Vertex-magic Total Labeling of Generalized Petersen Graphs and Convex Polytopes

Mirka MILLER
School of Information Technology and Mathematical Sciences
University of Ballarat, Australia
e-mail: m.miller@ballarat.edu.au

Martin BAČA
Department of Applied Mathematics
Technical University, Košice, Slovak Republic
e-mail: hollbaca@ccsun.tuke.sk

James A. MacDOUGALL
Department of Mathematics
The University of Newcastle, Australia
e-mail: jim.macdougall@newcastle.edu.au

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Abstract

A *vertex-magic total labeling* on a graph with v vertices and e edges is a one-to-one map taking the vertices and edges onto the integers $1, 2, \ldots, v+e$ with the property that the sum of the label on a vertex and the labels of its incident edges is constant, independent of the choice of vertex. We give vertex-magic total labelings for several classes of regular graphs. The paper concludes with several conjectures and open problems in the area.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let

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$e = |E|$ and $v = |V|$. A general reference for graph theoretic notions is [8]. In this paper we will deal only with connected graphs, although the concepts apply equally to graphs with more than one connected component.

A *labeling* for a graph is a map that takes graph elements to numbers (usually positive or non-negative integers). In this paper the domain is the set of all vertices and edges, giving rise to *total labelings*. Other labelings use the vertex set alone ('vertex labelings') or the edge set alone ('edge labelings'). The most complete recent survey of graph labelings is [3].

Various authors have introduced labelings that generalize the idea of a magic square. These are called *magic labelings* and readers are referred to [7] for a discussion of magic labelings and a standardisation of the terminology.

In [6] we introduced the notion of a *vertex-magic total labeling*. This is an assignment of the integers from 1 to $e + v$ to the vertices and edges of G so that at each vertex the vertex label and the labels on the edges incident at that vertex add to a fixed constant. More formally, a one-to-one map λ from $E \cup V$ onto the integers $\{1, 2, \cdots, e + v\}$ is a *vertex-magic total labeling* if there is a constant k so that for every vertex x,

$$\lambda(x) + \sum \lambda(xy) = k \tag{1}$$

where the sum is over all vertices y adjacent to x.

Set $M = e + v$ and let S_v be the sum of the vertex labels and S_e the sum of the edge labels. Clearly, since the labels are the numbers $1, 2, ..., M$, we have as the sum of all labels

$$S_v + S_e = \sum_1^M i = \binom{M+1}{2}.$$

At each vertex x_i we have $\lambda(x_i) + \sum \lambda(x_i y) = k$. We sum this over all v vertices x_i. This adds each vertex label once and each edge label twice, so that

$$S_v + 2S_e = vk. \tag{2}$$

Combining these two equations gives us

$$S_e + \binom{M+1}{2} = vk. \tag{3}$$

The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the e smallest labels or, at the other extreme, the e largest labels, or anything between. Consequently we have

$$\sum_{1}^{e} i \leq S_e \leq \sum_{v+1}^{M} i. \tag{4}$$

A similar result holds for S_v. Combining (3) and (4), we get

$$\binom{M+1}{2} + \binom{e+1}{2} \leq vk \leq 2\binom{M+1}{2} - \binom{v+1}{2} \tag{5}$$

which will give the range of feasible values for k.

It is clear from the definition of vertex-magic total labeling that when k is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. In this paper we give new vertex-magic total labelings of several classes of graphs. The paper concludes with a list of conjectures and open problems.

2 Labelings of generalized Petersen graphs

Suppose g is an edge-labeling of a graph, that is, a one-to-one map from E onto the integers $\{1, 2, \ldots, e\}$. Then the *weight* $w(x)$ of a vertex $x \in V$ is defined as the sum of labels assigned to all edges incident to x.

An edge-labeling g is said to be *consecutive* if the weights of all vertices constitute a set of consecutive integers. Two edge-labelings g and g' are said to be *complementary* if there is a constant c such that $w_g(x) + w_{g'}(x) = c$ for all $x \in V$.

Let $I = \{1, 2, \ldots, n\}$ be an index set. For simplicity, we use the convention that $x_{j,n+1} = x_{j,1}$ for $j = 1, 2, \ldots, 6$.

A *generalized Petersen graph* $P(n, m)$, $1 \leq m < \frac{n}{2}$, consists of an outer n-cycle $y_1, y_2, \ldots, y_n$, a set of n spokes $y_i x_i$, $1 \leq i \leq n$, and n inner edges $x_i x_{i+m}$, $1 \leq i \leq n$, with indices taken modulo n. The standard Petersen graph is the instance $P(5, 2)$.

$P(n,m)$ is regular of degree 3 and has $v = 2n$ vertices and $e = 3n$ edges; thus $M = 5n$. Using (5) we can readily determine the feasible values of k for the generalized Petersen graphs $P(n,m)$:

$$\frac{17n}{2} + 2 \leq k \leq \frac{23n}{2} + 2.$$

It was shown in [1] that for $n \geq 4$, n even and $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n,m)$ has a consecutive edge-magic labeling defined by the bijective mapping $g_1 : E(P(n,m)) \longrightarrow \{1,2,\ldots,3n\}$:

- $g_1(y_i y_{i+1}) = \frac{5n+i+1}{2}\delta(i) + \frac{n-i+2}{2}\delta(i+1)$
- $g_1(y_i x_i) = \frac{3n-i+1}{2}\delta(i) + [(n+\frac{i}{2}-2)\rho(i,4) + (\frac{n+i}{2}-2)\rho(6,i)]\delta(i+1)$
- $g_1(x_i x_{i+m}) = \frac{3n+i+1}{2}\delta(i) + [(2n-\frac{i}{2}+3)\rho(i,4) + (\frac{5n-i}{2}+3)\rho(6,i)]\delta(i+1)$
- for $i \in I$, where

$$\delta(x) = \begin{cases} 0 & \text{if } x \equiv 0 \bmod 2 \\ 1 & \text{if } x \equiv 1 \bmod 2 \end{cases} \tag{6}$$

$$\rho(x,y) = \begin{cases} 0 & \text{if } x > y \\ 1 & \text{if } x \leq y \end{cases} \tag{7}$$

The weights of vertices under the mapping g_1 constitute the sets

$W_1 = \{w_{g_1}(y_i) : i \in I\} = \{\frac{7n}{2}+2, \frac{7n}{2}+3, \ldots, \frac{9n}{2}+1\}$ and

$W_2 = \{w_{g_1}(x_i) : i \in I\} = \{\frac{9n}{2}+2, \frac{9n}{2}+3, \ldots, \frac{11n}{2}+1\}$.

Label the edges of the generalized Petersen graph $P(n,m)$ by the consecutive labeling g_1. If g_2 is the complementary vertex labeling with values in the set

$$\{|E|+1, |E|+2, \ldots, |E|+|V|\} = \{3n+1, 3n+2, \ldots, 5n\}$$

then the labelings g_1 and g_2 combine to give a vertex-magic total labeling of $P(n,m)$. Since the largest labels are assigned to the vertices, it is easily seen that the resulting magic constant $k = \frac{17n}{2} + 2$ is the largest possible.

Thus we have the following

Theorem 1 *For $n \geq 4$, n even and $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n,m)$ has a vertex-magic total labeling with $k = \frac{17n}{2} + 2$.*

In [6] it is proved that if a regular graph G possesses a vertex-magic total labeling λ with magic constant k, then G also has a dual labeling λ' having magic constant $k' = (r+1)(M+1) - k$. Since $P(n,m)$ is regular, it has a dual labeling. Hence

Corollary 1 *For $n \geq 4$, n even and $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n,m)$ has a vertex-magic total labeling with $k = \frac{23n}{2} + 2$.*

Proof. In this case the dual labeling g' is defined by

- $g_1'(u) = |E| + |V| + 1 - g_1(u)$ for any edge $u \in E(P(n,m))$,
- $g_2'(x) = |E| + |V| + 1 - g_2(x)$ for any vertex $x \in V(P(n,m))$.

Since the magic sum for g is $k = \frac{17n}{2} + 2$, then g' is a vertex-magic total labeling with magic sum $k' = 4M + 4 - k = \frac{23n}{2} + 2$. □

The prism D_n, $n \geq 3$, is a trivalent graph which can be defined as the cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. We note that the prism D_n is the generalized Petersen graph $P(n,1)$.

Corollary 2 *If n is even, $n \geq 4$, then the prism D_n has a vertex-magic total labeling with $k = \frac{17n}{2} + 2$ and another one with $k = \frac{23n}{2} + 2$.*

3 Labelings of some families of convex polytopes

In this section we shall investigate the graphs of two families of convex polytopes. First we consider the graphs R_n consisting of $2n$ 5-sided faces, n 6-sided faces and a pair of n-sided faces, embedded in the plane and labeled as in Figure 2. Using equation (5) where $v = 6n$ and $e = 9n$, we can determine the feasible values of the magic constant k for the graph R_n:

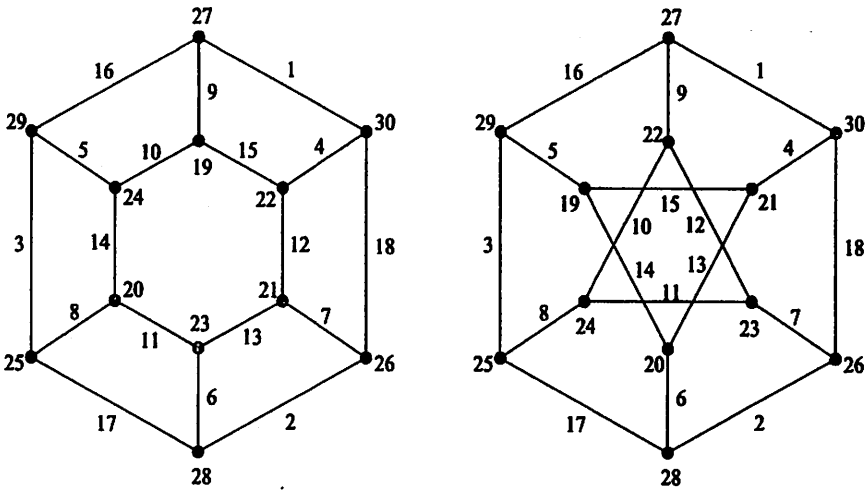

Figure 1: Labeling of $P(6,1)$ and $P(6,2)$.

$$\frac{51}{2}n+2 \leq k \leq \frac{69}{2}n+2.$$

The dual graph of R_n is a planar graph B_n which has been investigated in [2]. There it was shown that B_n has a face anti-magic labeling g, i.e., an edge labeling in which the sum around each face is a constant. To obtain a vertex-magic total labeling of R_n we make use of the labeling $h_1 : E(R_n) \longrightarrow \{1, 2, \ldots, 9n\}$ which is a modification of the labeling g from [2]. It is defined as follows (with δ and ρ as defined in (6) and (7)):

- $h_1(x_{1,i}x_{1,i+1}) = [(8n+i)\delta(i) + (n-i)\delta(i+1)]\rho(i, n-1) + n\rho(n,i)$
- $h_1(x_{1,i}x_{2,i}) = (\frac{5n}{2}+i)\delta(i) + (\frac{5n}{2}-i+1)\delta(i+1)$
- $h_1(x_{2,i}x_{3,i}) = (\frac{15n}{2}-i+1)\delta(i) + (\frac{9n}{2}-i+2)\delta(i+1)$
- $h_1(x_{3,i}x_{2,i+1}) = (\frac{13n}{2}+i)\delta(i) + (\frac{9n}{2}-i+1)\delta(i+1)$
- $h_1(x_{3,i}x_{4,i}) = (n+1)\rho(i,1) + [\frac{3n-i+3}{2}\delta(i) + (\frac{9n}{2}+i)\delta(i+1)]\rho(2,i)$
- $h_1(x_{4,i}x_{5,i}) = \frac{11n+i+1}{2}\delta(i) + (\frac{11n}{2}-i+1)\delta(i+1)$
- $h_1(x_{5,i}x_{4,i+1}) = \frac{13n-i+1}{2}\delta(i) + \frac{15n+i}{2}\delta(i+1)$

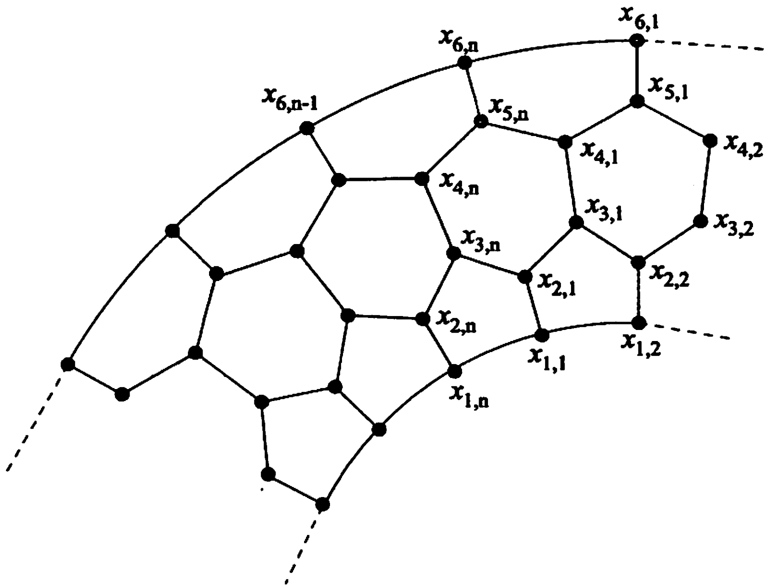

Figure 2: The convex polytope R_n.

- $h_1(x_{5,i}x_{6,i}) = (\frac{5n}{2} - i + 1)\delta(i) + (\frac{5n}{2} + i)\delta(i+1)$
- $h_1(x_{6,i}x_{6,i+1}) = (8n + i + 1)\delta(i) + (n - i + 1)\delta(i+1)$
- for $i \in I$.

The edge labeling h_1 is consecutive: the weights of vertices in turn assume the values $\frac{21n}{2} + 2, \frac{21n}{2} + 3, \ldots, \frac{33n}{2} + 1$.

If h_2 is the complementary vertex labeling with values in the set $\{|E(R_n)| + 1, |E(R_n)| + 2, \ldots, |E(R_n)| + |V(R_n)|\} = \{9n+1, 9n+2, \ldots, 15n\}$ then the labelings h_1 and h_2 combine to give a vertex-magic total labeling of R_n with the magic constant $k = \frac{51n}{2} + 2$ which is the minimum possible.

We have

Theorem 2 *For $n \geq 4$, n even, the plane graph R_n has a vertex-magic total labeling with $k = \frac{51n}{2} + 2$.*

Since R_n is regular, we have a dual labeling as before:

Corollary 3 *For $n \geq 4$, n even, the plane graph R_n has a vertex-magic total labeling with $k = \frac{69n}{2} + 2$.*

The final graphs we investigate are the *antiprisms* A_n, $n \geq 3$, a family of planar graphs that are regular of degree 4. These are Archimedean convex polytopes and, in particular, A_3 is the octahedron.

We will denote the vertex set of A_n by $V = \{x_i : i \in I\} \cup \{y_i : i \in I\}$ and the edge set by $E = \{(x_i x_{i+1}) : i \in I\} \cup \{(y_i y_{i+1}) : i \in I\} \cup \{(x_i y_i) : i \in I\} \cup \{(y_i x_{i+1}) : i \in I\}$ as in Figure 3.

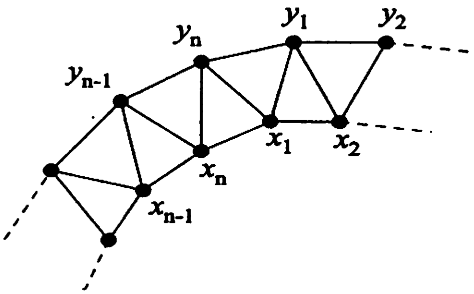

Figure 3: The antiprism A_n.

From (5) we get the range of feasible values for k:

$$\frac{26n+5}{2} \leq k \leq \frac{34n+5}{2}.$$

Theorem 3 *For $n \geq 4$, n even, the antiprism A_n has a vertex-magic total labeling with $k = 15n + 2$.*

Proof. We construct an edge labeling f_1 of A_n, $n = 2m$, $m \geq 2$, in the following way:

- $f_1(x_i x_{i+1}) = 6n\rho(i,1) + [(5n+i-1)\delta(i) + i\delta(i+1)]\rho(2,i)$
- $f_1(y_i y_{i+1}) = [(5n+i)\delta(i) + (2n+i+1)\delta(i+1)]\rho(i,n-1) + (2n+1)\rho(n,i)$

$$\begin{aligned} f_1(x_i y_i) \;=\; & (3n+1)\alpha(1,i,1) + (5n-2i+3)\alpha(2,i,m+1) \\ & + (3n+3)\alpha(m+2,i,m \\ & +2) + (5n-2i+3)\alpha(m+3,i,n-1) + (4n-1)\alpha(n,i,n) \end{aligned}$$

- $f_1(y_i x_{i+1}) = 2n - 2i + 1$

- for $i \in I$, where

$$\alpha(x,y,z) = \begin{cases} 1 & \text{if } x \le y \le z \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The edge labeling f_1 is a one-to-one map from $E(A_n)$ onto the set $\{i : i \in I\} \cup \{n+2j-1 : j = 1,2,\ldots,2n\} \cup \{5n+i : i \in I\}$. The weights of the vertices under the edge labeling f_1 constitute the set

$$W = \{w_{f_1}(x) : x \in V(A_n)\} = \{10n + 2j : j = 1,2,\ldots,2n\}.$$

If f_2 is the complementary vertex labeling with values in the set $\{n+2j : j = 1,2,\ldots,2n\}$ then the labelings f_1 and f_2 combine to give a vertex-magic total labeling of A_n with the magic constant $k = 15n+2$. □

Again by duality we have

Theorem 4 *For $n \geq 4$, n even, the antiprism A_n has a vertex-magic total labeling with $k = 15n+3$.*

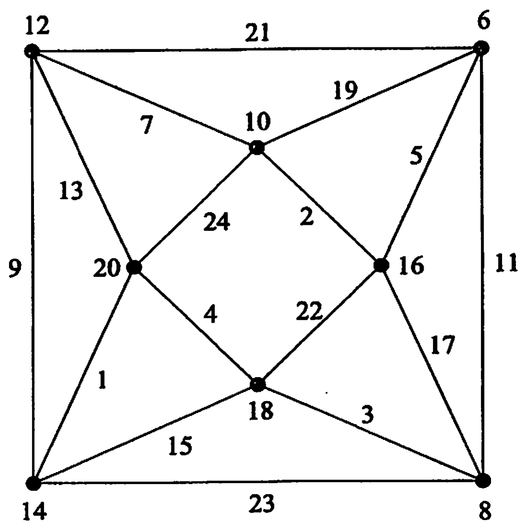

Figure 4: Labeling of the antiprism A_4.

4 Open problems

We have shown that there exist vertex-magic total labelings for the generalized Petersen graph $P(n, m)$ for $n \geq 4$, n even. We conjecture that

Conjecture 1 *There is a vertex-magic total labeling for the prism* $D_n = P(n, 1)$ *for all* $n \geq 3$.

More strongly,

Conjecture 2 *There is a vertex-magic total labeling for the generalized Petersen graph* $P(n, m)$ *for all* $n \geq 3$ *and* $2 \leq m < \frac{n}{2}$.

We have not yet found a construction that will produce a vertex-magic total labeling for the plane graph R_n for n odd. However, we suggest the following

Conjecture 3 *There is a vertex-magic total labeling for the plane graph* R_n *for all* $n \geq 3$.

Open Problem 1 *Find a vertex-magic total labeling for the antiprism* A_n *for all odd* $n \geq 3$.

The ladder L_n, $n \geq 3$, can be viewed as the cartesian product $P_2 \times P_n$ of a path on two vertices and a path on n vertices, or as a prism D_n with two edges deleted.

Open Problem 2 *Find a vertex-magic total labeling for the ladder* L_n.

added in proof: Conjectures 1 and 2 have been proved by Dan McQuillan.

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