Enumeration and Isomorphic Classification of Self-orthogonal Latin Squares

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Abstract

We enumerate the self-orthogonal Latin squares of orders 1 through 9 and discuss the nature of the isomorphism classes of each order. Furthermore, we consider the possibility of enlarging sets of self-orthogonal Latin squares to produce complete sets.

1 Introduction

In this section we will present some definitions and a brief history of certain aspects of Latin squares, complete sets of Latin squares, and the enumeration of Latin squares. The reader who is interested in a more detailed historical account may consult J.Dénes and A.D.Keedwell [5] and [6]. A Latin square of order n is an $n \times n$ matrix each of whose rows and columns is a permutation of a set of n elements. A Latin square is in *reduced form* when the symbols in the first row and first column are in natural order.

Latin squares and their enumeration were first studied by L.Euler [7] in 1779. He showed that the number of distinct, reduced squares of order 2 was 1, of order 3 was 1, of order 4 was 4, and of order 5 was 56. In 1890, M.Frolov [8] correctly calculated the number of reduced squares of order 6 to be 9408. He also incorrectly calculated the number of squares of order 7. In 1948, A.Sade [19] properly determined the number of distinct, reduced Latin squares of order 7 to be 16,942,080. In 1967, M.B.Wells [23] computed the number of distinct, reduced Latin squares of order 8 to be 535,281,401,856. S.E.Bammel and J.Rothstein [1] in 1975 calculated the number of reduced squares of order 9 to be 377,597,570,964,258,816. In 1995, B.D.McKay and E.Rogoyski [11] computed the number of reduced

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squares of order 10 to be 7,580,721,483,160,132,811,489,280. Recently, McKay and I.M.Wanless [12] calculated the number of reduced squares of order 11 to be 5,363,937,773,277,371,298,119,673,540,771,840. Let ℓ_n denote the number of distinct, reduced Latin squares of order n and let L_n denote the total number of distinct squares of order n. Since it is possible to permute the columns of a reduced square in $n!$ ways and to permute the last $n-1$ rows in $(n-1)!$ ways, $L_n = n!(n-1)!\ell_n$.

Two Latin squares of order n, $A = (a_{ij})$ and $B = (b_{ij})$, are *orthogonal* if the n^2 pairs (a_{ij}, b_{ij}) $(i,j = 1,2,\ldots,n)$ are distinct. A set $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of Latin squares of order n is *mutually orthogonal* provided A_i is orthogonal to A_j for each $i \neq j$.

General results on the construction of mutually orthogonal Latin squares (MOLS) were given by H.F.MacNeish [10] in 1922. For n a prime, he showed how to construct a set of $n-1$ mutually orthogonal Latin squares of order n. For any n, no larger set can exist, so a set of $n-1$ mutually orthogonal Latin squares of order n is called a *complete set of mutually orthogonal Latin squares.*

In 1938, R.C.Bose [2] proved that every projective plane of order n defines and is defined by a complete set of mutually orthogonal Latin squares of order n. Also he showed how to construct a set of p^r-1 mutually orthogonal Latin squares of order p^r when p is a prime and r is a positive integer. Independently and about the same time, W.L.Stevens [21] also presented a simple technique for constructing a set of p^r-1 mutually orthogonal Latin squares of order p^r. Earlier, in 1896, E.H.Moore [14] had solved the two-fold school-girl-system, SGS$[m-1,2,m]$ where $m > 2$ and $m = p^r$ where p is any prime. In his solution, Moore essentially constructed a complete set of MOLS of order p^r in the same manner as did Bose and Stevens. For more details see J.Dénes and A.D.Keedwell [6] page 379.

A Latin square which is orthogonal to its transpose is said to be a *self-orthogonal Latin square* (SOLS). The term "self-orthogonal" was introduced in 1970 by R.C.Mullin and E.Nemeth [15]; however, the problem of constructing a Latin square orthogonal to its transpose seems to have been considered first by S.K.Stein [20] in 1957. In 1973-74 R.K.Brayton, D.Coppersmith, and A.J.Hoffman [3] and [4] showed that there exists a self-orthogonal Latin square of order n for $n \neq 2,3,6$. We will restrict our attention to Latin squares of order n in which the entries are from the set $V_n = \{1,2,\ldots,n\}$. For A self-orthogonal, the main diagonal pairs from A, A^T are repeated pairs. Because of orthogonality they are distinct; consequently, the main diagonal of A must be a permutation of V_n. We will say that a self-orthogonal Latin square is *idempotent* when the entries on the main diagonal are in natural order. Let s_n denote the number of distinct, idempotent, self-orthogonal Latin squares of order n and let S_n denote the total number of self-orthogonal squares of order n. Since it is possible to permute the entries of a square in $n!$ ways, $S_n = n! s_n$.

2 Results

$n = 1$. There is only one Latin square of order 1—namely, (1). By definition (1) is self-orthogonal. Thus, $S_1 = s_1 = 1$.

$n = 2$. Since there is no Latin square of order 2 with main diagonal entries 1, 2, there is no self-orthogonal Latin square of order 2. Hence, $S_2 = s_2 = 0$.

$n = 3$. The only Latin square of order 3 with main diagonal entries 1, 2, 3 is

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Clearly, this square is not self-orthogonal; therefore, $S_3 = s_3 = 0$.

$n = 6$. In 1900, GṪarry [22] proved by exhaustive enumeration that there does not exist a pair of mutually orthogonal Latin squares of order 6. Consequently, there does not exist a self-orthogonal Latin square of order 6 and $S_6 = s_6 = 0$.

Let σ be any permutation of V_n and let A be any Latin square of order n. We will use σA to denote the isomorphic Latin square obtained by simultaneously permuting the rows, columns, and entries of A by σ. We will also say that the square B is *isomorphic* to the square A if there exists a permutation σ such that $B = \sigma A$. From this definition, it follows that (i) A is a SOLS if and only if σA is a SOLS and (ii) A is symmetric if and only if σA is symmetric. Furthermore, if $\{A, B, C, \ldots\}$ is a set of MOLS, then $\{\sigma A, \sigma B, \sigma C, \ldots\}$ is a set of MOLS.

$n = 4$. There are two idempotent SOLS of order 4,

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

and its transpose, A^T. Thus, $s_4 = 2$ and $S_4 = 48$. Both A and A^T belong to the same isomorphism class. There are $4! = 24$ permutations of V_4. The twelve even permutations map A onto itself while the twelve odd permutations map A onto A^T. Let

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

The set $\mathcal{A} = \{A, A^T, B\}$ is a complete set of mutually orthogonal Latin squares.

$n = 5$. There are twelve idempotent SOLS of order 5, so $s_5 = 12$ and $S_5 = 1440$. Furthermore, there are two isomorphism classes. One class contains the square

$$C = \begin{bmatrix} 1 & 3 & 2 & 5 & 4 \\ 4 & 2 & 5 & 1 & 3 \\ 5 & 4 & 3 & 2 & 1 \\ 3 & 5 & 1 & 4 & 2 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$$

and five distinct squares isomorphic to C. The other class contains C^T and five distinct squares isomorphic to it. There are $5! = 120$ permutations of V_5. Twenty isomorphisms of C yield C, and five disjoint sets of twenty isomorphisms each yield the other five SOLS isomorphic to C. The same results hold for C^T. Let

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \\ 2 & 4 & 1 & 5 & 3 \\ 5 & 3 & 4 & 1 & 2 \\ 4 & 5 & 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$$

The set $\mathcal{C} = \{C, C^T, D, E\}$ is a complete set of MOLS.

$n = 7$. There are 3840 idempotent SOLS of order 7, so $s_7 = 3840$ and $S_7 = 19,353,600$. There are eight isomorphism classes of SOLS. Four classes contain 120 isomorphic Latin squares while another four classes contain 840 isomorphic Latin squares. There are $7! = 5040$ permutations of V_7. If A and B are in a class consisting of 120 squares, then there are 42 permutations σ such that $\sigma A = B$. If A and B are in a class consisting of 840 squares, then there are 6 permutations σ such that $\sigma A = B$. Let

$$F = \begin{bmatrix} 1 & 3 & 2 & 5 & 4 & 7 & 6 \\ 4 & 2 & 7 & 1 & 6 & 5 & 3 \\ 5 & 6 & 3 & 7 & 1 & 2 & 4 \\ 6 & 7 & 5 & 4 & 3 & 1 & 2 \\ 7 & 4 & 6 & 2 & 5 & 3 & 1 \\ 2 & 1 & 4 & 3 & 7 & 6 & 5 \\ 3 & 5 & 1 & 6 & 2 & 4 & 7 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 1 & 5 & 4 & 7 & 6 & 3 & 2 \\ 7 & 2 & 5 & 3 & 1 & 4 & 6 \\ 6 & 4 & 3 & 1 & 2 & 7 & 5 \\ 3 & 6 & 2 & 4 & 7 & 5 & 1 \\ 2 & 3 & 7 & 6 & 5 & 1 & 4 \\ 5 & 7 & 1 & 2 & 4 & 6 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 7 \end{bmatrix}$$

The set $\{F, F^T, G, G^T\}$ is mutually orthogonal and one Latin square comes from each of the four classes containing 120 isomorphic squares. Let

$$H = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 2 & 6 & 7 & 4 \\ 2 & 4 & 1 & 7 & 3 & 5 & 6 \\ 5 & 3 & 6 & 1 & 7 & 4 & 2 \\ 4 & 7 & 2 & 6 & 1 & 3 & 5 \\ 7 & 6 & 4 & 5 & 2 & 1 & 3 \\ 6 & 5 & 7 & 3 & 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 7 & 3 & 5 & 6 \\ 3 & 1 & 5 & 2 & 6 & 7 & 4 \\ 4 & 7 & 2 & 6 & 1 & 3 & 5 \\ 5 & 3 & 6 & 1 & 7 & 4 & 2 \\ 6 & 5 & 7 & 3 & 4 & 2 & 1 \\ 7 & 6 & 4 & 5 & 2 & 1 & 3 \end{bmatrix}$$

The set $\mathcal{F} = \{F, F^T, G, G^T, H, I\}$ is a complete set of MOLS.

Given an isomorphism class containing 840 Latin squares, there is exactly one distinct class of 840 squares such that the Latin squares in this class are the transposes of the Latin squares in the given class. Furthermore, if A is in an isomorphism class of 840 squares and if σ is any permutation of V_7, then A^T is in a different class of 840 squares and $(\sigma A)^T = \sigma(A^T)$—hence, $\{A, A^T\}$ is a set of MOLS and so is $\{\sigma A, \sigma A^T\}$. Let A be any square from any class of 840 squares, the set $\{A, A^T\}$ is maximal—that is, there does not exist any Latin square B of order 7 such that $\{A, A^T, B\}$ is a set of MOLS.

$n = 8$. There are 103, 680 idempotent SOLS of order 8, so $s_8 = 103,680$ and $S_8 = 4,180,377,600$. There are eight isomorphism classes of SOLS.

The isomorphism class of

$$J = \begin{bmatrix} 1 & 3 & 2 & 5 & 6 & 7 & 8 & 4 \\ 4 & 2 & 7 & 8 & 1 & 3 & 5 & 6 \\ 5 & 4 & 3 & 6 & 8 & 1 & 2 & 7 \\ 3 & 1 & 8 & 4 & 7 & 2 & 6 & 5 \\ 7 & 6 & 4 & 2 & 5 & 8 & 1 & 3 \\ 8 & 7 & 5 & 1 & 3 & 6 & 4 & 2 \\ 2 & 8 & 6 & 3 & 4 & 5 & 7 & 1 \\ 6 & 5 & 1 & 7 & 2 & 4 & 3 & 8 \end{bmatrix}$$

contains 40,320 distinct Latin squares. The SOLS J^T is not in this class, so there is a second distinct isomorphism class containing 40,320 SOLS. The set $\{J, J^T\}$ is maximal in that there does not exist a Latin square X such that $\{J, J^T, X\}$ is a set of MOLS.

The SOLS

$$K = \begin{bmatrix} 1 & 3 & 2 & 5 & 6 & 7 & 8 & 4 \\ 4 & 2 & 8 & 1 & 3 & 5 & 6 & 7 \\ 5 & 7 & 3 & 6 & 8 & 4 & 1 & 2 \\ 6 & 8 & 7 & 4 & 1 & 2 & 3 & 5 \\ 8 & 6 & 4 & 7 & 5 & 3 & 2 & 1 \\ 2 & 4 & 1 & 8 & 7 & 6 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 & 8 & 7 & 6 \\ 7 & 5 & 6 & 3 & 2 & 1 & 4 & 8 \end{bmatrix}$$

does not belong to either of the two previous classes. The isomorphism class of K contains 5040 SOLS as does the distinct isomorphism class containing K^T. Let

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 8 & 7 & 6 & 5 \\ 3 & 4 & 1 & 2 & 6 & 5 & 8 & 7 \\ 4 & 3 & 2 & 1 & 7 & 8 & 5 & 6 \\ 5 & 8 & 6 & 7 & 1 & 3 & 4 & 2 \\ 6 & 7 & 5 & 8 & 3 & 1 & 2 & 4 \\ 7 & 6 & 8 & 5 & 4 & 2 & 1 & 3 \\ 8 & 5 & 7 & 6 & 2 & 4 & 3 & 1 \end{bmatrix}$$

The set $\{K, K^T, L\}$ is maximal.

The isomorphism class of

$$M = \begin{bmatrix} 1 & 3 & 4 & 5 & 6 & 7 & 8 & 2 \\ 4 & 2 & 5 & 7 & 1 & 8 & 6 & 3 \\ 7 & 6 & 3 & 1 & 8 & 5 & 2 & 4 \\ 8 & 5 & 2 & 4 & 7 & 3 & 1 & 6 \\ 3 & 8 & 7 & 6 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 8 & 3 & 6 & 5 & 7 \\ 6 & 1 & 8 & 3 & 2 & 4 & 7 & 5 \\ 5 & 7 & 6 & 2 & 4 & 1 & 3 & 8 \end{bmatrix}$$

does not belong to any of the previous four isomorphism classes and it contains 5760 distinct SOLS as does the distinct isomorphism class containing M^T. The set $\{M, M^T\}$ is maximal.

The isomorphism class of

$$N=\begin{bmatrix} 1&3&4&5&6&7&8&2\\ 5&2&7&1&8&4&6&3\\ 6&4&3&8&1&2&5&7\\ 7&8&5&4&2&1&3&6\\ 8&7&2&6&5&3&1&4\\ 2&5&8&3&7&6&4&1\\ 3&1&6&2&4&8&7&5\\ 4&6&1&7&3&5&2&8 \end{bmatrix}$$

does not belong to any of the previous six isomorphism classes and it contains 720 distinct SOLS as does the distinct isomorphism class containing N^T. Let

$$O=\begin{bmatrix} 1&4&5&6&7&8&2&3\\ 8&2&6&5&3&1&4&7\\ 2&8&3&7&6&4&1&5\\ 3&6&2&4&8&7&5&1\\ 4&1&7&3&5&2&8&6\\ 5&7&1&8&4&6&3&2\\ 6&3&8&1&2&5&7&4\\ 7&5&4&2&1&3&6&8 \end{bmatrix} \quad P=\begin{bmatrix} 1&6&7&8&2&3&4&5\\ 7&2&1&3&6&8&5&4\\ 8&5&3&1&4&7&2&6\\ 2&7&6&4&1&5&8&3\\ 3&4&8&7&5&1&6&2\\ 4&3&5&2&8&6&1&7\\ 5&8&4&6&3&2&7&1\\ 6&1&2&5&7&4&3&8 \end{bmatrix}$$

The SOLS O and P are in the isomorphism class of N while O^T and P^T are in the isomorphism class of N^T. Let

$$Q=\begin{bmatrix} 1&2&3&4&5&6&7&8\\ 2&1&5&8&3&7&6&4\\ 3&5&1&6&2&4&8&7\\ 4&8&6&1&7&3&5&2\\ 5&3&2&7&1&8&4&6\\ 6&7&4&3&8&1&2&5\\ 7&6&8&5&4&2&1&3\\ 8&4&7&2&6&5&3&1 \end{bmatrix}$$

The set $\{N, N^T, O, O^T, P, P^T, Q\}$ is a complete set of MOLS of order 8.

$n = 9$. There are 69,088,320 idempotent SOLS of order 9, so $s_9 =$ 69,088,320 and $S_9 =$ 25,070,769,561,600. There are 283 isomorphism classes of SOLS. Figure 1 summarizes the characteristics of the isomorphism classes based on the orbit length, whether or not the transpose of each square in the orbit occurs in the same orbit, and the type of maximal set associated with the orbit.

Four projective planes of order 9 have been known for a number of years and in 1991 DWHLam, GKolesova, and LThiel [9] proved there are no other projective planes of order 9. T.G. Room and P. B. Kirkpatrick [18] used the following notation for the four projective planes of order 9: the desarguesian plane is denoted by Φ, the translation plane by Ω, the dual of the translation plane by Ω^D, and the Hughes plane by Ψ. Affine planes are obtained from projective planes by selecting a line, called the line at infinity ℓ_∞, and then deleting this line and the points on it. The remaining lines and points are an affine plane. Deleting the line at infinity and the points on it, one affine plane is obtained from the desarguesian plane Φ and two non-isomorphic affine planes are obtained from each of the projective planes Ω, Ω^D, and Ψ. We number the affine planes as follows:

Affine Plane	Projective Plane	Line at Infinity
1	Φ	Any
2	Ω	$\ell_\infty = t$
3	Ω	$\ell_\infty \neq t$
4	Ω^D	$\ell_\infty \ I \ T$
5	Ω^D	$\ell_\infty \ \not{I} \ T$
6	Ψ	Real
7	Ψ	Complex

where t denotes the translation line, T denotes the translation point, I denotes incidence of a point and a line, and $\not{I}$ denotes its negation.

The set of SOLS which are members of complete sets of order 9 consist precisely of those SOLS in 18 orbits. One SOLS from each orbit—the square in the orbit which was found first by our search procedure—appears in Figure 2. In Figure 3 we present information regarding the total number of Latin squares orthogonal to each SOLS pair Z, Z^T in Figure 2, as well as orbit information. In Figure 4, we display complete set information and plane information for the same set of SOLS. From Figure 4, we observe that it is possible to represent any of the seven affine planes by a complete set of Latin squares which contains exactly 2 pairs of SOLS and also exactly 3 pairs of SOLS. We see when 3 pairs of SOLS appear in a complete set, the remaining two Latin squares are always a symmetric square and a square with constant diagonal. SOLS N, O, P, and Q belong to three different types of complete sets while R belongs to 78 different types of complete sets. R appears in complete sets representing the affine planes 1 and 2—3 times each and representing the affine planes 4 and 6—36 times each. R does not appear in any complete set representing the affine planes 3, 5, or 7.

Computer generation of SOLS. For $n = 4, 5$, and 7 it is easy to find all SOLS by creating a Latin square one row at a time; checking after adding the k-th row for the distinctness of the ordered pairs (a_{ij}, a_{ji}) for $i, j =$

1, 2, ... k and (i, i) for $i = k + 1, \ldots, n$; and back-tracking when necessary. For $n = 8$ and 9, at each stage, we add the k-th partial row to the right of the diagonal element a_{kk}—that is, the elements $a_{k\ell}$ for $\ell = k + 1, \ldots, n$— and add the k-th partial column below the same diagonal element—that is, $a_{\ell k}$ for $\ell = k + 1, \ldots, n$. Then we check for distinctnesss of the ordered pairs (a_{ij}, a_{ji}) for $i, j = 1, 2, \ldots k$; (a_{ij}, a_{ji}) for $i = 1, \ldots k$, $j = k + 1, \ldots n$; (a_{ij}, a_{ji}) for $i = k + 1, \ldots n$, $j = 1, \ldots k$; and (i, i) for $i = k + 1, \ldots, n$. Back-tracking was used when necessary.

Determination of plane type for $n = 9$. Once we find a complete set of MOLS of order 9, we construct the associated incidence matrix and determine the type of projective plane and affine plane which corresponds to the complete set from the incidence matrix itself. Using a different approach, P˙ J˙ Owens and DȦṖreece [16] and [17] consider the collection C of all complete sets of MOLS of order 9 in which the first row of each square is in ascending order. In [16] they define an equivalence relation on C and show that this relation partitions the collection C into 19 equivalence classes. Also in [16] they provide an algorithm for identifying the type of projective and affine plane associated with any complete set of MOLS in the collection C.

Almost Self-Orthogonal Latin Squares of Orders 2, 3, and 6

Many results have been obtained regarding orthogonal Latin squares with "holes"—that is, Latin squares which are orthogonal except for subsquares coincident in position. L. Euler, himself, gave an example of two 6 × 6 squares which were orthogonal except for a coincident 2 × 2 subsquare. Our rules for constructing almost self-orthogonal Latin squares are as follows:

1. Entries are from V_n.
2. Unfilled entries are designated by 0.
3. The main diagonal entries, in order, are $1, 2, \ldots, n$.
4. If the i, j entry is not 0, then the j, i entry is not 0.

$n = 2$. The best attainable almost SOLS is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The pair A, A^T has two of the four ordered pairs required for orthogonality.

$n = 3$. The best attainable almost SOLS is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The pair B, B^T has only three of the nine ordered pairs required for orthogonality.

$n = 6$. There are 84 best attainable almost SOLS with the same configuration as

$$C = \begin{bmatrix} 1 & 3 & 4 & 2 & 6 & 5 \\ 5 & 2 & 1 & 6 & 3 & 4 \\ 2 & 6 & 3 & 5 & 4 & 1 \\ 6 & 5 & 2 & 4 & 0 & 0 \\ 3 & 4 & 6 & 0 & 5 & 0 \\ 4 & 1 & 5 & 0 & 0 & 6 \end{bmatrix}$$

The pair C, C^T has thirty of the thirty-six ordered pairs required for orthogonality.

3 Summary

Let s_n denote the number of distinct, idempotent, self-orthogonal Latin squares of order n, then the total number of self-orthogonal squares of order n is $S_n = n!s_n$. Let ℓ_n denote the number of distinct, reduced Latin squares of order n, then the total number of distinct squares of order n is $L_n = n!(n-1)!\ell_n$. The following table summarizes the known values of s_n and ℓ_n.

n	s_n	Reference	ℓ_n	Reference
1	1		1	
2	0	[3]	1	[7]
3	0	[3]	1	[7]
4	2		4	[7]
5	12		56	[7]
6	0	[3]	9,408	[8]
7	3,840		16,942,080	[19]
8	103,680		535,281,401,856	[23]
9	69,088,320		377,597,570,964,258,816	[1]
10			7,580,721,483,160,132,811,489,280	[11]
11			5,363,937,773,277,371,298,119,673,540,771,840	[12]

For $n = 4$ all SOLS are members of complete sets of Latin squares consisting of the SOLS, its transpose, and a symmetric square. These complete sets are associated with the field plane of order four.

For $n = 5$ all SOLS are members of complete sets of Latin squares consisting of the SOLS, its transpose, a symmetric square, and one additional square. These complete sets are associated with the field plane of order five.

For $n = 7$, 87.5% of the SOLS have maximal set which consists of only two squares—the self-orthogonal square and its transpose. The remaining 12.5% of the SOLS are members of complete sets comprised of the SOLS, its transpose, another self-orthogonal square, its transpose, a symmetric Latin square, and a sixth square. These complete sets are associated with the field plane of order seven.

For $n = 8$, $88.\overline{8}\%$ of the SOLS have a maximal set with two squares—the self-orthogonal square and its transpose. The maximal set for $9.7\overline{2}\%$ of the SOLS contains three squares—the SOLS, its transpose, and a third Latin square. The remaining $1.3\overline{8}\%$ of the SOLS are members of complete sets consisting of three SOLS, their transposes, and a symmetric square. These complete sets are associated with the field plane of order eight.

For $n = 9$, 88.37% of the SOLS have a maximal set consisting of two squares—the SOLS and its transpose. The maximal set for 5.34% of the SOLS contains three squares—the SOLS, its transpose, and a third Latin square. The maximal set for 2.80% of the SOLS contains four squares—the SOLS, its transpose, and two other Latin squares. The remaining 3.49% of the SOLS are members of complete sets of MOLS.

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Orbit Length	Orbit Contains	Number of Orbits	I	II	III	IV	Complete Set
362,880	A and A^T	11	8	1	1	0	1
	A only	126	120	0	2	2	2
181,440	A and A^T	36	30	2	2	2	0
	A only	50	40	0	4	4	2
120,960	A and A^T	2	0	0	2	0	0
90,720	A and A^T	8	4	1	3	0	0
	A only	12	12	0	0	0	0
60,480	A and A^T	4	0	0	0	2	2
	A only	6	0	0	0	0	6
45,360	A and A^T	4	0	0	0	0	4
	A only	20	10	2	2	0	6
5,040	A and A^T	2	0	0	0	0	2
	A only	2	0	0	0	0	2

Type of Maximal Set: $I - \{A, A^T\}$, $II - \{A, A^T, S\}$, $III - \{A, A^T, N\}$, $IV - \{A, A^T, X, Y\}^*$

S–symmetric, N–nonsymmetric,

*X is symmetric if and only if Y has constant main diagonal.

Figure 1: Characteristics of Isomorphism Classes of Order 9

$$A = \begin{bmatrix} 1 & 3 & 2 & 5 & 6 & 7 & 8 & 9 & 4 \\ 4 & 2 & 8 & 3 & 7 & 9 & 5 & 6 & 1 \\ 5 & 7 & 3 & 9 & 4 & 8 & 2 & 1 & 6 \\ 6 & 8 & 7 & 4 & 2 & 1 & 9 & 3 & 5 \\ 7 & 1 & 9 & 8 & 5 & 3 & 6 & 4 & 2 \\ 3 & 5 & 1 & 2 & 9 & 6 & 4 & 7 & 8 \\ 9 & 4 & 6 & 1 & 8 & 2 & 7 & 5 & 3 \\ 2 & 9 & 5 & 6 & 1 & 4 & 3 & 8 & 7 \\ 8 & 6 & 4 & 7 & 3 & 5 & 1 & 2 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 4 & 5 & 2 & 7 & 8 & 9 & 6 \\ 4 & 2 & 5 & 9 & 8 & 3 & 1 & 6 & 7 \\ 7 & 9 & 3 & 2 & 6 & 1 & 5 & 4 & 8 \\ 6 & 1 & 7 & 4 & 3 & 8 & 9 & 2 & 5 \\ 8 & 4 & 9 & 7 & 5 & 2 & 6 & 3 & 1 \\ 9 & 5 & 8 & 3 & 1 & 6 & 4 & 7 & 2 \\ 5 & 6 & 2 & 8 & 4 & 9 & 7 & 1 & 3 \\ 2 & 7 & 1 & 6 & 9 & 5 & 3 & 8 & 4 \\ 3 & 8 & 6 & 1 & 7 & 4 & 2 & 5 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 2 & 5 & 4 & 7 & 8 & 9 & 6 \\ 4 & 2 & 6 & 7 & 3 & 9 & 5 & 1 & 8 \\ 6 & 8 & 3 & 2 & 1 & 5 & 9 & 7 & 4 \\ 2 & 5 & 9 & 4 & 7 & 8 & 3 & 6 & 1 \\ 8 & 9 & 7 & 6 & 5 & 1 & 4 & 3 & 2 \\ 3 & 4 & 8 & 1 & 9 & 6 & 2 & 5 & 7 \\ 9 & 1 & 5 & 8 & 6 & 4 & 7 & 2 & 3 \\ 7 & 6 & 4 & 9 & 2 & 3 & 1 & 8 & 5 \\ 5 & 7 & 1 & 3 & 8 & 2 & 6 & 4 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & 2 & 5 & 4 & 7 & 8 & 9 & 6 \\ 4 & 2 & 9 & 3 & 7 & 8 & 6 & 1 & 5 \\ 6 & 4 & 3 & 7 & 8 & 2 & 9 & 5 & 1 \\ 2 & 7 & 1 & 4 & 9 & 5 & 3 & 6 & 8 \\ 8 & 6 & 7 & 2 & 5 & 9 & 1 & 4 & 3 \\ 5 & 1 & 8 & 9 & 3 & 6 & 4 & 2 & 7 \\ 9 & 5 & 6 & 8 & 2 & 1 & 7 & 3 & 4 \\ 7 & 9 & 4 & 1 & 6 & 3 & 5 & 8 & 2 \\ 3 & 8 & 5 & 6 & 1 & 4 & 2 & 7 & 9 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 3 & 4 & 2 & 6 & 7 & 8 & 9 & 5 \\ 5 & 2 & 6 & 9 & 7 & 4 & 1 & 3 & 8 \\ 6 & 8 & 3 & 7 & 2 & 9 & 5 & 4 & 1 \\ 7 & 1 & 5 & 4 & 8 & 3 & 9 & 6 & 2 \\ 9 & 6 & 1 & 3 & 5 & 8 & 4 & 2 & 7 \\ 8 & 9 & 7 & 1 & 4 & 6 & 2 & 5 & 3 \\ 2 & 4 & 9 & 8 & 3 & 5 & 7 & 1 & 6 \\ 3 & 7 & 2 & 5 & 9 & 1 & 6 & 8 & 4 \\ 4 & 5 & 8 & 6 & 1 & 2 & 3 & 7 & 9 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 3 & 4 & 5 & 2 & 7 & 8 & 9 & 6 \\ 6 & 2 & 5 & 7 & 4 & 9 & 1 & 3 & 8 \\ 9 & 7 & 3 & 8 & 6 & 4 & 5 & 1 & 2 \\ 8 & 9 & 2 & 4 & 3 & 5 & 6 & 7 & 1 \\ 7 & 6 & 8 & 9 & 5 & 1 & 4 & 2 & 3 \\ 4 & 8 & 1 & 2 & 9 & 6 & 3 & 5 & 7 \\ 3 & 5 & 9 & 1 & 8 & 2 & 7 & 6 & 4 \\ 2 & 4 & 7 & 6 & 1 & 3 & 9 & 8 & 5 \\ 5 & 1 & 6 & 3 & 7 & 8 & 2 & 4 & 9 \end{bmatrix}$$

Figure 2. A Self-orthogonal Latin Square From Each Orbit Whose Squares Are Members of Complete Sets

$$G = \begin{bmatrix} 1 & 3 & 2 & 5 & 6 & 7 & 8 & 9 & 4 \\ 4 & 2 & 9 & 8 & 3 & 5 & 1 & 7 & 6 \\ 5 & 4 & 3 & 7 & 9 & 8 & 6 & 2 & 1 \\ 8 & 7 & 6 & 4 & 1 & 9 & 2 & 3 & 5 \\ 9 & 1 & 8 & 2 & 5 & 3 & 4 & 6 & 7 \\ 2 & 9 & 1 & 3 & 7 & 6 & 5 & 4 & 8 \\ 3 & 6 & 5 & 9 & 8 & 4 & 7 & 1 & 2 \\ 7 & 5 & 4 & 6 & 2 & 1 & 9 & 8 & 3 \\ 6 & 8 & 7 & 1 & 4 & 2 & 3 & 5 & 9 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 3 & 4 & 2 & 6 & 7 & 8 & 9 & 5 \\ 4 & 2 & 6 & 7 & 9 & 8 & 5 & 3 & 1 \\ 5 & 4 & 3 & 9 & 8 & 1 & 2 & 6 & 7 \\ 7 & 1 & 8 & 4 & 3 & 9 & 6 & 5 & 2 \\ 9 & 7 & 2 & 8 & 5 & 4 & 3 & 1 & 6 \\ 8 & 5 & 9 & 3 & 7 & 6 & 1 & 2 & 4 \\ 6 & 9 & 1 & 5 & 2 & 3 & 7 & 4 & 8 \\ 2 & 6 & 7 & 1 & 4 & 5 & 9 & 8 & 3 \\ 3 & 8 & 5 & 6 & 1 & 2 & 4 & 7 & 9 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 3 & 4 & 2 & 6 & 7 & 8 & 9 & 5 \\ 5 & 2 & 6 & 3 & 9 & 8 & 4 & 7 & 1 \\ 6 & 7 & 3 & 8 & 2 & 9 & 5 & 1 & 4 \\ 7 & 9 & 5 & 4 & 1 & 3 & 6 & 2 & 8 \\ 8 & 6 & 1 & 9 & 5 & 4 & 2 & 3 & 7 \\ 9 & 4 & 8 & 1 & 7 & 6 & 3 & 5 & 2 \\ 3 & 1 & 9 & 5 & 8 & 2 & 7 & 4 & 6 \\ 2 & 5 & 7 & 6 & 4 & 1 & 9 & 8 & 3 \\ 4 & 8 & 2 & 7 & 3 & 5 & 1 & 6 & 9 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 3 & 2 & 5 & 6 & 7 & 8 & 9 & 4 \\ 4 & 2 & 9 & 7 & 3 & 8 & 5 & 6 & 1 \\ 5 & 6 & 3 & 9 & 7 & 2 & 4 & 1 & 8 \\ 6 & 5 & 8 & 4 & 1 & 9 & 2 & 7 & 3 \\ 8 & 1 & 6 & 2 & 5 & 4 & 9 & 3 & 7 \\ 9 & 7 & 4 & 1 & 8 & 6 & 3 & 5 & 2 \\ 3 & 9 & 1 & 8 & 4 & 5 & 7 & 2 & 6 \\ 2 & 4 & 7 & 3 & 9 & 1 & 6 & 8 & 5 \\ 7 & 8 & 5 & 6 & 2 & 3 & 1 & 4 & 9 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 3 & 4 & 2 & 6 & 7 & 8 & 9 & 5 \\ 5 & 2 & 6 & 3 & 7 & 4 & 9 & 1 & 8 \\ 7 & 9 & 3 & 6 & 2 & 8 & 5 & 4 & 1 \\ 6 & 8 & 5 & 4 & 1 & 9 & 3 & 7 & 2 \\ 8 & 6 & 1 & 9 & 5 & 3 & 4 & 2 & 7 \\ 9 & 1 & 7 & 8 & 4 & 6 & 2 & 5 & 3 \\ 2 & 4 & 9 & 1 & 8 & 5 & 7 & 3 & 6 \\ 3 & 7 & 2 & 5 & 9 & 1 & 6 & 8 & 4 \\ 4 & 5 & 8 & 7 & 3 & 2 & 1 & 6 & 9 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 3 & 2 & 5 & 4 & 7 & 8 & 9 & 6 \\ 6 & 2 & 5 & 7 & 9 & 1 & 4 & 3 & 8 \\ 7 & 6 & 3 & 1 & 8 & 9 & 5 & 2 & 4 \\ 8 & 9 & 7 & 4 & 3 & 5 & 6 & 1 & 2 \\ 9 & 1 & 6 & 8 & 5 & 4 & 2 & 7 & 3 \\ 3 & 5 & 8 & 2 & 7 & 6 & 9 & 4 & 1 \\ 2 & 8 & 4 & 9 & 1 & 3 & 7 & 6 & 5 \\ 5 & 4 & 9 & 3 & 6 & 2 & 1 & 8 & 7 \\ 4 & 7 & 1 & 6 & 2 & 8 & 3 & 5 & 9 \end{bmatrix}$$

Figure 2. (Continued) A Self-orthogonal Latin Square From Each Orbit Whose Squares Are Members of Complete Sets

$$M = \begin{bmatrix} 1&3&2&5&6&7&8&9&4\\ 4&2&8&3&9&1&6&5&7\\ 7&5&3&9&8&4&2&6&1\\ 6&8&7&4&1&2&9&3&5\\ 3&1&6&2&5&9&4&7&8\\ 5&7&9&8&3&6&1&4&2\\ 9&4&5&6&2&8&7&1&3\\ 2&9&1&7&4&5&3&8&6\\ 8&6&4&1&7&3&5&2&9 \end{bmatrix}$$

$$N = \begin{bmatrix} 1&3&4&5&2&7&8&9&6\\ 6&2&5&7&9&4&1&3&8\\ 7&6&3&2&8&9&5&1&4\\ 8&9&7&4&3&5&6&2&1\\ 9&4&6&8&5&1&2&7&3\\ 3&5&8&1&7&6&9&4&2\\ 4&8&2&9&1&3&7&6&5\\ 5&1&9&3&6&2&4&8&7\\ 2&7&1&6&4&8&3&5&9 \end{bmatrix}$$

$$O = \begin{bmatrix} 1&3&2&5&6&7&8&9&4\\ 4&2&7&3&9&8&5&6&1\\ 7&1&3&9&8&4&6&5&2\\ 3&6&8&4&2&9&1&7&5\\ 9&7&4&8&5&3&2&1&6\\ 8&5&9&1&7&6&4&2&3\\ 6&9&5&2&4&1&7&3&8\\ 2&4&1&6&3&5&9&8&7\\ 5&8&6&7&1&2&3&4&9 \end{bmatrix}$$

$$P = \begin{bmatrix} 1&3&2&5&6&7&8&9&4\\ 4&2&6&9&7&8&3&5&1\\ 7&9&3&2&8&1&6&4&5\\ 9&7&8&4&2&3&5&1&6\\ 3&1&9&6&5&4&2&7&8\\ 8&4&5&7&9&6&1&2&3\\ 6&5&4&8&1&9&7&3&2\\ 2&6&1&3&4&5&9&8&7\\ 5&8&7&1&3&2&4&6&9 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1&3&4&5&2&7&8&9&6\\ 6&2&8&9&3&4&5&7&1\\ 9&4&3&7&8&2&6&1&5\\ 8&7&5&4&6&9&1&2&3\\ 7&9&6&2&5&1&3&4&8\\ 4&5&1&8&9&6&2&3&7\\ 3&1&9&6&4&8&7&5&2\\ 2&6&7&3&1&5&9&8&4\\ 5&8&2&1&7&3&4&6&9 \end{bmatrix}$$

$$R = \begin{bmatrix} 1&3&4&5&6&7&8&9&2\\ 5&2&6&9&7&4&3&1&8\\ 6&9&3&7&2&8&5&4&1\\ 7&1&2&4&8&3&9&6&5\\ 8&6&1&3&5&9&4&2&7\\ 9&8&7&1&4&6&2&5&3\\ 2&4&9&8&1&5&7&3&6\\ 3&7&5&2&9&1&6&8&4\\ 4&5&8&6&3&2&1&7&9 \end{bmatrix}$$

Figure 2. (Continued) A Self-orthogonal Latin Square From Each Orbit Whose Squares Are Members of Complete Sets

SOLS	Number LS Orthogonal Z and Z^T	Orbit Contains	Orbit Length
A	6	A only	45360
B	6	B only	45360
C	9	C and C^T	362880
D	9	D only	362880
E	10	E and E^T	60480
F	10	F and F^T	60480
G	14	G only	181440
H	14	H and H^T	45360
I	14	I and I^T	45360
J	54	J only	45360
K	54	K and K^T	45360
L	194	L only	60480
M	194	M only	60480
N	302	N only	5040
O	486	O and O^T	45360
P	486	P only	60480
Q	486	Q and Q^T	5040
R	63666	R and R^T	5040

Figure 3. Orbit Information for Complete Sets of Mutually Orthogonal Latin Squares of Order 9 which Contain SOLS

	Complete Set Information				Plane Information	
SOLS	Number SOLS in Set	SOLS in Orbit of	Symmetric LS	Constant Diagonal LS	Projective Plane	Affine Plane
A	2	A, A^T	No	No	Ω^D	5
B	2	B, B^T, J, J^T	No	No	Ω^D	5
C	3	C	Yes	Yes	Ψ	7
D	2	D, D^T	No	No	Ψ	7
E	3	E, P, P^T	Yes	Yes	Ψ	6
F	3	F	Yes	Yes	Ψ	6
G	2	G, G^T	No	No	Ω	3
H	3	H	Yes	Yes	Ω	3
I	3	I	Yes	Yes	Ω	3
J	2	J, J^T, B, B^T	No	No	Ω^D	5
K	3	K	Yes	Yes	Ω^D	5
L	2	L, L^T, P, P^T	No	No	Ψ	6
M	2	M, M^T	No	No	Ψ	6
N_1	3	N, N^T, R	Yes	Yes	Φ	1
N_2	2	N, N^T	No	No	Ω	2
N_3	2	N, N^T, R	No	No	Ω	2
O_1	3	O	Yes	Yes	Ω^D	4
O_2	2	O	No	No	Ψ	6
O_3	2	O	No	No	Ψ	6
P_1	3	P, P^T, E	Yes	Yes	Ψ	6
P_2	2	P, P^T	No	No	Ω^D	4
P_3	2	P, P^T, L, L^T	No	No	Ψ	6
Q_1	3	Q	Yes	Yes	Ω^D	4
Q_2	2	Q	No	No	Ψ	6
Q_3	2	Q	No	No	Ψ	6
R						
Number of Cases						
1	3	R, N, N^T	Yes	Yes	Φ	1
1	3	R	Yes	Yes	Ω	2
2	2	R, N, N^T	No	No	Ω	2
2	2	R	No	No	Φ	1
36	1	R	No	No	Ω^D	4
36	1	R	No	No	Ψ	6

Figure 4. Complete Set and Plane Information for Sets of Mutually Orthogonal Latin Squares of Order 9 which Contain SOLS