = k$ if the original code has only 0s in the coordinate to be deleted, and $k' = k - 1$ otherwise. Therefore, in the parent test of a McKay-type algorithm—after adding one coordinate via a new column in the parity check matrix—one should first check which coordinates are all-zero. In the test itself, only coordinates that are not all-zero should be considered.

Given $[n_1, k, d_1]$ and $[n_2, k, d_2]$ codes, an $[n_1 + n_2, k, d_1 + d_2]$ code can be obtained by concatenating their generator matrices. This is called *juxtaposition*, and leads to the following lemma.

Lemma 2 *The juxtaposition of a code with itself is a self-orthogonal code. If the original code is even, then the SO code obtained from juxtaposition is doubly-even.*

Proof. The inner product of two words of the juxtaposed code is $(c_i, c_i) \cdot (c_j, c_j) = 2c_i \cdot c_j \equiv 0 \pmod 2$, so it is self-orthogonal. Moreover, $\mathrm{wt}((c_i, c_i)) = 2\mathrm{wt}(c_i)$, which is divisible by 4 if $\mathrm{wt}(c_i)$ is even. □

We denote the juxtaposition of i copies of C by C^i. Any even number of copies of a code can be juxtaposed to create an SO code, for example, four copies of S_4, where S_4 is the only [4,3,2] code, gives S_4^4, an SO $[16, 3, 8]$ code. If the number of copies is a multiple of 4, the resulting code is doubly-even.

3 Optimal SO Codes of Dimension up to 3

For dimension 1, it is trivial that the only optimal SO codes are the even length repetition codes, i.e., $[2m, 1, 2m]$ codes for $m \geq 1$. For $k = 2$, the optimal codes have distance [13]

$$d_{max}(n, 2) = \left\lfloor \frac{2}{3} n \right\rfloor,$$

so even-distance optimal linear codes only exist for $n = 3m$ or $n = 3m+1$. Further, when m is odd, an optimal code of length $3m$ is equivalent to one with codewords of the form

$$(1\ldots1,1\ldots1,0\ldots0),\ (1\ldots1,0\ldots0,1\ldots1),\ \text{and}\ (0\ldots0,1\ldots1,1\ldots1)$$

where each block contains m bits. Thus this code is not self-orthogonal. Therefore, optimal linear codes with dimension $k = 2$ are self-orthogonal only for $n = 6m$, $6m+1$, or $6m+4$ with $d_{max}(n,2) = 4m$, $4m$, or $4m+2$, respectively. For $n = 6m+2$ the optimal SO code has codewords of the form

$$(1\ldots1,1\ldots1,0\ldots0),\ (1\ldots1,0\ldots0,1\ldots1),\ \text{and}\ (0\ldots0,1\ldots1,1\ldots1)$$

where the last block contains $2m+2$ bits and the other two blocks contain $2m$ bits. For $n = 6m+3$ and $6m+5$ the optimal SO code is obtained by the optimal $[6m+2,2,2m]$ and $[6m+4,2,2m+2]$ SO code, respectively, by adding a zero coordinate.

Denote by M the generator matrix of the [7,3,4] simplex code, whose ith column is the binary representation of i, $i = 1,2,\ldots,7$. To construct every code C of dimension 3 without zero coordinates, we use l_i copies of the ith column of M. Note that if $l_i > 1$, then the dual code $C^{\perp}$ has minimum weight 2. We can say that the code C is defined by the set $\{l_1, l_2, \ldots, l_7\}$. The length of C is $l_1+l_2+\cdots+l_7$ and its nonzero weights are $l_4+l_5+l_6+l_7$, $l_2+l_3+l_6+l_7$, $l_2+l_3+l_4+l_5$, $l_1+l_3+l_5+l_7$, $l_1+l_2+l_4+l_7$, $l_1+l_2+l_5+l_6$, and $l_1+l_3+l_4+l_6$.

Lemma 3 *The code C is a self-orthogonal code iff $l_1 \equiv l_2 \equiv \cdots \equiv l_7 \equiv n \pmod 2$.*

Proof. The code C is self-orthogonal iff

$$l_6+l_7 \equiv l_4+l_5 \equiv l_2+l_3 \equiv l_5+l_7 \equiv l_1+l_3 \equiv l_3+l_7 \equiv 0 \pmod 2 \iff$$
$$l_1 \equiv l_2 \equiv \cdots \equiv l_7 \pmod 2.$$

For the length of the code we have $n = l_1+\cdots+l_7 \equiv 7l_1 \equiv l_1 \pmod 2$. □

Using the Griesmer bound $n \geq d + \lfloor \frac{d}{2} \rfloor + \lfloor \frac{d}{4} \rfloor$ we obtain that $n \geq 7s$ when $d = 4s$, and $n \geq 7s+4$ when $d = 4s+2$ [11].

The automorphism group of the [7,3,4] simplex code is $GL(3,2)$ which is 2-transitive, so we can take $l_1 \geq l_2 \geq l_j, j = 3,\ldots,7$ and $l_4 \geq l_j, j = 5,6,7$

[20]. Since $l_1+l_2+\dots+l_7=n=7s+r\leq 7l_1$ we have $l_1\geq s$ and $l_1\geq s+1$ when $r=1,2,\dots,6$. Let us denote $l_2+l_3=a_1,\ l_4+l_5=a_2,\ l_6+l_7=a_3$.

If $d=4s$ then $a_2+a_3=7s+r-l_1-a_1\geq 4s$ which implies that $a_1\leq 3s+r-l_1$. Similarly, $a_2\leq 3s+r-l_1$ and $a_3\leq 3s+r-l_1$. Hence $4s\leq a_2+a_3\leq 6s+2r-2l_1$ and therefore $l_1\leq s+r$.

- $n=7s$. In this case $l_1=s$ and since $l_2+\dots+l_7=6s$ we have $l_2=l_3=\dots=l_7=s$. Hence a unique self-orthogonal $[7s,3,4s]$ code exists and it is obtained by taking s times the [7,3,4] simplex code [13].

- $n=7s+1$. In this case $l_1=s+1$, therefore $a_1+a_2+a_3=6s$ and $a_i+a_j=4s, 1\leq i<j\leq 3$. Hence $a_1=a_2=a_3=2s$. We are looking for SO codes and therefore $l_i\neq s$. Since $l_2\geq l_3,\ l_4\geq l_5$, and $l_1+l_3+l_5+l_7\geq 4s$, it follows that $l_2=l_4=l_7=s+1$ and $l_3=l_5=l_6=s-1$. Hence there exist two $[7s+1,3,4s]$ self-orthogonal codes. The first one is constructed from the $[7s,3,4s]$ code by adding a zero coordinate. The weight enumerator of the second code is $W(y)=1+6y^{4s}+y^{4s+4}$.

- $n=7s+2$. In this case $l_1=s+1$ or $s+2$. Since $l_1\equiv 7s+2\equiv s \pmod 2$ we have $l_1=s+2$. Hence $a_1=a_2=a_3=2s$. If $s=1$ we obtain only one possibility, namely $l_2=\dots=l_7=s$. So a unique self-orthogonal $[9,3,4]$ code without zero coordinate exists and its weight enumerator is $W=1+3y^4+4y^6$. When $s\geq 2$ we have two more possibilities, namely $l_2=s+2,\ l_3=s-2,\ l_4=s+2,\ l_5=s-2,\ l_6=s+2,\ l_7=s-2$ and $l_2=s+2,\ l_3=s-2,\ l_4=l_5=l_6=l_7=s$. Thus there are exactly five inequivalent self-orthogonal $[7s+2,3,4s]$ codes for $s\geq 2$. Two of these are trivial extensions of the SO $[7s+1,3,4s]$ codes, while the other three codes have weight enumerators $W(y)=1+5y^{4s}+2y^{4s+4}$, $W(y)=1+3y^{4s}+4y^{4s+2}$, and $W(y)=1+6y^{4s}+y^{4s+8}$.

- $n=7s+3$. Using the inequalities for l_1 and Lemma 2 we obtain $l_1=s+1$ or $s+3$. When $l_1=s+3$ we have $l_2+\dots+l_7=6s$ and $a_j\leq 2s, j=1,2,3$. Hence $a_1=a_2=a_3=2s$, and we obtain four inequivalent possibilities:

$$l_2=l_4=l_7=s+3,\ l_3=l_5=l_6=s-3\ (s\geq 3);$$

$$l_2 = s+3,\ l_3 = s-3,\ l_4 = l_7 = s+1,\ l_5 = l_6 = s-1\ (s \geq 3);$$
$$l_2 = l_4 = l_7 = s+1,\ l_3 = l_5 = l_6 = s-1;$$
$$\text{or}\ \ l_2 = l_4 = l_6 = s+1,\ l_3 = l_5 = l_7 = s-1.$$

When $l_1 = s+1$ we have $l_2 + \cdots + l_7 = 6s+2 \leq 6l_2$ and so $l_2 = s+1$. Since $a_1 + a_2 + a_3 = 6s+2$ and $a_i \geq 4s$, $\{a_1, a_2, a_3\} = \{2s+2, 2s+2, 2s-2\}$ or $\{2s+2, 2s, 2s\}$. If $a_i = 2s+2$ then $l_{2i} = l_{2i+1} = s+1$ and if $a_i = 2s$ then $\{l_{2i}, l_{2i+1}\} = \{s+1, s-1\}$, so we have the following two possibilities

$$l_1 = l_2 = l_3 = l_4 = l_5 = l_6 = s+1,\ l_7 = s-3\ (s \geq 3);$$

$$l_1 = l_2 = l_3 = l_4 = l_5 = s+1,\ l_6 = l_7 = s-1.$$

Hence there exist six self-orthogonal $[7s+3, 3, 4s]$ codes without zero bits for $s \geq 3$ with weight enumerators

$$\begin{array}{ll} W_1 = 1 + 2y^{4s} + 4y^{4s+2} + y^{4s+4}, & W_2 = 1 + 3y^{4s} + 3y^{4s+2} + y^{4s+6}, \\ W_3 = 1 + 4y^{4s} + 3y^{4s+4}, & W_4 = 1 + 4y^{4s} + 3y^{4s+4}, \\ W_5 = 1 + 5y^{4s} + y^{4s+4} + y^{4s+8}, & W_6 = 1 + 6y^{4s} + y^{4s+12}. \end{array}$$

There exist three $[10, 3, 4]$ and three $[17, 3, 8]$ self-orthogonal codes without zero columns and their weight enumerators are W_1, W_2, and W_3 for the corresponding values of s.

- $n = 7s+4$. If $d = 4s+2$ then $a_2 + a_3 = 7s + r - l_1 - a_1 \geq 4s+2$ which implies that $a_1 \leq 3s + r - 2 - l_1$. Similarly, $a_2 \leq 3s + r - 2 - l_1$ and $a_3 \leq 3s + r - 2 - l_1$. Hence $4s + 2 \leq a_2 + a_3 \leq 6s + 2r - 4 - 2l_1$, thus $l_1 \leq s + r - 3$. In our case $s + 1 \leq l_1 \leq s+1$ and so $l_1 = s+1$. But $s + 1 \not\equiv 7s + 4 \pmod 2$. Therefore this case is impossible and no self-orthogonal $[7s+4, 3, 4s+2]$ code exists. When $d = 4s$ we obtain the following cases:

$l_1 = l_2 = s+2,\ l_3 = l_4 = l_5 = l_6 = l_7 = s:$
$W = 1 + y^{4s} + 4y^{4s+2} + 2y^{4s+4};$

$l_1 = l_2 = l_4 = s+2,\ l_5 = l_6 = l_3 = s,\ l_7 = s-2:$
$W = 1 + 3y^{4s} + 4y^{4s+4}(s \geq 2);$

$l_1 = l_2 = l_4 = s+2,\ l_3 = l_5 = s,\ l_6 = s-2,\ l_7 = s:$
$W = 1 + 2y^{4s} + 3y^{4s+2} + y^{4s+4} + y^{4s+6}\ (s \geq 2);$

$l_1 = l_2 = l_4 = s+2,\ l_3 = l_5 = s,\ l_6 = s-4,\ l_7 = s+2:$
$W = 1 + 4y^{4s} + 2y^{4s+4} + y^{4s+8}\ (s \geq 4);$

$l_1 = l_2 = l_4 = s+2,\ l_3 = s,\ l_5 = s-2,\ l_6 = s-2,\ l_7 = s+2:$
$W = 1 + 2y^{4s} + 4y^{4s+2} + y^{4s+8}\ (s \geq 2);$

$l_1 = a_2 = l+4,\ l_3 = s-4,\ l_4 = s+2,\ l_5 = l_6 = s-2,\ l_7 = s+2:$
$W = 1 + 5y^{4s} + y^{4s+4} + y^{4s+12}\ (s \geq 4);$

$l_1 = s+4,\ l_2 = s+2,\ l_3 = s-2,\ l_4 = s+2,\ l_5 = s-2,\ l_6 = s,\ l_7 = s:$
$W = 1 + 4y^{4s} + 2y^{4s+4} + y^{4s+8}\ (s \geq 2);$

$l_1 = s+4,\ l_2 = s+2,\ l_4 = s+2,\ l_3 = l_5 = l_6 = s-2,\ l_7 = s+2:$
$W = 1 + 3y^{4s} + 3y^{4s+2} + y^{4s+10}(s \geq 2);$

$l_1 = s+4,\ l_2 = l_3 = l_4 = l_5 = l_6 = l_7 = s:$
$W = 1 + 3y^{4s} + 4y^{4s+4};$

$l_1 = s+4,\ l_2 = s+2,\ l_4 = l_5 = l_6 = l_7 = s,\ l_3 = s-2:$
$W = 1 + 3y^{4s} + 2y^{4s+2} + 2y^{4s+6}\ (s \geq 2);$

$l_1 = l_2 = s+4,\ l_4 = l_5 = l_6 = l_7 = s,\ l_3 = s-4:$
$W = 1 + 5y^{4s} + 2y^{4s+8}\ (s \geq 4);$

$l_1 = l_2 = l_4 = l_7 = s+4,\ l_3 = l_5 = l_6 = s-4:$
$W = 1 + 6y^{4s} + y^{4s+16}\ (s \geq 4).$

So the number of inequivalent self-orthogonal $[7s+4, 3, 4s]$ codes without zero coordinates is two when $s = 1$, eight when $s = 2$, eight when $s = 3$, and twelve when $s \geq 4$.

- $n = 7s+5$, $d = 4s+2$. In this case $s+1 \leq l_1 \leq s+2$ and therefore $l_1 = s+1 \equiv 7s+5 \pmod 2$. Since $l_i \leq s+1, i = 2, 3, \ldots, 7$ and $l_2 + \cdots l_7 = 6s+4$, it follows that all $l_j = s+1$ except one and since $GL(3,2)$ is transitive, we can take $l_7 = s-1$, $l_1 = \cdots = l_6 = s+1$. Hence a unique $[7s+5, 3, 4s+2]$ self-orthogonal code exists and its weight enumerator is $W = 1 + 4y^{4s+2} + 3y^{4s+4}$.

- $n = 7s+6$, $d = 4s+2$. In this case $s+1 \leq l_1 \leq s+3$ and therefore $l_1 = s+2 \equiv 7s+6 \pmod 2$. Hence $a_i \leq 2s+2, i = 1, 2, 3$. As

$a_1+a_2+a_3 = 6s+4$, one of the numbers a_1, a_2, a_3 is $2s$ and the other two are equal to $2s+2$. If $a_i = 2s+2$ then $\{l_{2i}, l_{2i+1}\} = \{s+2, s\}$ and if $a_i = 2s$ then $\{l_{2i}, l_{2i+1}\} = \{s+2, s-2\}$ or $l_{2i} = l_{2i+1} = s$. So we obtain the following inequivalent possibilities:

$$l_1 = l_2 = l_4 = s+2,\ l_3 = l_5 = l_6 = l_7 = s:$$
$$W = 1 + 6y^{4s+2} + y^{4s+6};$$
$$l_1 = l_2 = l_4 = l_7 = s+2,\ l_3 = l_5 = s,\ l_6 = s-2:$$
$$W = 1 + 4y^{4s+2} + 2y^{4s+4} + y^{4s+8} \quad (s \geq 2).$$

Thus there is a unique $[13,3,6]$ and two inequivalent $[7s+6,3,4s+2]$ $(s \geq 2)$ SO codes without zero coordinates.

Note that we could use the same approach to classify the self-orthogonal codes of dimension greater than 3, but the calculations would then be much more complicated and, as the length of the simplex code of dimension k is $2^k - 1$, the number of cases to consider would be much larger.

4 Optimal Codes of Length Less Than 12

In this section, we present the classification of optimal self-orthogonal (SO) codes with length $n < 12$ and dimension $k > 3$. From now on, all results have been obtained by computer using the approaches outlined in Section 2. For the parameters considered it is straightforward to determine (again, by computer) the minimum weight of the classified codes and thereby the highest minimum weight.

For length 8, the [8,4,4] extended Hamming code is the only linear code with these parameters. For length 9, exactly three self-orthogonal [9,4] codes exist and only the code obtained from H_8 by adding a zero coordinate has minimum weight 4 [20].

For length 10, there is a $[10,4,4]$ code obtained by doubly extending the $[8,4,4]$ code with zero coordinates. In addition, there are three other codes given by (here and in the sequel, matrices denoted by A are parts of a generator matrix of the form $[I\ A]$)

$$A_{10,4,1} = \begin{bmatrix} 111000 \\ 110100 \\ 101100 \\ 011111 \end{bmatrix},\quad A_{10,4,2} = \begin{bmatrix} 111000 \\ 110100 \\ 110010 \\ 110001 \end{bmatrix},\quad A_{10,4,3} = \begin{bmatrix} 111000 \\ 110100 \\ 110010 \\ 101111 \end{bmatrix},$$

with weight enumerators $W_{10,4,1} = 1 + 7y^4 + 7y^6 + y^{10}$, $W_{10,4,2} = 1 + 10y^4 + 5y^8$, $W_{10,4,3} = 1 + 6y^4 + 8y^6 + y^8$.

The orders of their automorphism groups are 1008, 3840 and 384, respectively. Note that the first code is obtained by adding the all-one vector to the trivial extension of the [7,3,4] simplex code, the second code is a juxtaposition of the [5,4,2] even code with itself, and the first three rows of $A_{10,4,3}$ give S_4^2. There are two self-dual codes of this length and both have minimum weight 2.

For length 11, we have two [11,4,4] SO codes without zero coordinates. They have generator matrices

$$G_{11,4,1} = \begin{bmatrix} 01110100000 \\ 10101100000 \\ 10110011100 \\ 00000011011 \end{bmatrix} \text{ and } G_{11,4,2} = \begin{bmatrix} 11100010000 \\ 11010100000 \\ 10111000000 \\ 00000001111 \end{bmatrix}$$

and weight enumerators $W_{11,4,1} = 1 + 4y^4 + 8y^6 + 3y^8$ and $W_{11,4,2} = 1 + 8y^4 + 7y^8$. The orders of their automorphism groups are 192 and 4032, respectively. The second code is the direct sum of the [7,3,4] and [4,1,4] codes.

There is a unique [11,5,4] SO code without zero coordinates. It has weight enumerator $W_{11,5} = 1 + 10y^4 + 16y^6 + 5y^8$ and its automorphism group is $Z_2^4 \cdot S_5$ [20].

5 Length 12

There is a unique [12,4,6] linear code but it is not self-orthogonal [14]. There are ten SO [12,4,4] codes given by

$$A_{12,4,1} = \begin{bmatrix} 11100000 \\ 11010000 \\ 11001000 \\ 01111111 \end{bmatrix}, A_{12,4,2} = \begin{bmatrix} 11100000 \\ 01110000 \\ 11011100 \\ 01101011 \end{bmatrix}, A_{12,4,3} = \begin{bmatrix} 11100000 \\ 10110000 \\ 11010000 \\ 01111111 \end{bmatrix},$$

$$A_{12,4,4} = \begin{bmatrix} 11101100 \\ 11011100 \\ 11000010 \\ 01110011 \end{bmatrix}, A_{12,4,5} = \begin{bmatrix} 11100000 \\ 11010000 \\ 10111100 \\ 01110011 \end{bmatrix}, A_{12,4,6} = \begin{bmatrix} 11111000 \\ 01100100 \\ 00011010 \\ 01111001 \end{bmatrix},$$

$$A_{12,4,7} = \begin{bmatrix} 11000100 \\ 10111100 \\ 10110110 \\ 10000101 \end{bmatrix}, A_{12,4,8} = \begin{bmatrix} 11000100 \\ 10111100 \\ 10000110 \\ 10000101 \end{bmatrix}, A_{12,4,9} = \begin{bmatrix} 11100000 \\ 00011100 \\ 01100010 \\ 01100001 \end{bmatrix},$$

$$A_{12,4,10} = \begin{bmatrix} 11001000 \\ 00110100 \\ 00110010 \\ 01001001 \end{bmatrix}.$$

Their weight enumerators are

$$\begin{aligned} W_{12,4,1} &= 1 + 6y^4 + 9y^8, & W_{12,4,2} &= 1 + 3y^4 + 7y^6 + 4y^8 + y^{10}, \\ W_{12,4,3} &= 1 + 7y^4 + 7y^8 + y^{12}, & W_{12,4,4} &= 1 + 2y^4 + 8y^6 + 5y^8, \\ W_{12,4,5} &= 1 + 3y^4 + 8y^6 + 3y^8 + y^{12}, & W_{12,4,6} &= 1 + 3y^4 + 8y^6 + 3y^8 + y^{12}, \\ W_{12,4,7} &= 1 + 4y^4 + 6y^6 + 3y^8 + 2y^{10}, & W_{12,4,8} &= 1 + 6y^4 + 4y^6 + y^8 + 4y^{10}, \\ W_{12,4,9} &= 1 + 7y^4 + 7y^8 + y^{12}, & W_{12,4,10} &= 1 + 6y^4 + 9y^8, \end{aligned}$$

and the orders of their automorphism groups are 192, 24, 168, 64, 48, 48, 12, 24, 24, and 72, respectively.

There are six SO $[12, 5, 4]$ codes, one of which is the trivial extension of the $[11, 5, 4]$ code above, plus

$$A_{12,5,1} = \begin{bmatrix} 1110000 \\ 1101000 \\ 1011000 \\ 0111000 \\ 0000111 \end{bmatrix}, A_{12,5,2} = \begin{bmatrix} 1110000 \\ 1101000 \\ 1011000 \\ 0111110 \\ 0111101 \end{bmatrix}, A_{12,5,3} = \begin{bmatrix} 1110000 \\ 1101000 \\ 1100100 \\ 1100010 \\ 1100001 \end{bmatrix},$$

$$A_{12,5,4} = \begin{bmatrix} 1110000 \\ 1101000 \\ 1100100 \\ 1011110 \\ 1011101 \end{bmatrix}, A_{12,5,5} = \begin{bmatrix} 1110000 \\ 1101000 \\ 1011110 \\ 1011101 \\ 1011011 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{12,5,1} &= 1 + 15y^4 + 15y^8 + y^{12}, & W_{12,5,2} &= 1 + 8y^4 + 14y^6 + 7y^8 + 2y^{10}, \\ W_{12,5,3} &= 1 + 15y^4 + 15y^6 + y^{12}, & W_{12,5,4} &= 1 + 7y^4 + 16y^6 + 7y^8 + y^{12}, \\ W_{12,5,5} &= 1 + 6y^4 + 16y^6 + 9y^8. \end{aligned}$$

The orders of their automorphism groups are 32256, 1344, 46080, 1536 and 1152, respectively. The code generated by $A_{12,5,1}$ is the direct sum of the [8, 4, 4] SD code and the [4, 1, 4] repetition code. The code generated by $A_{12,5,3}$ is the juxtaposition of two copies of the [6, 5, 2] single-parity check (SPC) code. The only other optimal SO code is the [12, 6, 4] self-dual code [20].

6 Length 13

The highest minimum weight among the self-orthogonal [13,4] codes is 4 and there are six inequivalent [13,4,4] SO codes without zero coordinates generated by

$$A_{13,4,1} = \begin{bmatrix} 011100000 \\ 110100000 \\ 111011110 \\ 000001101 \end{bmatrix}, \quad A_{13,4,2} = \begin{bmatrix} 011111000 \\ 110001000 \\ 111110110 \\ 000000111 \end{bmatrix},$$

$$A_{13,4,3} = \begin{bmatrix} 111000000 \\ 011100000 \\ 101111000 \\ 000011111 \end{bmatrix}, \quad A_{13,4,4} = \begin{bmatrix} 111000000 \\ 011111000 \\ 110100110 \\ 101010101 \end{bmatrix},$$

$$A_{13,4,5} = \begin{bmatrix} 000111000 \\ 111000000 \\ 110000100 \\ 011000111 \end{bmatrix}, \quad A_{13,4,6} = \begin{bmatrix} 111000000 \\ 101100000 \\ 110100000 \\ 000011111 \end{bmatrix}.$$

They have weight enumerators

$$W_{13,4,1} = 1 + 4y^4 + 11y^8, \qquad W_{13,4,2} = 1 + 2y^4 + 6y^6 + 5y^8 + 2y^{10},$$
$$W_{13,4,3} = 1 + 3y^4 + 5y^6 + 4y^8 + 3y^{10}, \qquad W_{13,4,4} = 1 + y^4 + 7y^6 + 6y^8 + y^{10},$$
$$W_{13,4,5} = 1 + 4y^4 + 4y^6 + 3y^8 + 4y^{10}, \qquad W_{13,4,6} = 1 + 7y^4 + y^6 + 7y^{10}.$$

The orders of their automorphism groups are 1152, 192, 1152, 192, 3456 and 120960, respectively.

There are 11 [13,5,4] SO codes but only five of them do not have zero coordinates. These codes have generator matrices

$$A_{13,5,1} = \begin{bmatrix} 11100000 \\ 11011111 \\ 10111111 \\ 01110000 \\ 01101000 \end{bmatrix}, \quad A_{13,5,2} = \begin{bmatrix} 11100000 \\ 11011111 \\ 10111111 \\ 01110000 \\ 01101110 \end{bmatrix},$$

$$A_{13,5,3} = \begin{bmatrix} 11111000 \\ 11000111 \\ 10111111 \\ 01111100 \\ 01100010 \end{bmatrix}, \quad A_{13,5,4} = \begin{bmatrix} 11100000 \\ 11010000 \\ 11001110 \\ 00001101 \\ 00001011 \end{bmatrix},$$

$$A_{13,5,5} = \begin{bmatrix} 11100000 \\ 00011100 \\ 11000010 \\ 00011001 \\ 00010101 \end{bmatrix}.$$

The corresponding weight enumerators are

$$\begin{aligned} W_{13,5,1} &= 1 + 10y^4 + 21y^8, \\ W_{13,5,2} &= 1 + 6y^4 + 12y^6 + 9y^8 + 4y^{10}, \\ W_{13,5,3} &= 1 + 4y^4 + 14y^6 + 11y^8 + 2y^{10}, \\ W_{13,5,4} &= 1 + 6y^4 + 12y^6 + 9y^8 + 4y^{10}, \\ W_{13,5,5} &= 1 + 10y^4 + 21y^8, \end{aligned}$$

and their automorphism groups have orders 1920, 192, 96, 144, and 1008, respectively.

There are two SO $[13,6,4]$ codes, one of which is the trivial extension of the $[12,6,4]$ SD code, and the other is the code $\overline{D}_{13}$ from [20] with weight enumerator $W_{13,6} = 1 + 10y^4 + 28y^6 + 21y^8 + 4y^{10}$, and automorphism group order 4032.

7 Length 14

There exist two SO [14,4,6] codes. The first one is constructed by adding the all-one vector to the [14,3,8] code. For the second code

$$A_{14,4} = \begin{bmatrix} 1101110000 \\ 1110001100 \\ 0011101010 \\ 1011110011 \end{bmatrix},$$

and its weight enumerator is $1 + 6y^6 + 7y^8 + 2y^{10}$.

The highest possible minimum weight of the SO codes of length 14 and dimension 5 is 4, and exactly 22 inequivalent SO [14,5,4] codes without zero coordinates exist. There exist twelve [14,6,4] SO codes. Two of these are trivial extensions of the [13,6,4] SO codes, one code is a juxtaposition of the [7,6,2] even code with itself, one is a direct sum of two copies of the [7,3,4] code, and one is a direct sum of the [6,2,4] and the [8,4,4] SO codes. The other seven have generator matrices

$$A_{14,6,1} = \begin{bmatrix} 11100000 \\ 11011111 \\ 10111111 \\ 01110000 \\ 01101000 \\ 01100100 \end{bmatrix}, A_{14,6,2} = \begin{bmatrix} 11100000 \\ 11011111 \\ 10111111 \\ 01110000 \\ 01101000 \\ 01100111 \end{bmatrix}, A_{14,6,3} = \begin{bmatrix} 11100000 \\ 11011100 \\ 10111100 \\ 01110000 \\ 01101011 \\ 00001110 \end{bmatrix},$$

$$A_{14,6,4} = \begin{bmatrix} 11100000 \\ 11011100 \\ 11011010 \\ 10100111 \\ 10110000 \\ 00001110 \end{bmatrix}, A_{14,6,5} = \begin{bmatrix} 11100000 \\ 11011100 \\ 11011010 \\ 10100111 \\ 10110000 \\ 11011001 \end{bmatrix}, A_{14,6,6} = \begin{bmatrix} 11111000 \\ 11110100 \\ 11110010 \\ 11000001 \\ 10100001 \\ 01100001 \end{bmatrix},$$

$$A_{14,6,7} = \begin{bmatrix} 11111000 \\ 11110100 \\ 11110010 \\ 11000001 \\ 10100001 \\ 10010001 \end{bmatrix},$$

and weight enumerators

$$\begin{aligned}
W_{14,6,1} &= 1 + 15y^4 + 47y^8 + y^{12}, \\
W_{14,6,2} &= 1 + 10y^4 + 21y^6 + 21y^8 + 10y^{10} + y^{14}, \\
W_{14,6,3} &= 1 + 7y^4 + 24y^6 + 23y^8 + 8y^{10} + y^{12}, \\
W_{14,6,4} &= 1 + 6y^4 + 25y^6 + 25y^8 + 6y^{10} + y^{14}, \\
W_{14,6,5} &= 1 + 6y^4 + 24y^6 + 25y^8 + 8y^{10}, \\
W_{14,6,6} &= 1 + 10y^4 + 21y^6 + 21y^8 + 10y^{10} + y^{14}, \\
W_{14,6,7} &= 1 + 9y^4 + 24y^6 + 19y^8 + 8y^{10} + 3y^{12}.
\end{aligned}$$

There exists a unique self-dual [14,7,4] code [20].

8 Length 15

The [15,4,8] simplex code is the only code with these parameters [5]. A unique self-orthogonal [15,5,6] code exists. It has generator matrix

$$A_{15,5} = \begin{bmatrix} 1111111000 \\ 1111000100 \\ 1110111111 \\ 1001110010 \\ 0101101001 \end{bmatrix}$$

and weight enumerator $1 + 10y^6 + 15y^8 + 6y^{10}$.

There exist 25 SO [15,6,4] codes. Twelve are trivial extensions of SO codes with a smaller length and dimension 6, one is a direct sum of the [11,5,4] and [4,1,4] SO codes, and another is a direct sum of the [8,3,4] and [7,3,4] SO codes. The remaining eleven codes have generator matrices

$$A_{15,6,1} = \begin{bmatrix} 111000000 \\ 110111110 \\ 101111110 \\ 011100000 \\ 011010000 \\ 000001101 \end{bmatrix}, \quad A_{15,6,2} = \begin{bmatrix} 111000000 \\ 110111000 \\ 101111000 \\ 011100000 \\ 011010110 \\ 000000111 \end{bmatrix},$$

$$A_{15,6,3} = \begin{bmatrix} 111000000 \\ 110111000 \\ 110110100 \\ 101001110 \\ 101100000 \\ 101001101 \end{bmatrix}, \quad A_{15,6,4} = \begin{bmatrix} 111000000 \\ 110111000 \\ 110110100 \\ 101001110 \\ 101100000 \\ 011100011 \end{bmatrix},$$

$$A_{15,6,5} = \begin{bmatrix} 111110000 \\ 111101000 \\ 111100100 \\ 110000010 \\ 101000010 \\ 000111101 \end{bmatrix}, \quad A_{15,6,6} = \begin{bmatrix} 111110000 \\ 111101000 \\ 111100100 \\ 110000010 \\ 101000010 \\ 111100001 \end{bmatrix},$$

$$A_{15,6,7} = \begin{bmatrix} 111000000 \\ 110100000 \\ 000011100 \\ 000011010 \\ 000010110 \\ 101101111 \end{bmatrix}, \quad A_{15,6,8} = \begin{bmatrix} 111000000 \\ 110100000 \\ 000011100 \\ 101111000 \\ 011100000 \\ 101110111 \end{bmatrix},$$

$$A_{15,6,9} = \begin{bmatrix} 111000000 \\ 110111000 \\ 101111000 \\ 000110100 \\ 000101100 \\ 000011111 \end{bmatrix}, \quad A_{15,6,10} = \begin{bmatrix} 111000000 \\ 110111110 \\ 101111110 \\ 000110001 \\ 000101001 \\ 000100101 \end{bmatrix},$$

$$A_{15,6,11} = \begin{bmatrix} 111110000 \\ 111101110 \\ 100011110 \\ 011001101 \\ 011001011 \\ 010011000 \end{bmatrix}.$$

Their weight enumerators are

$$\begin{aligned}
W_{15,6,1} &= 1 + 11y^4 + 47y^8 + 5y^{12},\\
W_{15,6,2} &= 1 + 7y^4 + 16y^6 + 23y^8 + 16y^{10} + y^{12},\\
W_{15,6,3} &= 1 + 5y^4 + 20y^6 + 23y^8 + 12y^{10} + 3y^{12},\\
W_{15,6,4} &= 1 + 4y^4 + 20y^6 + 25y^8 + 12y^{10} + 2y^{12},\\
W_{15,6,5} &= 1 + 6y^4 + 16y^6 + 25y^8 + 16y^{10},\\
W_{15,6,6} &= 1 + 9y^4 + 16y^6 + 19y^8 + 16y^{10} + 3y^{12},\\
W_{15,6,7} &= 1 + 10y^4 + 49y^8 + 4y^{12},\\
W_{15,6,8} &= 1 + 8y^4 + 16y^6 + 21y^8 + 16y^{10} + 2y^{12},\\
W_{15,6,9} &= 1 + 6y^4 + 20y^6 + 21y^8 + 12y^{10} + 4y^{12},\\
W_{15,6,10} &= 1 + 9y^4 + 51y^8 + 3y^{12},\\
W_{15,6,11} &= 1 + 3y^4 + 20y^6 + 27y^8 + 12y^{10} + y^{12}.
\end{aligned}$$

There are ten inequivalent self-orthogonal codes of length 15 and dimension 7 but only four of them have minimum weight 4. These are the codes D_{14} with an added zero coordinate, $A_8 \oplus \overline{A}_7$, $\overline{E}_{15}$, and $\overline{F}_{15}$ (see [20]).

9 Length 16

There are three SO $[16,4,8]$ codes [5], one of which is the trivial extension of the $[15,4,8]$ code given above. The second is the juxtaposition of two copies of the $[8,4,4]$ code with weight enumerator $W_{16,4,1} = 1 + 14y^8 + y^{16}$. The third code is also constructed from copies of the $[8,4,4]$ code, but in this case it is the juxtaposition of the $[8,4,4]$ codes with generator matrices

$$G_{8,4} = \begin{bmatrix} I & I+J \end{bmatrix} \quad \text{and} \quad G'_{8,4} = \begin{bmatrix} 1111 & 0000 \\ S_4 & S_4 \end{bmatrix}$$

where J is the all-one matrix. It has weight enumerator $W_{16,4,2} = 1 + 13y^8 + 2y^{12}$.

There is one SO $[16,5,8]$ code and this is the Reed-Muller code RM(1,4) [14]. There is one SO $[16,6,6]$ code which can be obtained by adding the all-one vector to the SO [15,5,6] code and its weight enumerator is $W_{16,6} = 1 + 16y^6 + 30y^8 + 16y^{10} + y^{16}$.

There are twenty SO [16,7,4] codes without zero coordinates. Eleven of them are obtained by adding the all-one vector to the [12,6,4] self-dual code and the [15,6,4] codes $C_{15,6,1}$, $C_{15,6,2}$, $C_{15,6,4}$, $C_{15,6,6}$, $C_{15,6,7}$, $C_{15,6,8}$,

$C_{15,6,10}$, $C_{15,6,11}$. The other nine codes are given by

$$A_{16,7,1} = \begin{bmatrix} 111000000 \\ 110111000 \\ 101111000 \\ 011100000 \\ 011010000 \\ 011001110 \\ 000000111 \end{bmatrix}, \quad A_{16,7,2} = \begin{bmatrix} 111000000 \\ 110100000 \\ 110010000 \\ 000001110 \\ 000001101 \\ 000001011 \\ 101110111 \end{bmatrix}, \quad A_{16,7,3} = \begin{bmatrix} 111110000 \\ 110001110 \\ 101111110 \\ 011001110 \\ 010101000 \\ 010011000 \\ 000000111 \end{bmatrix},$$

$$A_{16,7,4} = \begin{bmatrix} 111110000 \\ 110001110 \\ 101111110 \\ 011001110 \\ 010101000 \\ 000000111 \\ 000111101 \end{bmatrix}, \quad A_{16,7,5} = \begin{bmatrix} 111110000 \\ 110001110 \\ 101111110 \\ 011001110 \\ 010101000 \\ 010011000 \\ 000111101 \end{bmatrix}, \quad A_{16,7,6} = \begin{bmatrix} 111000000 \\ 110111110 \\ 101111110 \\ 011100000 \\ 011011100 \\ 000011001 \\ 000010101 \end{bmatrix},$$

$$A_{16,7,7} = \begin{bmatrix} 111110000 \\ 111101000 \\ 111011000 \\ 110000100 \\ 110000010 \\ 000111000 \\ 101000111 \end{bmatrix}, \quad A_{16,7,8} = \begin{bmatrix} 111000000 \\ 110111110 \\ 101111110 \\ 011100000 \\ 011010000 \\ 000001101 \\ 000001011 \end{bmatrix}, \quad A_{16,7,9} = \begin{bmatrix} 111110000 \\ 111101000 \\ 111011000 \\ 110000100 \\ 110000010 \\ 000111000 \\ 110000001 \end{bmatrix}.$$

Their weight enumerators are

$$\begin{aligned}
W_{16,7,1} &= 1 + 11y^4 + 26y^6 + 47y^8 + 36y^{10} + 5z^{12} + 2y^{14}, \\
W_{16,7,2} &= 1 + 13y^4 + 99y^8 + 15y^{12}, \\
W_{16,7,3} &= 1 + 7y^4 + 34y^6 + 47y^8 + 28y^{10} + 9y^{12} + 2y^{14}, \\
W_{16,7,4} &= 1 + 5y^4 + 32y^6 + 51y^8 + 32y^{10} + 7y^{12}, \\
W_{16,7,5} &= 1 + 6y^4 + 33y^6 + 49y^8 + 30y^{10} + 8y^{12} + y^{14}, \\
W_{16,7,6} &= 1 + 9y^4 + 24y^6 + 51y^8 + 40y^{10} + 3y^{12}, \\
W_{16,7,7} &= 1 + 10y^4 + 25y^6 + 49y^8 + 38y^{10} + 4y^{12} + y^{14}, \\
W_{16,7,8} &= 1 + 13y^4 + 99y^8 + 15y^{12}, \\
W_{16,7,9} &= 1 + 13y^4 + 28y^6 + 43y^8 + 32y^{10} + 7y^{12} + 4y^{14}.
\end{aligned}$$

Three inequivalent self-dual [16,8,4] codes exist [20]. One of them is doubly-even and the other two are singly-even.

$C_{15,6,10}$, $C_{15,6,11}$. The other nine codes are given by

$$A_{16,7,1} = \begin{bmatrix} 111000000 \\ 110111000 \\ 101111000 \\ 011100000 \\ 011010000 \\ 011001110 \\ 000000111 \end{bmatrix}, \quad A_{16,7,2} = \begin{bmatrix} 111000000 \\ 110100000 \\ 110010000 \\ 000001110 \\ 000001101 \\ 000001011 \\ 101110111 \end{bmatrix}, \quad A_{16,7,3} = \begin{bmatrix} 111110000 \\ 110001110 \\ 101111110 \\ 011001110 \\ 010101000 \\ 010011000 \\ 000000111 \end{bmatrix},$$

$$A_{16,7,4} = \begin{bmatrix} 111110000 \\ 110001110 \\ 101111110 \\ 011001110 \\ 010101000 \\ 000000111 \\ 000111101 \end{bmatrix}, \quad A_{16,7,5} = \begin{bmatrix} 111110000 \\ 110001110 \\ 101111110 \\ 011001110 \\ 010101000 \\ 010011000 \\ 000111101 \end{bmatrix}, \quad A_{16,7,6} = \begin{bmatrix} 111000000 \\ 110111110 \\ 101111110 \\ 011100000 \\ 011011100 \\ 000011001 \\ 000010101 \end{bmatrix},$$

$$A_{16,7,7} = \begin{bmatrix} 111110000 \\ 111101000 \\ 111011000 \\ 110000100 \\ 110000010 \\ 000111000 \\ 101000111 \end{bmatrix}, \quad A_{16,7,8} = \begin{bmatrix} 111000000 \\ 110111110 \\ 101111110 \\ 011100000 \\ 011010000 \\ 000001101 \\ 000001011 \end{bmatrix}, \quad A_{16,7,9} = \begin{bmatrix} 111110000 \\ 111101000 \\ 111011000 \\ 110000100 \\ 110000010 \\ 000111000 \\ 110000001 \end{bmatrix}.$$

Their weight enumerators are

$$\begin{aligned}
W_{16,7,1} &= 1 + 11y^4 + 26y^6 + 47y^8 + 36y^{10} + 5z^{12} + 2y^{14}, \\
W_{16,7,2} &= 1 + 13y^4 + 99y^8 + 15y^{12}, \\
W_{16,7,3} &= 1 + 7y^4 + 34y^6 + 47y^8 + 28y^{10} + 9y^{12} + 2y^{14}, \\
W_{16,7,4} &= 1 + 5y^4 + 32y^6 + 51y^8 + 32y^{10} + 7y^{12}, \\
W_{16,7,5} &= 1 + 6y^4 + 33y^6 + 49y^8 + 30y^{10} + 8y^{12} + y^{14}, \\
W_{16,7,6} &= 1 + 9y^4 + 24y^6 + 51y^8 + 40y^{10} + 3y^{12}, \\
W_{16,7,7} &= 1 + 10y^4 + 25y^6 + 49y^8 + 38y^{10} + 4y^{12} + y^{14}, \\
W_{16,7,8} &= 1 + 13y^4 + 99y^8 + 15y^{12}, \\
W_{16,7,9} &= 1 + 13y^4 + 28y^6 + 43y^8 + 32y^{10} + 7y^{12} + 4y^{14}.
\end{aligned}$$

Three inequivalent self-dual [16,8,4] codes exist [20]. One of them is doubly-even and the other two are singly-even.

10 Length 17

There are five SO $[17,4,8]$ codes [5], three of which are trivial extensions of the $[16,4,8]$ codes given above. The remaining two codes are given by

$$A_{17,4,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1110100100011 \end{bmatrix}, \quad A_{17,4,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101111 \end{bmatrix},$$

with weight enumerators

$$W_{17,4,1} = 1 + 11y^8 + 4y^{12} \text{ and } W_{17,4,2} = 1 + 7y^8 + 8y^{10},$$

respectively.

There are five linear $[17,5,8]$ codes, but only two of them are self-orthogonal. These are the codes denoted by $a1$ and c in the online electronic table [14].

Exactly three inequivalent SO [17,6,6] codes exist. One of them is the trivial extension of the unique [16,6,6] code, the others are

$$A_{17,6,1} = \begin{bmatrix} 11111110000 \\ 11000001110 \\ 10111001000 \\ 01110100100 \\ 11100010001 \\ 00000110111 \end{bmatrix}, \quad A_{17,6,2} = \begin{bmatrix} 11111110000 \\ 11000001110 \\ 10111001000 \\ 01110100100 \\ 11100010001 \\ 00010011101 \end{bmatrix},$$

with weight enumerators

$$W_{17,6,1} = 1 + 12y^6 + 25y^8 + 20y^{10} + 6y^{12}$$
$$W_{17,6,2} = 1 + 13y^6 + 25y^8 + 18y^{10} + 6y^{12} + y^{14}.$$

The highest minimum weight for the SO [17,7] codes is 4 and there are 37 SO [17,7,4] codes without zero coordinates.

There are seven inequivalent [17,8,4] SO codes, three of which are trivial extensions of the three self-dual [16,8,4] codes. The other four codes do not have zero coordinates and they are denoted in [20] by $\overline{I}_{17}^{(1)}$, $\overline{I}_{17}^{(2)}$, $\overline{I}_{17}^{(3)}$, and $\overline{H}_{17}$.

11 Length 18

There are 15 SO $[18,4,8]$ codes [5], five of which are trivial extensions of the $[17,4,8]$ codes given above, and one code is obtained by adding the all-one vector to the trivial extension of the [14,3,8] code. The remaining nine codes are given by

$$A_{18,4,1} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11110000000111 \end{bmatrix}, \quad A_{18,4,2} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101010000111 \end{bmatrix},$$

$$A_{18,4,3} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100111 \end{bmatrix}, \quad A_{18,4,4} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11101001001100 \\ 11011001000011 \end{bmatrix},$$

$$A_{18,4,5} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11101001001100 \\ 11010100100011 \end{bmatrix}, \quad A_{18,4,6} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11101001001100 \\ 10011101101011 \end{bmatrix},$$

$$A_{18,4,7} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11110000001110 \\ 11001101101101 \end{bmatrix}, \quad A_{18,4,8} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001100001110 \\ 11000001101101 \end{bmatrix},$$

$$A_{18,4,9} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001100001110 \\ 10101011101101 \end{bmatrix},$$

with weight enumerators

$$W_{18,4,1} = 1 + 10y^8 + 4y^{12} + y^{16},$$
$$W_{18,4,2} = 1 + 9y^8 + 6y^{12},$$
$$W_{18,4,3} = 1 + 7y^8 + 6y^{10} + 2y^{14},$$
$$W_{18,4,4} = 1 + 10y^8 + 6y^{10} + 4y^{12} + y^{16},$$
$$W_{18,4,5} = 1 + 9y^8 + 6y^{12},$$
$$W_{18,4,6} = 1 + 6y^8 + 7y^{10} + y^{12} + y^{16},$$
$$W_{18,4,7} = 1 + 6y^8 + 8y^{10} + y^{16},$$
$$W_{18,4,8} = 1 + 9y^8 + 6y^{12},$$
$$W_{18,4,9} = 1 + 5y^8 + 8y^{10} + 2y^{12},$$

respectively. There are seven SO [18, 5, 8] codes [5], two of which are trivial extensions of the [17, 5, 8] codes above. The other five codes are given by

$$A_{18,5,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1110100100011 \end{bmatrix}, A_{18,5,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011111 \end{bmatrix},$$

$$A_{18,5,3} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 0011110110010 \\ 1010101110001 \end{bmatrix}, A_{18,5,4} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 0011110110010 \\ 1010101101111 \end{bmatrix},$$

$$A_{18,5,5} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101110010 \\ 1010110101001 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{18,5,1} &= 1 + 21y^8 + 10y^{12}, & W_{18,5,2} &= 1 + 15y^8 + 15y^{10} + y^{18}, \\ W_{18,5,3} &= 1 + 22y^8 + 8y^{12} + y^{16}, & W_{18,5,4} &= 1 + 14y^8 + 16y^{10} + y^{16}, \\ W_{18,5,5} &= 1 + 21y^8 + 10y^{12}, \end{aligned}$$

respectively. The second code is obtained by adding the all-one vector to the trivial extension of the [15,4,8] simplex code.

There are two SO $[18,6,8]$ codes [10] with weight enumerators

$$W_{18,6,1} = 1 + 46y^8 + 16y^{12} + y^{16} \text{ and } W_{18,6,2} = 1 + 45y^8 + 18y^{12},$$

respectively. Both can be obtained from the extended Golay code.

There are three inequivalent SO [18,7,6] codes, one of which is obtained by adding the all-one vector to the second [18,6,8] code. The other two codes are given by

$$A_{18,7,1} = \begin{bmatrix} 11111000000 \\ 11000111000 \\ 10100100110 \\ 11110110100 \\ 11101101010 \\ 01111111110 \\ 00000011111 \end{bmatrix}, \quad A_{18,7,2} = \begin{bmatrix} 11111000000 \\ 11110111000 \\ 10001111110 \\ 11101100101 \\ 11011010011 \\ 10010110010 \\ 00101100110 \end{bmatrix},$$

and have weight enumerators

$$\begin{aligned} W_{18,7,1} &= 1 + 20y^6 + 46y^8 + 40y^{10} + 16y^{12} + 4y^{14} + y^{16}, \\ W_{18,7,2} &= 1 + 19y^6 + 45y^8 + 42y^{10} + 18y^{12} + 3y^{14}, \end{aligned}$$

respectively.

12 Length 19

There are 17 SO $[19,5,8]$ codes [5], seven of which are trivial extensions of the $[18,5,8]$ codes given above. The remaining ten codes are given by

$$A_{19,5,1} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 11110000000111 \end{bmatrix}, \quad A_{19,5,2} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100111 \end{bmatrix},$$

$$A_{19,5,3} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 11101001000011 \end{bmatrix}, \ A_{19,5,4} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 11000001011011 \end{bmatrix},$$

$$A_{19,5,5} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 01101011011011 \end{bmatrix}, \ A_{19,5,6} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11101001000110 \\ 11010101000101 \end{bmatrix},$$

$$A_{19,5,7} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11101001000110 \\ 10110011000101 \end{bmatrix}, \ A_{19,5,8} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11101001000110 \\ 10011011011101 \end{bmatrix},$$

$$A_{19,5,9} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011110 \\ 01011011011101 \end{bmatrix}, \ A_{19,5,10} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001100001110 \\ 11000001101101 \\ 10101011011011 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned}
W_{19,5,1} &= 1 + 18y^8 + 12y^{12} + y^{16}, \\
W_{19,5,2} &= 1 + 15y^8 + 10y^{10} + 6y^{14}, \\
W_{19,5,3} &= 1 + 18y^8 + 12y^{12} + y^{16}, \\
W_{19,5,4} &= 1 + 17y^8 + 14y^{12}, \\
W_{19,5,5} &= 1 + 13y^8 + 12y^{10} + 2y^{12} + 4y^{14}, \\
W_{19,5,6} &= 1 + 18y^8 + 12y^{12} + y^{16}, \\
W_{19,5,7} &= 1 + 17y^8 + 14y^{12}, \\
W_{19,5,8} &= 1 + 11y^8 + 14y^{10} + 4y^{12} + 2y^{14}, \\
W_{19,5,9} &= 1 + 10y^8 + 16y^{10} + 4y^{12} + y^{16}, \\
W_{19,5,10} &= 1 + 9y^8 + 16y^{10} + 6y^{12},
\end{aligned}$$

respectively.

There are four SO $[19, 6, 8]$ codes [5], two of which are trivial extensions of the $[18, 6, 8]$ codes given above. The remaining two codes are given by

$$A_{19,6,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 0110101110010 \\ 1001101110001 \end{bmatrix}, \quad A_{19,6,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 0110101110010 \\ 1011001100011 \end{bmatrix},$$

with weight enumerators

$$W_{19,6,1} = 1 + 38y^8 + 24y^{12} + y^{16} \quad \text{and} \quad W_{19,6,2} = 1 + 37y^8 + 26y^{12},$$

respectively. Note that both codes are doubly-even. There is only one SO $[19, 7, 8]$ code [10], and it is a subcode of the punctured Golay code. It has weight enumerator

$$W_{19,7} \quad = \quad 1 + 78y^8 + 48y^{12} + y^{16}.$$

A unique self-orthogonal [19,8,6] code exists. It has generator matrix

$$A_{19,8} = \begin{bmatrix} 11111000000 \\ 11000111000 \\ 10100111110 \\ 11110110111 \\ 11101110001 \\ 01111100110 \\ 00000011111 \\ 00101110100 \end{bmatrix}$$

and weight enumerator $1 + 28y^6 + 78y^8 + 88y^{10} + 48y^{12} + 12y^{14} + y^{16}$.

There are fourteen inequivalent [19,9,4] codes. Two of these are obtained as trivial extensions of the two self-dual [18,9,4] codes. The other twelve codes do not have zero coordinates (for details see [20]).

13 Length 20

There are 27 SO $[20,6,8]$ codes [5], four of which are trivial extensions of the $[19,6,8]$ codes given above. The remaining 23 codes are given by

$$A_{20,6,1} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 10010110111000 \\ 11110000000111 \end{bmatrix}, \quad A_{20,6,2} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 10010110111000 \\ 01101011100111 \end{bmatrix},$$

$$A_{20,6,3} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 11101001000011 \end{bmatrix}, \quad A_{20,6,4} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 11111101011011 \end{bmatrix},$$

$$A_{20,6,5} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10010110111011 \end{bmatrix}, \quad A_{20,6,6} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10101010000111 \end{bmatrix},$$

$$A_{20,6,7} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 11000111010111 \end{bmatrix}, \quad A_{20,6,8} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 11101001000110 \\ 11010101000101 \end{bmatrix},$$

$$A_{20,6,9} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 11101001000110 \\ 10010110111101 \end{bmatrix}, \quad A_{20,6,10} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 00111101100100 \\ 10101011100010 \\ 01011011100001 \end{bmatrix},$$

$$A_{20,6,11} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 00111101100100 \\ 10101011100010 \\ 10101101010001 \end{bmatrix}, \quad A_{20,6,12} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 00111101100100 \\ 10101011100010 \\ 11000001011101 \end{bmatrix},$$

$$A_{20,6,13} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 00111101100100 \\ 10101011100010 \\ 01101011011101 \end{bmatrix}, \quad A_{20,6,14} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 00111101100100 \\ 10101011011110 \\ 01011011011101 \end{bmatrix},$$

$$A_{20,6,15} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 11001011010001 \end{bmatrix}, \quad A_{20,6,16} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 00111011010001 \end{bmatrix},$$

$$A_{20,6,17} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 10111001001001 \end{bmatrix}, \quad A_{20,6,18} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 10101100001101 \end{bmatrix},$$

$$A_{20,6,19} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 10001001001111 \end{bmatrix}, \quad A_{20,6,20} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 01111001001111 \end{bmatrix},$$

$$A_{20,6,21} = \begin{bmatrix} 1111111000000 \\ 1111000111000 0 \\ 11001101101000 \\ 11101001000110 \\ 11010101000101 \\ 10011011011011 \end{bmatrix}, \quad A_{20,6,22} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011110 \\ 01011011011101 \\ 01100111011011 \end{bmatrix},$$

$$A_{20,6,23} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001100001110 \\ 11000001101101 \\ 10101011011011 \\ 01010110110111 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned}
W_{20,6,1} &= 1 + 46y^8 + 16y^{12} + y^{16}, \\
W_{20,6,2} &= 1 + 30y^8 + 16y^{10} + 16y^{14} + y^{16}, \\
W_{20,6,3} &= 1 + 32y^8 + 28y^{12} + 3y^{16}, \\
W_{20,6,4} &= 1 + 31y^8 + 31y^{12} + y^{20}, \\
W_{20,6,5} &= 1 + 25y^8 + 21y^{10} + 6y^{12} + 10y^{14} + y^{18}, \\
W_{20,6,6} &= 1 + 31y^8 + 30y^{12} + 2y^{16}, \\
W_{20,6,7} &= 1 + 25y^8 + 20y^{10} + 6y^{12} + 12y^{14}, \\
W_{20,6,8} &= 1 + 30y^8 + 32y^{12} + y^{16}, \\
W_{20,6,9} &= 1 + 21y^8 + 25y^{10} + 10y^{12} + 6y^{14} + y^{18}, \\
W_{20,6,10} &= 1 + 34y^8 + 24y^{12} + 5y^{16}, \\
W_{20,6,11} &= 1 + 32y^8 + 28y^{12} + 3y^{16}, \\
W_{20,6,12} &= 1 + 30y^8 + 32y^{12} + y^{16}, \\
W_{20,6,13} &= 1 + 22y^8 + 24y^{10} + 8y^{12} + 8y^{14} + y^{16}, \\
W_{20,6,14} &= 1 + 18y^8 + 32y^{10} + 8y^{12} + 5y^{16}, \\
W_{20,6,15} &= 1 + 32y^8 + 28y^{12} + 3y^{16}, \\
W_{20,6,16} &= 1 + 31y^8 + 31y^{12} + y^{20}, \\
W_{20,6,17} &= 1 + 31y^8 + 30y^{12} + 2y^{16}, \\
W_{20,6,18} &= 1 + 30y^8 + 32y^{12} + y^{16}, \\
W_{20,6,19} &= 1 + 29y^8 + 34y^{12}, \\
W_{20,6,20} &= 1 + 21y^8 + 24y^{10} + 10y^{12} + 8y^{14}, \\
W_{20,6,21} &= 1 + 18y^8 + 28y^{10} + 12y^{12} + 4y^{14} + y^{16}, \\
W_{20,6,22} &= 1 + 16y^8 + 32y^{10} + 12y^{12} + 3y^{16}, \\
W_{20,6,23} &= 1 + 15y^8 + 32y^{10} + 15y^{12} + y^{20},
\end{aligned}$$

respectively.

There are five SO $[20, 7, 8]$ codes [5], one of which is the trivial extension of the $[19, 7, 8]$ code, and another can be constructed by adding the all-one vector to the nineteenth [20,6,8] code given above. The remaining three codes are given by

$$A_{20,7,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1001101110001 \end{bmatrix}, \; A_{20,7,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 0110101110010 \\ 1100011101010 \\ 1010011110001 \end{bmatrix},$$

$$A_{20,7,3} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 0110101110010 \\ 1001101110001 \\ 1100101000111 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{20,7,1} &= 1 + 66y^8 + 56y^{12} + 5y^{16}, \\ W_{20,7,2} &= 1 + 64y^8 + 60y^{10} + 3y^{16}, \\ W_{20,7,3} &= 1 + 62y^8 + 64y^{12} + y^{16}, \end{aligned}$$

respectively.

There is one SO $[20, 8, 8]$ code [10]. It has weight enumerator

$$W_{20,8} = 1 + 130y^8 + 120y^{12} + 5y^{16}.$$

A unique self-orthogonal [20,9,6] code exists. It has generator matrix

$$A_{20,9} = \begin{bmatrix} 11111000000 \\ 11000111000 \\ 10100100110 \\ 11110011001 \\ 11101000111 \\ 01001010110 \\ 00110101111 \\ 00110011100 \\ 00011101010 \end{bmatrix}$$

and weight enumerator $W_{20,9} = 1+40y^6+130y^8+176y^{10}+120y^{12}+40y^{14}+5y^{16}$.

There are seven inequivalent [20,10,4] self-dual codes [20].

14 Length 21

There is one SO $[21,4,10]$ code with

$$A_{21,4} = \begin{bmatrix} 11111111100000000 \\ 11110000011111000 \\ 10001110011100110 \\ 11101101011010101 \end{bmatrix}$$

and weight enumerator $W_{21,4} = 1 + 7y^{10} + 7y^{12} + y^{14}$.

There are 20 SO $[21,7,8]$ codes [5], five of which are trivial extensions of the $[20,7,8]$ codes given above. The remaining 15 codes are given by

$$A_{21,7,1} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 10010110111000 \\ 01101011100100 \\ 10101010000111 \end{bmatrix}, \; A_{21,7,2} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 10010110111000 \\ 01101011100100 \\ 11000111010111 \end{bmatrix},$$

$$A_{21,7,3} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 11000111010100 \\ 11010101000011 \end{bmatrix}, \ A_{21,7,4} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 11000111010100 \\ 10010110111011 \end{bmatrix},$$

$$A_{21,7,5} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10011011100010 \\ 10100111100001 \end{bmatrix}, \ A_{21,7,6} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11101001000110 \\ 11010101000101 \\ 10011011011011 \\ 01000011111111 \end{bmatrix},$$

$$A_{21,7,7} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10011011100010 \\ 10101010000111 \end{bmatrix}, \ A_{21,7,8} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10011011100010 \\ 11000111010111 \end{bmatrix},$$

$$A_{21,7,9} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10110011000110 \\ 10001111000101 \end{bmatrix}, \ A_{21,7,10} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10110011000110 \\ 10010110111011 \end{bmatrix},$$

$$A_{21,7,11} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10110011000110 \\ 11101001000011 \end{bmatrix}, \; A_{21,7,12} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 11101001000110 \\ 11010101000101 \\ 10110011000011 \end{bmatrix},$$

$$A_{21,7,13} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 10111001001001 \\ 01010110000111 \end{bmatrix}, \; A_{21,7,14} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011100100 \\ 10101101010010 \\ 11001011010001 \\ 01111001001111 \end{bmatrix},$$

$$A_{21,7,15} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 10101011011000 \\ 01101011100100 \\ 10011011100010 \\ 10110011000101 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned}
W_{21,7,1} &= 1 + 54y^8 + 64y^{12} + 9y^{16},\\
W_{21,7,2} &= 1 + 46y^8 + 32y^{10} + 16y^{12} + 32y^{14} + y^{16},\\
W_{21,7,3} &= 1 + 54y^8 + 64y^{12} + 9y^{16},\\
W_{21,7,4} &= 1 + 45y^8 + 33^{10} + 18y^{12} + 30y^{14} + y^{18},\\
W_{21,7,5} &= 1 + 54y^8 + 64y^{12} + 9y^{16},\\
W_{21,7,6} &= 1 + 28y^8 + 56y^{10} + 28y^{12} + 8^{14} + 7y^{16},\\
W_{21,7,7} &= 1 + 50y^8 + 72y^{12} + 5y^{16},\\
W_{21,7,8} &= 1 + 38y^8 + +40y^{10} + 24y^{12} + 24y^{14} + 5y^{16},\\
W_{21,7,9} &= 1 + 52y^8 + 68^{10} + 7y^{16},\\
W_{21,7,10} &= 1 + 37y^8 + 41y^{10} + 26y^{12} + 22y^{14} + y^{18},\\
W_{21,7,11} &= 1 + 50y^8 + 72y^{12} + 5y^{16},\\
W_{21,7,12} &= 1 + 48y^8 + 76y^{12} + 3y^{16},\\
W_{21,7,13} &= 1 + 48y^8 + 76y^{12} + 3y^{16},\\
W_{21,7,14} &= 1 + 32y^8 + 48y^{10} + 28y^{12} + 16y^{14} + 3y^{16},\\
W_{21,7,15} &= 1 + 52y^8 + 68^{10} + 7y^{16},
\end{aligned}$$

respectively. There are three SO $[21, 8, 8]$ codes [5], one of which is the trivial extension of the $[20, 8, 8]$ code given above. The other two codes are given by

$$A_{21,8,1} = \begin{bmatrix} 1111111000000\\ 1111000111000\\ 1100110110100\\ 1010101101100\\ 1001011011100\\ 0110101110010\\ 1100011101010\\ 1010011110001 \end{bmatrix}, \quad A_{21,8,2} = \begin{bmatrix} 1111111000000\\ 1111000111000\\ 1100110110100\\ 1010101101100\\ 0110101110010\\ 1100011101010\\ 1010011110001\\ 0110110101001 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned}
W_{21,8,1} &= 1 + 106y^8 + 136y^{12} + 13y^{16},\\
W_{21,8,2} &= 1 + 102y^8 + 144y^{12} + 9y^{16},
\end{aligned}$$

respectively. There is one SO $[21, 9, 8]$ code [10]. It has weight enumerator

$$W_{21,9} = 1 + 210y^8 + 280y^{12} + 21y^{16}.$$

A unique self-orthogonal [21,10,6] code exists. It can be obtained as a subcode of the "shorter Golay code" g_{22} [22, Table X]. Its weight enumerator is $1 + 56y^6 + 210y^8 + 336y^{10} + 280y^{12} + 120y^{14} + 21y^{16}$.

15 Length 22

There are seven [22,4,10] codes, one of which is the trivial extension of the [21, 4, 10] code given above. The other six codes are given by

$$A_{22,4,1} = \begin{bmatrix} 101100111111110000 \\ 010111110001110000 \\ 001011110010001110 \\ 110000000111110101 \end{bmatrix}, \quad A_{22,4,2} = \begin{bmatrix} 111001111111100000 \\ 111110001000011100 \\ 100011001011111110 \\ 100110001111100001 \end{bmatrix},$$

$$A_{22,4,3} = \begin{bmatrix} 111111110111111100 \\ 011110001111100000 \\ 011101000100011110 \\ 100000000111111101 \end{bmatrix}, \quad A_{22,4,4} = \begin{bmatrix} 111110001111100100 \\ 111101110101011000 \\ 100011100001111010 \\ 111001110111100111 \end{bmatrix},$$

$$A_{22,4,5} = \begin{bmatrix} 111001111111000100 \\ 011110001000111100 \\ 111001100100011010 \\ 100110001111000101 \end{bmatrix}, \quad A_{22,4,6} = \begin{bmatrix} 011111110111100000 \\ 111111001100010000 \\ 100111000011101100 \\ 111000000111100011 \end{bmatrix},$$

and have weight enumerators

$$W_{22,4,1} = 1 + 6y^{10} + 6y^{12} + 2y^{14} + y^{16}, \quad W_{22,4,2} = 1 + 6y^{10} + 7y^{12} + y^{14} + y^{18},$$
$$W_{22,4,3} = 1 + 6y^{10} + 7y^{12} + y^{14} + y^{18}, \quad W_{22,4,4} = 1 + 5y^{10} + 7y^{12} + 3y^{14},$$
$$W_{22,4,5} = 1 + 6y^{10} + 6y^{12} + 2y^{14} + y^{16}, \quad W_{22,4,6} = 1 + 7y^{10} + 7y^{12} + y^{22}.$$

There are three SO [22, 9, 8] codes [5], one of which is the trivial extension of the [21, 9, 8] code given above. The other two codes are given by

$$A_{22,9,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 1010011110001 \end{bmatrix}, \quad A_{22,9,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 1010011110001 \\ 0110110101001 \end{bmatrix},$$

15 Length 22

There are seven [22,4,10] codes, one of which is the trivial extension of the $[21, 4, 10]$ code given above. The other six codes are given by

$$A_{22,4,1} = \begin{bmatrix} 101100111111110000 \\ 010111110001110000 \\ 001011110010001110 \\ 110000000111110101 \end{bmatrix}, \quad A_{22,4,2} = \begin{bmatrix} 111001111111100000 \\ 111110001000011100 \\ 100011001011111110 \\ 100110001111100001 \end{bmatrix},$$

$$A_{22,4,3} = \begin{bmatrix} 111111101111111100 \\ 011110001111100000 \\ 011101000100011110 \\ 100000000111111101 \end{bmatrix}, \quad A_{22,4,4} = \begin{bmatrix} 111110001111100100 \\ 111101110101011000 \\ 100011100001111010 \\ 111001110111100111 \end{bmatrix},$$

$$A_{22,4,5} = \begin{bmatrix} 111001111111000100 \\ 011110001000111100 \\ 111001100100011010 \\ 100110001111000101 \end{bmatrix}, \quad A_{22,4,6} = \begin{bmatrix} 011111110111100000 \\ 111111001100010000 \\ 100111000011101100 \\ 111000000111100011 \end{bmatrix},$$

and have weight enumerators

$$W_{22,4,1} = 1 + 6y^{10} + 6y^{12} + 2y^{14} + y^{16}, \quad W_{22,4,2} = 1 + 6y^{10} + 7y^{12} + y^{14} + y^{18},$$
$$W_{22,4,3} = 1 + 6y^{10} + 7y^{12} + y^{14} + y^{18}, \quad W_{22,4,4} = 1 + 5y^{10} + 7y^{12} + 3y^{14},$$
$$W_{22,4,5} = 1 + 6y^{10} + 6y^{12} + 2y^{14} + y^{16}, \quad W_{22,4,6} = 1 + 7y^{10} + 7y^{12} + y^{22}.$$

There are three SO $[22, 9, 8]$ codes [5], one of which is the trivial extension of the $[21, 9, 8]$ code given above. The other two codes are given by

$$A_{22,9,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 1010011110001 \end{bmatrix}, \quad A_{22,9,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 1010011110001 \\ 0110110101001 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{22,9,1} &= 1 + 170y^8 + 296y^{12} + 45y^{16}, \\ W_{22,9,2} &= 1 + 162y^8 + 312y^{12} + 37y^{16}, \end{aligned}$$

respectively. There is one SO $[22, 10, 8]$ code [10]. It has weight enumerator

$$W_{22,10} = 1 + 330y^8 + 616y^{12} + 77y^{16},$$

and is a doubly-even subcode of the punctured Golay code.

The "shorter Golay code" g_{22} is the only self-dual [22,11,6] code [22, Table X].

16 Length 23

There is one SO $[23, 4, 12]$ code given by

$$A_{23,4} = \begin{bmatrix} 1111111111100000000 \\ 1111110000011111000 \\ 1110001110011100110 \\ 1001101101011010101 \end{bmatrix},$$

with weight enumerator $W_{23,4} = 1 + 14y^{12} + y^{16}$. This is the juxtaposition of the $[15, 4, 8]$ code and the $[8, 4, 4]$ code.

Exactly two SO [23,5,10] codes exist. They are given by

$$A_{23,5,1} = \begin{bmatrix} 111111111000000000 \\ 111100000111110000 \\ 110011110111101110 \\ 001011001110011100 \\ 000000101101111011 \end{bmatrix}, \quad A_{23,5,2} = \begin{bmatrix} 111111111000000000 \\ 111100000111110000 \\ 110011110111101110 \\ 001011001110011100 \\ 100010101101010011 \end{bmatrix},$$

and have weight enumerators

$$\begin{aligned} W_{23,5,1} &= 1 + 11y^{10} + 14y^{12} + 4y^{14} + y^{16} + y^{18}, \\ W_{23,5,2} &= 1 + 10y^{10} + 14y^{12} + 6y^{14} + y^{16}. \end{aligned}$$

There are three SO $[23, 10, 8]$ codes [5], one which is the trivial extension of the [22, 10, 8] code given above. The other two codes are given by

$$A_{23,10,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 0001111100110 \\ 0101011110001 \end{bmatrix}, \quad A_{23,10,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 1010011110001 \\ 0110110101001 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{23,10,1} &= 1 + 266y^8 + 616y^{12} + 141y^{16}, \\ W_{23,10,2} &= 1 + 250y^8 + 648y^{12} + 125y^{16}, \end{aligned}$$

respectively. There is one [23, 11, 8] code [10]. It has weight enumerator

$$W_{23,11} = 1 + 506y^8 + 1288y^{12} + 253y^{16},$$

and is a doubly-even subcode of the Golay code.

17 Length 24

All the optimal SO codes of length 24 are subcodes of the extended binary Golay code. There are five SO [24, 4, 12] codes, one of which is the trivial extension of the [23, 4, 12] code given above. The remaining codes are given by

$$A_{24,4,1} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11100011100111001100 \\ 00011111100111000011 \end{bmatrix}, \quad A_{24,4,2} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11100011100111001100 \\ 10011011010111000011 \end{bmatrix},$$

$$A_{24,4,3,1} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11100011100111001100 \\ 11010011010110100011 \end{bmatrix}, \quad A_{24,4,3,2} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11110011000110001110 \\ 11001100110001101101 \end{bmatrix},$$

of the [22, 10, 8] code given above. The other two codes are given by

$$A_{23,10,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 0001111100110 \\ 0101011110001 \end{bmatrix}, \quad A_{23,10,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 1010011110001 \\ 0110110101001 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{23,10,1} &= 1 + 266y^8 + 616y^{12} + 141y^{16}, \\ W_{23,10,2} &= 1 + 250y^8 + 648y^{12} + 125y^{16}, \end{aligned}$$

respectively. There is one [23, 11, 8] code [10]. It has weight enumerator

$$W_{23,11} = 1 + 506y^8 + 1288y^{12} + 253y^{16},$$

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17 Length 24

All the optimal SO codes of length 24 are subcodes of the extended binary Golay code. There are five SO [24, 4, 12] codes, one of which is the trivial extension of the [23, 4, 12] code given above. The remaining codes are given by

$$A_{24,4,1} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11100011100111001100 \\ 00011111100111000011 \end{bmatrix}, \quad A_{24,4,2} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11100011100111001100 \\ 10011011010111000011 \end{bmatrix},$$

$$A_{24,4,3,1} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11100011100111001100 \\ 11010011010110100011 \end{bmatrix}, \quad A_{24,4,3,2} = \begin{bmatrix} 11111111111000000000 \\ 11111100000111110000 \\ 11110011000110001110 \\ 11001100110001101101 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{24,4,1} &= 1 + 14y^{12} + y^{24}, \\ W_{24,4,2} &= 1 + 13y^{12} + y^{16} + y^{24}, \\ W_{24,4,3} &= 1 + 12y^{12} + 3y^{16}. \end{aligned}$$

The first code is the juxtaposition of three copies of the [8, 4, 4] code. The second is the juxtaposition of two copies of the [8, 4, 4] code and $G'_{8,4}$. The third is the juxtaposition of two copies of the optimal [12, 4, 6] code. There is one SO [24, 5, 12] code [13] given by

$$A_{24,5} = \begin{bmatrix} 1111111111100000000 \\ 1111110000011111000 \\ 1110001110011100110 \\ 1001101101011010101 \\ 0101011010110101011 \end{bmatrix},$$

with weight enumerator $W_{24,5} = 1 + 28y^{12} + 3y^{16}$.

There is one SO [24, 6, 10] code given by

$$A_{24,6} = \begin{bmatrix} 111111111000000000 \\ 111100000111110000 \\ 100011100111001100 \\ 111011010110101010 \\ 110110101101101001 \\ 001101100100010111 \end{bmatrix},$$

with weight enumerator

$$W_{24,6} = 1 + 18y^{10} + 28y^{12} + 12y^{14} + 3y^{16} + 2y^{18}.$$

There are four SO [24, 11, 8] codes, one of which is the trivial extension of the [24, 11, 8] code given above. The other codes are given by

$$A_{24,11,1} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 0001111100110 \\ 1011100010110 \\ 0101011110001 \end{bmatrix}, \; A_{24,11,2} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 0001111100110 \\ 0101011110001 \\ 1001110101001 \end{bmatrix},$$

$$A_{24,11,3} = \begin{bmatrix} 1111111000000 \\ 1111000111000 \\ 1100110110100 \\ 1010101101100 \\ 1001011011100 \\ 0110101110010 \\ 1100011101010 \\ 0101110011010 \\ 1010011110001 \\ 0110110101001 \\ 0011101011001 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{24,11,1} &= 1 + 407y^8 + 1232y^{12} + 407y^{16} + y^{24}, \\ W_{24,11,2} &= 1 + 378y^8 + 1288y^{12} + 381y^{16}, \\ W_{24,11,3} &= 1 + 375y^8 + 1296y^{12} + 375y^{16} + y^{24}, \end{aligned}$$

respectively. The first eight rows of all three matrices generate the unique [20, 8, 8] code given previously. The remaining three rows generate the optimal SO [15, 3, 8] code in the first two cases, and the optimal SO [14, 3, 8] code in the third case. All three [24, 11, 12] codes can be obtained by expurgating codewords from the [24, 12, 8] Golay code. In particular, the second code is obtained by deleting the all-one codeword.

There is one SD [24, 12, 8] code, namely the extended Golay code g_{24}.

18 Length 25

There are eleven SO [25,4,12] codes, five of which are trivial extensions of the SO [24,4,12] codes. The remaining six codes are given by

$$A_{25,4,1} = \begin{bmatrix} 111111111111011100000 \\ 111111110000100011000 \\ 111110000000111100110 \\ 011111000111100000011 \end{bmatrix}, \quad A_{25,4,2} = \begin{bmatrix} 111111010110000110000 \\ 111111101001111001000 \\ 111000011001111110110 \\ 011100110111111000011 \end{bmatrix},$$

$$A_{25,4,3} = \begin{bmatrix} 111111110111011011000 \\ 111111001000100111000 \\ 111100111000111000100 \\ 100000001111011011011 \end{bmatrix}, \quad A_{25,4,4} = \begin{bmatrix} 111111010110011000000 \\ 111111101001100100000 \\ 111000011001111011110 \\ 011100110111100011011 \end{bmatrix},$$

$$A_{25,4,5} = \begin{bmatrix} 111111010100010011000 \\ 111111101011101111000 \\ 011110011011110000100 \\ 101110110111100011111 \end{bmatrix}, \quad A_{25,4,6} = \begin{bmatrix} 111111001111100000000 \\ 111111110000011100000 \\ 100000011001111011110 \\ 011100101110011000011 \end{bmatrix},$$

and have weight enumerators

$$\begin{aligned} W_{25,4,1} &= 1 + 11y^{12} + 3y^{16} + y^{20}, & W_{25,4,2} &= 1 + 7y^{12} + 7y^{14} + y^{18}, \\ W_{25,4,3} &= 1 + 10y^{12} + 5y^{16}, & W_{25,4,4} &= 1 + 6y^{12} + 8y^{14} + y^{16}, \\ W_{25,4,5} &= 1 + 10y^{12} + 5y^{16}, & W_{25,4,6} &= 1 + 10y^{12} + 5y^{16}, \end{aligned}$$

respectively. There are three SO [25,5,12] codes, one of which is the trivial extension of the [24,5,12] code. The other two codes have generator matrices

$$A_{25,5,1} = \begin{bmatrix} 11110011111000110000 \\ 11111100000111110000 \\ 01110011100111001100 \\ 01011000110001111110 \\ 10111111111111111111 \end{bmatrix}, \quad A_{25,5,2} = \begin{bmatrix} 01110111111000110000 \\ 11111000100111110000 \\ 01110111000111001100 \\ 11010010010001111110 \\ 00111110011111110111 \end{bmatrix},$$

and weight enumerators $W_{25,5,1} = 1 + 25y^{12} + 5y^{16} + y^{20}$ and $W_{25,5,2} = 1 + 24y^{12} + 7y^{16}$, respectively.

18 Length 25

There are eleven SO [25,4,12] codes, five of which are trivial extensions of the SO [24,4,12] codes. The remaining six codes are given by

$$A_{25,4,1} = \begin{bmatrix} 11111111111011100000 \\ 11111111000010001100 0 \\ 11111000000011110011 0 \\ 01111100011110000001 1 \end{bmatrix}, \quad A_{25,4,2} = \begin{bmatrix} 11111101011000011000 0 \\ 11111110100111100100 0 \\ 11100001100111111011 0 \\ 01110011011111100001 1 \end{bmatrix},$$

$$A_{25,4,3} = \begin{bmatrix} 11111111011101101100 0 \\ 11111100100010011100 0 \\ 11110011100011100010 0 \\ 10000000111101101101 1 \end{bmatrix}, \quad A_{25,4,4} = \begin{bmatrix} 11111101011001100000 0 \\ 11111110100110010000 0 \\ 11100001100111101111 0 \\ 01110011011110001101 1 \end{bmatrix},$$

$$A_{25,4,5} = \begin{bmatrix} 11111101010001001100 0 \\ 11111110101110111100 0 \\ 01111001101111000010 0 \\ 10111011011110001111 1 \end{bmatrix}, \quad A_{25,4,6} = \begin{bmatrix} 11111100111110000000 0 \\ 11111111000001110000 0 \\ 10000001100111101111 0 \\ 01110010111001100001 1 \end{bmatrix},$$

and have weight enumerators

$$\begin{aligned} W_{25,4,1} &= 1 + 11y^{12} + 3y^{16} + y^{20}, & W_{25,4,2} &= 1 + 7y^{12} + 7y^{14} + y^{18}, \\ W_{25,4,3} &= 1 + 10y^{12} + 5y^{16}, & W_{25,4,4} &= 1 + 6y^{12} + 8y^{14} + y^{16}, \\ W_{25,4,5} &= 1 + 10y^{12} + 5y^{16}, & W_{25,4,6} &= 1 + 10y^{12} + 5y^{16}, \end{aligned}$$

respectively. There are three SO [25,5,12] codes, one of which is the trivial extension of the [24,5,12] code. The other two codes have generator matrices

$$A_{25,5,1} = \begin{bmatrix} 11110011111000110000 \\ 11111100000111110000 \\ 01110011100111001100 \\ 01011000110001111110 \\ 10111111111111111111 \end{bmatrix}, \quad A_{25,5,2} = \begin{bmatrix} 01110111111000110000 \\ 11111000100111110000 \\ 01110111000111001100 \\ 11010010010001111110 \\ 00111110011111110111 \end{bmatrix},$$

and weight enumerators $W_{25,5,1} = 1 + 25y^{12} + 5y^{16} + y^{20}$ and $W_{25,5,2} = 1 + 24y^{12} + 7y^{16}$, respectively.

Ten inequivalent SO [25,11,8] codes exist. Four of them are the trivial extensions of the SO [24,11,8] codes, and the other six are given by

$$A_{25,11,1} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11101111111111 \\ 11011001001111 \\ 10011100111110 \\ 10110010101111 \\ 10010111010001 \\ 01010111101110 \\ 01110100011111 \\ 01011010110001 \\ 00111011011000 \end{bmatrix}, \; A_{25,11,2} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11101111111111 \\ 11011001001111 \\ 10011100111110 \\ 10110010101111 \\ 10010111010001 \\ 01010111101110 \\ 01110100011111 \\ 01011010110001 \\ 00111101100001 \end{bmatrix},$$

$$A_{25,11,3} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11001011010100 \\ 10110100101100 \\ 10100111011000 \\ 10011010111000 \\ 01100111100100 \\ 01000100101111 \\ 01101010000111 \\ 00000111010111 \end{bmatrix}, \; A_{25,11,4} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11001011010100 \\ 10110100101100 \\ 10100111011000 \\ 10011010111000 \\ 01100111100100 \\ 01110010011111 \\ 01011100110111 \\ 00101001111111 \end{bmatrix},$$

$$A_{25,11,5} = \begin{bmatrix} 11111110000000 \\ 11110001111111 \\ 11101001110000 \\ 11011111111100 \\ 11010100001110 \\ 11100011001100 \\ 10001110110001 \\ 00010100111101 \\ 00100101001111 \\ 00010011101110 \\ 01110101101000 \end{bmatrix}, \; A_{25,11,6} = \begin{bmatrix} 11111110000000 \\ 11110001110000 \\ 11001101101000 \\ 11100010010110 \\ 10001111000110 \\ 10110011000101 \\ 10011100001101 \\ 01111000010101 \\ 01110010100011 \\ 01101010111000 \\ 00011110101010 \end{bmatrix},$$

and have weight enumerators

$$\begin{aligned}
W_{25,11,1} &= 1+266y^8+240y^{10}+616y^{12}+672y^{14}+141y^{16}+112y^{18},\\
W_{25,11,2} &= 1+330y^8+176y^{10}+616y^{12}+672y^{14}+77y^{16}+176y^{18},\\
W_{25,11,3} &= 1+298y^8+1208y^{12}+525y^{16}+16y^{20},\\
W_{25,11,4} &= 1+250y^8+256y^{10}+648y^{12}+640y^{14}+125y^{16}+128y^{18},\\
W_{25,11,5} &= 1+274y^8+1248y^{12}+517y^{16}+8y^{20},\\
W_{25,11,6} &= 1+270y^8+1260y^{12}+505y^{16}+12y^{20},
\end{aligned}$$

respectively. A unique self-orthogonal [25,12,8] code exists and it is the trivial extension of the Golay code g_{24}.

19 Optimal SO Codes of Lengths Greater Than 25

A summary of the results given in this and the previous sections is presented in Tables 1 and 2. Note that there are no self-orthogonal codes with $k > n/2$. For each length $n \leq 40$ and dimension $3 \leq k \leq 10$, two entries are given: the first is the maximum minimum distance for an SO $[n, k]$ code and the second is the number of corresponding codes without zero coordinates. Note that in some cases all optimal SO $[n, k, d]$ codes have zero coordinates. For the parameters considered here, these are the codes with parameters [9,4,4], [25,12,8], and [33,6,16]. Entries that could not be computed with a reasonable amount of CPU time are empty.

Generator matrices of the inequivalent linear [26,6,12], [27,7,12], [30,9,12], [31,10,12], and [36,8,16] are presented in the electronic table [14]. All these codes are doubly-even and therefore they are optimal SO codes.

The simplex $[2^k-1, k, 2^{k-1}]$ code is the unique code with its parameters. Taking s copies of the simplex code, we obtain the unique linear $[(2^k - 1)s, k, 2^{k-1}s]$ code. For $k \geq 3$ these codes are self-orthogonal. In our table, these are the codes with parameters [7,3,4], [14,3,8], [21,3,12], [28,3,16], [35,3,20], [15,4,8], [30,4,16], and [31,5,16]. Adding the all-one vector to the simplex code we obtain the Reed-Muller code $RM(1, k-1)$. The code $RM(1,4)$ has parameters [16,5,8], and $RM(1,5)$ is the code with parameters [32,6,16].

Table 1: Classification of binary self-orthogonal codes, $3 \leq k \leq 6$

$n \backslash k$	3		4		5		6	
7	4*	1						
8	4*	1	4*	1				
9	4*	1	4	0				
10	4	3	4*	3	2	2		
11	4	2	4	2	4*	1		
12	6*	1	4	10	4*	5	4*	1
13	6	1	4	6	4	5	4*	1
14	8*	1	6	2	4	22	4	10
15	8*	1	8*	1	6	1	4	13
16	8*	3	8*	2	8*	1	6*	1
17	8	3	8*	2	8*	1	6	2
18	8	8	8*	10	8*	5	8*	2
19	10*	1	8	12	8*	10	8*	2
20	10	2	8	50	8	50	8*	23
21	12*	1	10*	1	8	101	8*	57
22	12*	1	10	6	8	417	8	416
23	12*	3	12*	1	10	2	8	1729
24	12	6	12*	4	12*	1	10*	1
25	12	8	12*	6	12*	2	10	27
26	14*	1	12	27	12*	13	12*	2
27	14	2	12	48	12	60	12*	24
28	16*	1	14*	1	12	345	12*	383
29	16*	1	14	3	12	1507	12	4468
30	16*	3	16*	1	14	3	12	
31	16	6	16*	2	16*	1	12	
32	16	12	16*	9	16*	3	16*	1
33	18*	1	16*	16	16*	8	16	0
34	18	2	16	71	16*	63	16*	15
35	20*	1	16	152	16*	380	16*	362
36	20*	1	18*	2	16	2876	16*	20392
37	20*	3	18	8	16	17705	16	
38	20	6	20*	1	18*	3	16	
39	20	12	20*	4	18	50	16	
40	22*	1	20*	17	20*	3	18*	41

* Optimal as linear codes

? The existence of a [35, 10, 13] code is unesolved

Table 2: Classification of binary self-orthogonal codes, $7 \leq k \leq 10$

$n\backslash k$	7		8		9		10	
14	4*	1						
15	4	3						
16	4	20	4	3				
17	4	37	4	4				
18	6	3	4	45	4	2		
19	8*	1	6	1	4	12		
20	8*	4	8*	1	6	1	4	7
21	8*	15	8*	2	8*	1	6	1
22	8*	117	8*	16	8*	2	8*	1
23	8	848	8*	104	8*	12	8*	2
24	8	9839	8*	1824	8*	124	8*	16
25	8	96560	8	37625	8*	1891	8*	60
26	10	26	8		8		8*	1689
27	12*	1	10*	1	8		8	
28	12*	61	10	43579	8		8	
29	12*	5694	12*	73	10		8	
30	12*		12*		12*	9	10	
31	12		12*		12*		12*	2
32	12		12		12*		12*	
33	12		12		12*		12*	
34	14	5399	12		12		12*	
35	16*	4	14		12		12^7	
36	16*	7484	16*	2	14*		12	
37	16*		16*		14		14*	
38	16*		16*		16*	47	14*	
39	16		16*		16*		14	
40	16		16*		16*		16*	

* Optimal as linear codes

A unique SO [27,8,10] code exists and it is given by

$$A_{27,8} = \begin{bmatrix} 1111111110000000000 \\ 1111110001111100000 \\ 1110001001110011000 \\ 1101101101111011111 \\ 1011101011101110110 \\ 1011001111010101100 \\ 0110011001001100011 \\ 0101100011011010001 \end{bmatrix}.$$

Its weight enumerator is $W_{27,8} = 1+36y^{10}+82y^{12}+72y^{14}+39y^{16}+20y^{18}+6y^{20}$.

A unique SO [28,4,14] code exists and it is given by

$$A_{28,4} = \begin{bmatrix} 111111001111100110110000 \\ 111111110000011001110000 \\ 111100001100011110001110 \\ 011000111101111000111011 \end{bmatrix},$$

Its weight enumerator is $W_{28,4} = 1 + 8y^{14} + 7y^{16}$.

There are four SO [29,4,14] codes. One is the trivial extension of the [28,4,14] code. The other three codes are given by

$$A_{29,4,1} = \begin{bmatrix} 1111111101011100011100000 \\ 1111111110100011100011000 \\ 0111100001100011111100100 \\ 1011100011011111100000111 \end{bmatrix},$$

$$A_{29,4,2} = \begin{bmatrix} 1111111111011100011000000 \\ 1111111100100011100100000 \\ 0111100011100011111011100 \\ 1011110011011111100001111 \end{bmatrix},$$

$$A_{29,4,3} = \begin{bmatrix} 1111110000111111100011000 \\ 1111111111000000011111000 \\ 0011001100111100011011100 \\ 1100001111011111100011111 \end{bmatrix},$$

and have weight enumerators

$$\begin{aligned} W_{29,4,1} &= 1 + 7y^{14} + 6y^{16} + y^{18} + y^{20}, \\ W_{29,4,2} &= 1 + 7y^{14} + 7y^{16} + y^{22}, \\ W_{29,4,3} &= 1 + 6y^{14} + 7y^{16} + 2y^{18}. \end{aligned}$$

There are three SO [30,5,14] codes. One is obtained by adding the all-one vector to the SO [28,4,14] code and has weight enumerator $1 + 15y^{14} + 15y^{16} + y^{30}$. The other two codes are given by

$$A_{30,5,1} = \begin{bmatrix} 1111101111010011011100000 \\ 0111110001101100111100000 \\ 1111101110101100100011100 \\ 1001100011100011100110110 \\ 1010011101111100011001011 \end{bmatrix},$$

$$A_{30,5,2} = \begin{bmatrix} 0110011111011100011100000 \\ 1111100011100011111000000 \\ 0110011100100011100111100 \\ 1111001110011001110001110 \\ 0011111000011111111010111 \end{bmatrix},$$

and have weight enumerators $W_{30,5,1} = 1 + 14y^{14} + 15y^{16} + 2y^{22}$ and $W_{30,5,2} = 1 + 12y^{14} + 15y^{16} + 4y^{18}$, respectively.

There are three SO [31,4,16] codes. One of them is the trivial extension of the [30,4,16] code and the other two codes are given by

$$A_{31,4,1} = \begin{bmatrix} 110111011101011100011100000 \\ 111111111110100011100000000 \\ 010111000001100011111111100 \\ 100111000001101111100000111 \end{bmatrix},$$

$$A_{31,4,2} = \begin{bmatrix} 111111110011011100011000000 \\ 111111111100100011100100000 \\ 011110000011100011111011100 \\ 101111001111011111100001111 \end{bmatrix},$$

with weight enumerators $W_{31,4,1} = 1 + 14y^{16} + y^{24}$ and $W_{31,4,2} = 1 + 13y^{16} + 2y^{20}$, respectively.

There are four inequivalent SO [32,5,16] codes. One is the trivial extension of the [31,5,16] simplex code, another is the juxtaposition of two copies of the [16,5,8] code. The other two codes are generated by

$$A_{32,5,1} = \begin{bmatrix} 1111100011110111000111000000 \\ 0111111100011000111111100000 \\ 1111100011101000111000111 00 \\ 101101101001111010010011010 \\ 0011111111110011111111111111 \end{bmatrix},$$

$$A_{32,5,2} = \begin{bmatrix} 0111100111110111000111000000 \\ 1111111000111000111111000000 \\ 011110011100100011100111100 \\ 110011001110011001110001110 \\ 0011111111000011111111010111 \end{bmatrix},$$

and weight enumerators $W_{32,5,1} = 1 + 29y^{16} + 2y^{24}$ and $W_{32,5,2} = 1 + 27y^{16} + 4y^{20}$, respectively.

There are four SO [35,7,16] codes, all of which are doubly-even with weight enumerators

$$\begin{aligned} W_{35,7,1} &= W_{35,7,4} = 1 + 84y^{16} + 35y^{20} + 7y^{24} + y^{28}, \\ W_{35,7,2} &= W_{35,7,3} = 1 + 85y^{16} + 32y^{20} + 10y^{24}, \end{aligned}$$

and are given by

$$A_{35,7,1} = \begin{bmatrix} 1101111110000011101111110000000 \\ 0011111111111000110011100000 00 \\ 01111100000111100001011111000 \\ 1100101010100011101011011100 \\ 0000110001011011101110110110 \\ 1110010110011101000110010011 \\ 1101000011110000111110110001 \end{bmatrix},$$

$$A_{35,7,2} = \begin{bmatrix} 1101111110000011101111110000000 \\ 0011111111111000110011100000 00 \\ 01111100000111100001011111000 \\ 1100101010100011101011011100 \\ 1110110111100100101000100110 \\ 1101000011110000111110110001 \\ 0000100011011111101110000111 \end{bmatrix},$$

$$A_{35,7,3} = \begin{bmatrix} 1101111100001110111110000000 \\ 0011111111110001100111000000 \\ 0111110000111100001011111000 \\ 1100101010100011101011011100 \\ 1101101111100101010000100110 \\ 1110001101011011000110100011 \\ 1101000011110000111110110001 \end{bmatrix},$$

$$A_{35,7,4} = \begin{bmatrix} 1101111100001110111110000000 \\ 0011111111110001100111000000 \\ 0111110000111100001011111000 \\ 0010111010101011011100011100 \\ 1110100101101011000110110010 \\ 1101000011110000111110110001 \\ 0000110001010011110101110111 \end{bmatrix}.$$

There are two SO [36,4,18] codes, which can be obtained as a juxtaposition of the SO [28,4,14] and [8,4,4] codes. For the first code we concatenate the all-one codeword of the Hamming [8,4,4] code with a codeword of weight 16, and for the second code with a codeword of weight 14, so the resulting weight enumerators are $1 + 8y^{18} + 6y^{20} + y^{24}$ and $1 + 7y^{18} + 7y^{20} + y^{22}$, respectively.

A unique SO [38,4,20] code exists and it is a juxtaposition of the [30,4,16] and [8,4,4] codes.

There are three SO [38,5,18] codes generated by

$$A_{38,5,1} = \begin{bmatrix} 111111001111110110011001110000000 \\ 111111110000101001100111110000000 \\ 011111001111011001100110001111000 \\ 100011000011111000011110010011110 \\ 001100111100111111100001101100101 \end{bmatrix},$$

$$A_{38,5,2} = \begin{bmatrix} 011111011111110110011001110000000 \\ 111111100001011001100111110000000 \\ 011111011110101001100110001111000 \\ 111100011011000001111000010110110 \\ 110011100111000110000111101011101 \end{bmatrix},$$

$$A_{38,5,3} = \begin{bmatrix} 111111001011101110011011011000000 \\ 111111110101110001100110110100000 \\ 111111001010010001100101001011100 \\ 100011000111011000011100111100110 \\ 101100111001011111100010101111011 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{38,5,1} &= 1 + 14y^{18} + 14y^{20} + y^{22} + y^{24} + y^{30}, \\ W_{38,5,2} &= 1 + 13y^{18} + 14y^{20} + 2y^{22} + y^{24} + y^{26}, \\ W_{38,5,3} &= 1 + 12y^{18} + 14y^{20} + 4y^{22} + y^{24}. \end{aligned}$$

There are five SO [39,4,20] codes. One of them has a zero coordinate, and the other four are generated by

$$A_{39,4,1} = \begin{bmatrix} 0111111001111101011110000111000000 0 \\ 111111110111110100001111000000000 00 \\ 01111100010000011000011111111111000 \\ 00111111000000011011111111000000 1111 \end{bmatrix},$$

$$A_{39,4,2} = \begin{bmatrix} 11111110011100110111100001110000000 \\ 11111111111111001000011110000000000 \\ 01111100000000111000011111111111000 \\ 10111111000000111101111111110000011111 \end{bmatrix},$$

$$A_{39,4,3} = \begin{bmatrix} 11111111110011001111000111000000000 \\ 11111111111110011000011100011000000 0 \\ 01111100000011110000111111001111000 \\ 10111111100111100011111110000000111111 \end{bmatrix},$$

$$A_{39,4,4} = \begin{bmatrix} 11111111110001110111000111000000000 \\ 11111111111110001000111000110000000 \\ 01111000000001111000111111001111100 \\ 10111111000111111011111110000000011111 \end{bmatrix},$$

with weight enumerators $W_{39,4,1} = 1 + 14y^{20} + y^{32}$, $W_{39,4,2} = 1 + 13y^{20} + y^{24} + y^{28}$, and $W_{39,4,3} = W_{39,4,4} = 1 + 12y^{20} + 3y^{24}$, respectively.

There are three SO [40,5,20] codes generated by

$$A_{40,5,1} = \begin{bmatrix} 0011011111100000111111111111111100000000 \\ 1111111111111111000000001100110000000000 \\ 1100011100000011110000001111110111111100 \\ 00111111000100111111001100000110110110 \\ 0011000000001111110000111111111101110001 \end{bmatrix},$$

$$A_{40,5,2} = \begin{bmatrix} 0011011111100000111111111111111100000000 \\ 1111111111111111000000001100110000000000 \\ 1100101111000110011101111000000111111100 \\ 0011110010101101010110110110000111110010 \\ 1111101111011001011101110111111011101111 \end{bmatrix},$$

$$A_{40,5,3} = \begin{bmatrix} 1001111110001001111111111111110000000000 \\ 1111100111111110111011101001000110000000 \\ 0010110000001111111011000001111111111000 \\ 1011101010111110010000100110001101001110 \\ 1101110010000001110011001110000110111101 \end{bmatrix},$$

with weight enumerators

$$\begin{aligned} W_{40,5,1} &= 1 + 28y^{20} + 2y^{24} + y^{32}, \\ W_{40,5,2} &= 1 + 27y^{20} + 3y^{24} + y^{28}, \\ W_{40,5,3} &= 1 + 26y^{20} + 5y^{24}. \end{aligned}$$

As the problem of whether a linear [39,10,16] code exists has not been solved until now, we give the answer in the following lemma.

Lemma 4 *No linear* $[39, 10, 16]$ *code exists.*

Proof. Jaffe [14] proved that if a [39,10,16] code exists it is doubly-even with weight enumerator $W(y) = 1 + 367y^{16} + 414y^{20} + 240y^{24} + 2y^{28}$. Then its residual code with respect to a codeword of weight 28 is a linear $[11, 9, 2]$ even code. Exactly four inequivalent [11,9,2] even codes without zero coordinates exist but none of them can be extended to a doubly-even [39,10,16] code. □

This lemma gives that the highest possible weight of a SO [39,10] code is 14, but the number of inequivalent SO [39,10,14] codes is too large so they could not be computed with a reasonable amount of CPU time.

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