Three-fold Kirkman Packing Designs $\mathrm{KPD}_3(\{4, s^*\}, v)$

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Abstract: A three-fold Kirkman packing design $\mathrm{KPD}_3(\{4, s^*\}, v)$ is a three-fold resolvable packing with maximum possible number of parallel classes, each containing one block of size s and all other blocks of size 4. This article investigates the spectra of three-fold Kirkman packing design $\mathrm{KPD}_3(\{4, s^*\}, v)$ for $s = 5$ and 6, and we show that it contains all positive integers $v \equiv s - 4 \pmod 4$ with $v \geq 17$ if $s = 5$, and $v \geq 26$ if $s = 6$.

1 Introduction

Let X be a set of v points. A *packing* of X is a collection of subsets of X (called *blocks*) such that any pair of distinct points from X occurs together in at most λ blocks in the collection. Let K be a set of positive integers. Denote by $\mathrm{P}_\lambda(K, v)$ a packing on v points with block sizes all in K. A packing is called *resolvable* if its block set admits a partition into *parallel classes*, each parallel class being a partition of the point set X. A *Kirkman packing*, denoted $\mathrm{KP}_\lambda(K, v)$, is a resolvable $\mathrm{P}_\lambda(K, v)$.

A *Kirkman packing design*, denoted $\mathrm{KPD}_\lambda(\{w, s^*\}, v)$, is a resolvable packing of a v-set by the maximum possible number $m(v)$ of parallel classes, each containing one block of size s and all other blocks of size w. We usually

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write $\text{KPD}_\lambda(w, v)$ instead of $\text{KPD}_\lambda(\{w, w^*\}, v)$. In [4, 8, 14], such a design is also called Kirkman school project when $\lambda = 1, w = 3$ and $s \in \{2, 4\}$.

Kirkman packing designs have been studied by many researchers and found to have a number of applications. Especially, Cao and Du [4] have given some applications in threshold schemes. There are also some known results on the existence of the Kirkman packing designs $\text{KPD}_\lambda(\{w, s^*\}, v)$. When $\lambda = 1, w = 3$ and $s \in \{2, 4, 5\}$, the spectrum problem for Kirkman packing designs $\text{KPD}(\{3, s^*\}, v)$ has been almost completely solved by Černý, Horák and Wallis [6], Phillips, Wallis and Rees [14], Colbourn and Ling [8], Cao and Zhu [5], finally Cao and Du [4]. When $\lambda = 2, w = 3$ and $s \in \{4, 5\}$, the spectrum problem for two-fold Kirkman packing designs $\text{KPD}_2(\{3, s^*\}, v)$ has been completely solved by Zhang and Du [9, 20]. When $\lambda = 1, w = 4$ and $s \in \{5, 6\}$, the spectrum problem for Kirkman packing designs $\text{KPD}(\{4, s^*\}, v)$ has been partly solved by Cao and Du [4]. In this article, we shall be restricting our attention to three-fold Kirkman packing design $\text{KPD}_3(\{4, s^*\}, v)$ for $s = 5$ and 6.

Some simple computation shows:

Lemma 1.1 *Suppose $v \geq 17$. If there exists a $KPD_3(\{4, s^*\}, v)$, then $m(v) \leq n(v)$, where*

$$n(v) = \begin{cases} v - 3, & v \equiv 1 \ (mod\ 4) \text{ and } s = 5, \\ v - 5, & v \equiv 2 \ (mod\ 4) \text{ and } s = 6. \end{cases}$$

This article investigates the existence of three-fold Kirkman packing design $\text{KPD}_3(\{4, s^*\}, v)$ for $s = 5$ and 6, and the following results will be proved.

Theorem 1.2 *There exists a $KPD_3(\{4, 5^*\}, v)$ containing $v - 3$ parallel classes for every $v \equiv 1 \ (mod\ 4)$ and $v \geq 17$.*

Theorem 1.3 *There exists a $KPD_3(\{4, 6^*\}, v)$ containing $v - 5$ parallel classes for every $v \equiv 2 \ (mod\ 4)$ and $v \geq 26$.*

2 Preliminaries

In this section we shall define some of the auxiliary designs and establish some of the fundamental results which will be used later. The reader is referred to [7] for more information on designs, and, in particular, group divisible designs and frames.

Let K and M be sets of positive integers. A *group divisible design* (GDD) $GD(K, \lambda, M; v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$ where

1. X is a v-set (of points),
2. $\mathcal{G}$ is a collection of nonempty subsets of X (called groups) with cardinality in M and which partition X,

3. $\mathcal{B}$ is a collection of subsets of X (called blocks) with cardinality at least two in K,

4. no block intersects any group in more than one point,

5. each pair set $\{x, y\}$ of points not contained in a group is contained in exactly λ blocks.

The type of the GDD $(X, \mathcal{G}, \mathcal{B})$ is the multiset of sizes $|G|$ of the $G \in \mathcal{G}$ and we usually use the "exponential" notation for its description: type $1^i 2^j 3^k \cdots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on.

A GDD $(K, \lambda, M; v)$ is resolvable if the blocks of $\mathcal{B}$ can be partitioned into parallel classes.

We shall denote by $GD(k, \lambda, m; v)$ a $GD(\{k\}, \lambda, \{m\}; v)$. We shall sometimes refer to a $GD(K, \lambda, M; v)$, as a (K, λ)-GDD. A *transversal design* $TD[k; n]$ is a $(k, 1)$-GDD of type n^k; it is *idempotent* if it contains a parallel class of blocks. It is well known that a $TD[k; n]$ is equivalent to $k - 2$ mutually orthogonal Latin squares of order n.

For the transversal design, we have the following existence result.

Theorem 2.1 ([1, 2, 7, 19]) (1) *There exists an idempotent $TD[6; v]$ for any positive integer $v \geq 7$, except possibly $v \in E_1$, where $E_1 = \{10, 14, 18, 22, 26, 30, 60\}$.*

(2) *There exists an idempotent $TD[7; v]$ for any positive integer $v \geq 7$, except possibly $v \in E_1 \cup E_2$, where $E_2 = \{12, 15, 20, 21, 28, 33, 34, 35, 38, 39, 42, 44, 46, 51, 52, 54, 62\}$.*

(3) *There exists a $TD[q + 1; q]$ for any prime power q.*

We shall use the following Wilson's Fundamental Construction (WFC) for GDDs. Before stating it, we define a *weighting* of a GDD $(X, \mathcal{G}, \mathcal{B})$ to be any mapping $w : X \to Z^+ \cup \{0\}$.

Lemma 2.2 ([18]) *Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a GDD and let $w : X \to Z^+ \cup \{0\}$ be a weighting of the GDD. For every $x \in X$, let S_x be the multiset of $w(x)$ copies of x. For each block $B \in \mathcal{B}$, assume a k-GDD of type $\{S_x : x \in B\}$ is given. Then there is a k-GDD of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.*

A GDD $(X, \mathcal{G}, \mathcal{B})$ is called *frame resolvable* if its block set $\mathcal{B}$ admits a partition into holey parallel classes, each holey parallel class being a partition of $X - G_j$ for some $G_j \in \mathcal{G}$. We shall denote by (K, λ)-frame a frame resolvable (K, λ)-GDD. The groups in a (K, λ)-frame are often referred to as holes. It is well known that to each group G_j there are exactly $|G_j|$ holey parallel classes that partition $X - G_j$.

For the $(4, 3)$-frame, we have the following existence result.

Theorem 2.3 (1) ([10, 11, 12]) *There exists a (4,3)-frame of type h^u if and only if $u \geq 5$ and $h(u-1) \equiv 0 \pmod 4$, except possibly where $h \equiv 2 \pmod 4$ and*

1. $h \in \{2\} \cup \{n : 10 \leq n \leq 3476\}$ *and* $u \in \{23, 27\}$;
2. $h = 6$ *and* $u \in \{7, 23, 27, 39, 47\}$.

(2) ([3, 13, 15, 17]) *There exists a (4,3)-frame of type g^5m^1 if and only if $g \equiv 0 \pmod 4$, $m \equiv 0 \pmod 4$, and $0 \leq m \leq 4/3g$.*

The main technique that we will be using throughout the remainder of the article is a variant of Stinson's "Filling in Holes" construction. In applying the "Filling in Holes" construction, we will require (4, 3)-frames in which the blocks we not necessarily all of the same size. To get these, we use the following recursive construction.

Lemma 2.4 ([16]) *Suppose that there is a K-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_m^{t_m}$ and that for each $k \in K$ there is a (4,3)-frame of type h^k. Then there is a (4,3)-frame of type $(hg_1)^{t_1}(hg_2)^{t_2} \cdots (hg_m)^{t_m}$.*

Finally, as the "Filling in Holes" construction will generally involve adjoining more than one infinite point to a (4, 3)-frame, we will require the notation of an incomplete three-fold Kirkman packing design. Let $s = 5$ and $w = 3$ or 5, or $s = 6$ and $w = 5$ or 6, and $v \equiv s - 4 \pmod 4$, an *incomplete three-fold Kirkman packing design*, $\mathrm{IKPD}_3(\{4, s^*\}, v, w)$, is a triple $(X, Y, \mathcal{B})$ where X is a set of v elements, Y is a subset of X of size w (Y is called the hole) and $\mathcal{B}$ is a collection of subsets of X (blocks), each of size 4 or s, such that

1. $|Y \cap B| \leq 1$ for all $B \in \mathcal{B}$,
2. any pair of distinct elements in X occur together either in Y or in at most three blocks,
3. $\mathcal{B}$ admits a partition $v - w$ parallel classes on X, each of which contains one block of size s, and a further $w - a$ holey parallel classes of quadruple on $X \setminus Y$, where $a = 3$ if $(s, w) = (5, 5)$, $a = 5$ if $(s, w) = (6, 6)$, and $a = w$ if $(s, w) = (5, 3)$ and $(6, 5)$.

It is obvious that an $\mathrm{IKPD}_3(\{4, s^*\}, v, w)$ is a $\mathrm{KPD}_3(\{4, s^*\}, v)$ indeed if $(s, w) \in \{(5, 3), (6, 5)\}$. If $s = w \in \{5, 6\}$, we can obtain a $\mathrm{KPD}_3(\{4, s^*\}, v)$ from an $\mathrm{IKPD}_3(\{4, s^*\}, v, w)$ by adding a new block which contains all the points in the hole to the holey parallel classes.

For our purpose we need the following incomplete three-fold Kirkman packing designs.

Lemma 2.5 *There is an $IKPD_3(\{4, 5^*\}, 25, 5)$.*

Proof: Take the point set $X = (Z_5 \times \{0,1,2,3\}) \cup Y$, where $Y = \{\infty_1, \cdots, \infty_5\}$. Let $B_1 = \{0_0, 0_1, 0_2, 0_3\}$ and $B_2 = \{0_0, 0_1, 1_2, 1_3\}$. B_1 and B_2 together generate the required two holey parallel classes by $+1$ modulo 5. The other 20 parallel classes will be generated modulo 5 from four initial parallel classes P_1, P_2, P_3 and P_4. The blocks of the initial parallel classes are listed below.

P_1: $0_0 1_0 0_1 2_1 3_1$ $2_0 3_0 1_1 \infty_1$ $4_0 0_2 1_2 \infty_2$ $4_1 2_2 0_3 \infty_3$ $3_2 4_2 1_3 \infty_4$
$2_3 3_3 4_3 \infty_5$

P_2: $0_0 1_1 0_2 0_3 3_3$ $0_1 1_3 2_3 \infty_1$ $1_0 2_0 4_3 \infty_2$ $2_1 3_2 4_2 \infty_3$ $3_0 4_1 1_2 \infty_4$
$4_0 3_1 2_2 \infty_5$

P_3: $0_0 3_1 2_2 4_2 3_3$ $3_0 1_2 2_3 \infty_1$ $1_1 3_2 0_3 \infty_2$ $2_0 4_1 4_3 \infty_3$ $1_0 4_0 1_3 \infty_4$
$0_1 2_1 0_2 \infty_5$

P_4: $0_0 0_2 2_2 1_3 4_3$ $1_1 1_2 4_2 \infty_1$ $0_1 4_1 3_3 \infty_2$ $1_0 3_0 2_3 \infty_3$ $2_1 3_1 0_3 \infty_4$
$2_0 4_0 3_2 \infty_5$ □

Lemma 2.6 *There is an* $IKPD_3(\{4, 5^*\}, v, 5)$ *for* $v \in \{29, 37, 45, 53\}$.

Proof: Take the point set $X = (Z_t \times \{0,1\}) \cup Y$, where $t = (v-5)/2$, $Y = \{\infty_1, \cdots, \infty_5\}$. Let $B_1 = \{0, t/4, t/2, 3t/4\}$. $B_1 \times \{0\}$ and $B_1 \times \{1\}$ together generate a holey parallel class by $+1$ modulo t. $B_2 = \{0_0, 0_1, (t/2)_0, (t/2)_1\}$ generates another holey parallel class by $+1$ modulo t. The $v - 5$ parallel classes will be generated modulo t from two initial parallel classes P_1 and P_2. The blocks of the initial parallel classes are listed below.

$v = 29$:
P_1: $0_0 2_0 4_0 6_1 8_1$ $1_0 3_0 6_0 10_1$ $5_0 8_0 1_1 \infty_1$ $7_0 11_0 0_1 \infty_2$ $9_0 2_1 5_1 \infty_3$
$10_0 7_1 9_1 \infty_4$ $3_1 4_1 11_1 \infty_5$

P_2: $0_0 1_0 0_1 1_1 11_1$ $11_0 2_1 6_1 9_1$ $7_0 4_1 8_1 \infty_1$ $8_0 3_1 10_1 \infty_2$ $2_0 3_0 5_1 \infty_3$
$4_0 9_0 7_1 \infty_4$ $5_0 6_0 10_0 \infty_5$

$v = 37$:
P_1: $0_0 2_0 5_0 9_1 14_1$ $1_0 4_0 8_0 13_1$ $3_0 7_0 10_0 15_1$ $6_0 0_1 5_1 11_1$ $9_0 11_0 1_1 \infty_1$
$12_0 13_0 3_1 \infty_2$ $14_0 2_1 4_1 \infty_3$ $15_0 6_1 12_1 \infty_4$ $7_1 8_1 10_1 \infty_5$

P_2: $0_0 1_0 0_1 1_1 2_1$ $4_0 3_1 6_1 15_1$ $11_0 5_1 8_1 14_1$ $6_0 7_0 12_0 10_1$ $2_0 4_1 13_1 \infty_1$
$13_0 7_1 11_1 \infty_2$ $8_0 14_0 9_1 \infty_3$ $9_0 15_0 12_1 \infty_1$ $3_0 5_0 10_0 \infty_5$

$v = 45$:
P_1: $0_0 3_0 7_0 12_1 18_1$ $1_0 5_0 10_0 16_1$ $2_0 6_0 11_0 17_1$ $4_0 12_0 13_0 0_1$
$8_0 1_1 6_1 13_1$ $9_0 2_1 8_1 15_1$ $14_0 15_0 3_1 \infty_1$ $16_0 17_0 4_1 \infty_2$
$18_0 5_1 7_1 \infty_3$ $19_0 9_1 11_1 \infty_4$ $10_1 14_1 19_1 \infty_5$

P_2: $0_0 2_0 0_1 1_1 2_1$ $1_0 5_1 13_1 17_1$ $15_0 9_1 12_1 18_1$ $18_0 11_1 14_1 15_1$
$6_0 13_0 19_0 3_1$ $11_0 14_0 17_0 16_1$ $3_0 4_1 7_1 \infty_1$ $5_0 8_1 19_1 \infty_2$
$4_0 12_0 6_1 \infty_3$ $7_0 9_0 10_1 \infty_4$ $8_0 10_0 16_0 \infty_5$

$v = 53$:

$P_1:$	$0_0 3_0 7_0 12_1 18_1$	$1_0 6_0 12_0 19_1$	$2_0 5_0 9_0 14_1$	$4_0 10_0 14_0 0_1$
	$8_0 11_0 16_0 22_1$	$13_0 2_1 8_1 15_1$	$15_0 1_1 5_1 10_1$	$17_0 3_1 4_1 6_1$
	$18_0 19_0 11_1 \infty_1$	$20_0 21_0 13_1 \infty_2$	$22_0 7_1 16_1 \infty_3$	$23_0 21_1 23_1 \infty_4$
	$9_1 17_1 20_1 \infty_5$			
$P_2:$	$0_0 1_0 0_1 2_1 3_1$	$2_0 1_1 5_1 6_1$	$4_0 7_1 11_1 21_1$	$18_0 9_1 14_1 22_1$
	$22_0 4_1 13_1 20_1$	$6_0 13_0 15_0 10_1$	$8_0 17_0 19_0 16_1$	$9_0 14_0 23_0 15_1$
	$10_0 8_1 18_1 \infty_1$	$11_0 12_1 19_1 \infty_2$	$3_0 16_0 23_1 \infty_3$	$12_0 20_0 17_1 \infty_4$
	$5_0 7_0 21_0 \infty_5$			

□

Lemma 2.7 *There is an* $IKPD_3(\{4,5^*\},v,5)$ *for* $v \in \{33,41\}$.

Proof: Take the point set $X = (Z_t \times \{0,1\}) \cup Y$, where $t = (v-5)/2$, $Y = \{\infty_1, \cdots, \infty_5\}$. $B_1 = \{0_0, 0_1, (t/2)_0, (t/2)_1\}$ and $B_2 = \{0_0, 1_1, (t/2)_0, (t/2+1)_1\}$ together generate the required two holey parallel classes by +1 modulo t. The $v-5$ parallel classes will be generated modulo t from two initial parallel classes P_1 and P_2. The blocks of the initial parallel classes are listed below.

$v = 33:$

$P_1:$	$1_0 3_0 6_0 11_0 0_1$	$0_0 4_0 8_0 13_1$	$2_0 1_1 6_1 12_1$	$5_0 7_0 2_1 \infty_1$
	$9_0 10_0 3_1 \infty_2$	$12_0 4_1 5_1 \infty_3$	$13_0 8_1 11_1 \infty_4$	$7_1 9_1 10_1 \infty_5$
$P_2:$	$2_0 0_1 4_1 8_1 13_1$	$1_0 7_0 3_1 5_1$	$3_0 4_0 7_1 9_1$	$0_0 2_1 10_1 \infty_1$
	$10_0 1_1 11_1 \infty_2$	$5_0 6_0 6_1 \infty_3$	$9_0 12_0 12_1 \infty_4$	$8_0 11_0 13_0 \infty_5$

$v = 41:$

$P_1:$	$0_0 2_0 5_0 10_0 0_1$	$1_0 4_0 8_0 13_1$	$3_0 6_0 7_0 1_1$	$9_0 2_1 7_1 14_1$
	$11_0 3_1 4_1 5_1$	$12_0 13_0 8_1 \infty_1$	$14_0 15_0 6_1 \infty_2$	$16_0 12_1 15_1 \infty_3$
	$17_0 10_1 16_1 \infty_4$	$9_1 11_1 17_1 \infty_5$		
$P_2:$	$0_0 0_1 3_1 7_1 17_1$	$1_0 7_0 13_0 15_1$	$15_0 1_1 4_1 12_1$	$2_0 4_0 5_1 10_1$
	$10_0 12_0 14_1 16_1$	$11_0 8_1 13_1 \infty_1$	$17_0 2_1 6_1 \infty_2$	$3_0 8_0 9_1 \infty_3$
	$6_0 14_0 11_1 \infty_4$	$5_0 9_0 16_0 \infty_5$		

□

Lemma 2.8 *There is an* $IKPD_3(\{4,5^*\},v,3)$ *for* $v \in \{17,21,25,29\}$.

Proof: For the case $v = 21$, we take the point set $(Z_9 \times \{0,1\}) \cup \{\infty_1, \infty_2, \infty_3\}$. The required 18 parallel classes will be generated modulo 9 from two initial parallel classes P_1 and P_2. The blocks of the initial parallel classes are listed below. For the other values of v, take the point set $X = Z_t \cup \{\infty_1, \infty_2, \infty_3\}$, $t = v - 3$. The $v-3$ parallel classes will be generated modulo t from an initial parallel class. The blocks of the initial parallel class for each X are listed in Appendix I.

$P_1:$	$0_0 1_0 0_1 1_1 2_1$	$2_0 4_1 7_1 8_1$	$3_0 5_0 6_0 \infty_1$	$4_0 8_0 6_1 \infty_2$	$7_0 3_1 5_1 \infty_3$
$P_2:$	$0_0 1_0 4_0 4_1 8_1$	$2_0 5_0 7_0 1_1$	$2_1 5_1 7_1 \infty_1$	$3_0 0_1 6_1 \infty_2$	$6_0 8_0 3_1 \infty_3$

□

Lemma 2.9 *There is an* $IKPD_3(\{4,6^*\},v,6)$ *for* $v \in \{30,38,46,54\}$.

Proof: Take the point set $X = (Z_t \times \{0,1\}) \cup Y$, where $t = (v-6)/2$, $Y = \{\infty_1, \cdots, \infty_6\}$. Let $B = \{0, t/4, t/2, 3t/4\}$. $B \times \{0\}$ and $B \times \{1\}$ together generate the required holey parallel class by +1 modulo t. The $v-6$ parallel classes will be generated modulo t from two initial parallel classes P_1 and P_2. The blocks of the initial parallel classes are listed below.

$v = 30:$

$P_1:$ $0_0 3_0 5_1 7_1 9_1 11_1$ $1_0 2_0 1_1 \infty_1$ $4_0 11_0 2_1 \infty_2$ $5_0 10_0 6_1 \infty_3$
$6_0 7_0 4_1 \infty_4$ $8_0 9_0 10_1 \infty_5$ $0_1 3_1 8_1 \infty_6$

$P_2:$ $0_0 2_0 4_0 6_0 8_1 11_1$ $3_0 1_1 6_1 \infty_1$ $7_0 0_1 7_1 \infty_2$ $9_0 9_1 10_1 \infty_3$
$10_0 4_1 5_1 \infty_4$ $11_0 2_1 3_1 \infty_5$ $1_0 5_0 8_0 \infty_6$

$v = 38:$

$P_1:$ $0_0 5_0 7_1 9_1 11_1 13_1$ $1_0 2_0 3_0 0_1$ $4_0 1_1 2_1 8_1$ $6_0 7_0 5_1 \infty_1$
$8_0 12_0 12_1 \infty_2$ $9_0 14_0 3_1 \infty_3$ $10_0 15_0 6_1 \infty_4$ $11_0 13_0 14_1 \infty_5$
$4_1 10_1 15_1 \infty_6$

$P_2:$ $0_0 3_0 7_0 10_0 12_1 15_1$ $5_0 8_0 15_0 0_1$ $9_0 3_1 10_1 11_1$ $1_0 4_1 7_1 \infty_1$
$2_0 2_1 9_1 \infty_2$ $11_0 1_1 14_1 \infty_3$ $12_0 5_1 6_1 \infty_4$ $13_0 8_1 13_1 \infty_5$
$4_0 6_0 14_0 \infty_6$

$v = 46:$

$P_1:$ $0_0 3_1 6_1 9_1 12_1 16_1$ $1_0 4_1 8_1 10_1$ $2_0 0_1 1_1 2_1$ $3_0 4_0 5_0 5_1$
$6_0 8_0 10_0 13_1$ $7_0 11_0 15_1 \infty_1$ $12_0 19_0 17_1 \infty_2$ $13_0 18_0 14_1 \infty_3$
$16_0 17_0 11_1 \infty_4$ $9_0 15_0 19_1 \infty_5$ $14_0 7_1 18_1 \infty_6$

$P_2:$ $0_0 3_0 6_0 11_0 14_0 19_1$ $5_0 12_0 16_0 18_1$ $2_0 9_0 16_1 17_1$ $1_0 13_0 3_1 11_1$
$10_0 2_1 7_1 9_1$ $4_0 1_1 13_1 \infty_1$ $8_0 0_1 5_1 \infty_2$ $15_0 6_1 15_1 \infty_3$
$18_0 4_1 12_1 \infty_4$ $19_0 10_1 14_1 \infty_5$ $7_0 17_0 8_1 \infty_6$

$v = 54:$

$P_1:$ $0_0 4_1 8_1 12_1 16_1 21_1$ $1_0 4_0 7_0 10_1$ $2_0 3_0 5_0 0_1$ $6_0 8_0 1_1 2_1$
$9_0 10_0 3_1 5_1$ $13_0 6_1 7_1 13_1$ $19_0 15_1 17_1 18_1$ $11_0 15_0 19_1 \infty_1$
$12_0 17_0 20_1 \infty_2$ $14_0 21_0 23_1 \infty_3$ $18_0 20_0 22_1 \infty_4$ $16_0 23_0 14_1 \infty_5$
$22_0 9_1 11_1 \infty_6$

$P_2:$ $0_0 5_0 10_0 14_0 18_0 21_1$ $1_0 8_0 16_0 17_0$ $2_0 1_1 8_1 11_1$ $13_0 4_1 18_1 23_1$
$6_0 19_0 6_1 20_1$ $11_0 23_0 0_1 13_1$ $12_0 22_0 3_1 12_1$ $3_0 9_1 15_1 \infty_1$
$4_0 5_1 16_1 \infty_2$ $7_0 14_1 17_1 \infty_3$ $15_0 7_1 22_1 \infty_4$
$21_0 2_1 10_1 \infty_5$ $9_0 20_0 19_1 \infty_6$ □

Lemma 2.10 *There is an* $IKPD_3(\{4,6^*\},v,6)$ *for* $v \in \{34,42\}$.

Proof: Take the point set $X = (Z_t \times \{0,1\}) \cup Y$, where $t = (v-6)/2$, $Y = \{\infty_1, \cdots, \infty_6\}$. $B = \{0_0, 0_1, (t/2)_0, (t/2)_1\}$ generates the required holey parallel class by +1 modulo t. The $v-6$ parallel classes will be generated modulo t from two initial parallel classes P_1 and P_2. The blocks of the initial parallel classes are listed below.

$v = 34:$

$P_1:$ $0_0 2_1 4_1 6_1 8_1 11_1$ $1_0 2_0 3_0 0_1$ $4_0 5_0 1_1 \infty_1$ $6_0 9_0 5_1 \infty_2$
$7_0 11_0 3_1 \infty_3$ $8_0 12_0 10_1 \infty_4$ $10_0 9_1 12_1 \infty_5$ $13_0 7_1 13_1 \infty_6$

$P_2:$ $0_0 2_0 5_0 7_0 10_0 5_1$ $8_0 1_1 2_1 11_1$ $3_0 7_1 8_1 \infty_1$ $4_0 9_1 10_1 \infty_2$
$11_0 6_1 12_1 \infty_3$ $13_0 0_1 3_1 \infty_4$ $1_0 9_0 4_1 \infty_5$ $6_0 12_0 13_1 \infty_6$

$v = 42:$

$P_1:$ $0_0 3_1 6_1 9_1 12_1 16_1$ $1_0 2_0 3_0 0_1$ $4_0 5_0 7_0 1_1$ $6_0 2_1 4_1 8_1$
$8_0 13_0 13_1 \infty_1$ $9_0 15_0 10_1 \infty_2$ $10_0 16_0 11_1 \infty_3$ $12_0 17_0 17_1 \infty_4$
$11_0 14_1 15_1 \infty_5$ $14_0 5_1 7_1 \infty_6$

$P_2:$ $0_0 3_0 6_0 10_0 14_0 17_1$ $1_0 5_1 9_1 16_1$ $2_0 4_1 12_1 14_1$ $4_0 9_0 11_0 10_1$
$8_0 0_1 13_1 \infty_1$ $12_0 1_1 2_1 \infty_2$ $15_0 3_1 8_1 \infty_3$ $17_0 6_1 7_1 \infty_4$ □
$5_0 13_0 15_1 \infty_5$ $7_0 16_0 11_1 \infty_6$

Lemma 2.11 *There is an $IKPD_3(\{4, 6^*\}, v, 5)$ for $v \in \{26, 30\}$.*

Proof: Take the point set $X = Z_t \cup Y$, $t = v - 5$, $Y = \{\infty_1, \cdots, \infty_5\}$. The $v-5$ parallel classes will be generated modulo t from an initial parallel class. The blocks of the initial parallel class for each X are listed in Appendix II. □

3 Main results

In this section, we shall prove our main results. For our purpose we need the following "Filling in Holes" construction.

Theorem 3.1 *Suppose*

1. *there is a (4,3)-frame of type $g_1 g_2 \cdots g_m$,*
2. *there is an $IKPD_3(\{4, s^*\}, g_i + w, w)$ for every $i < m$,*
3. *there is a $KPD_3(\{4, s^*\}, g_m + w)$.*

Then there is a $KPD_3(\{4, s^*\}, w + \sum_{1 \le i \le m} g_i)$.

Proof: We start with a (4,3)-frame of type $g_1 g_2 \cdots g_m$ $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G} = \{G_1, G_2, \cdots, G_m\}$ and $|G_i| = g_i$ $(1 \le i \le m)$. For $i < m$, there are g_i frame parallel classes missing the group G_i, and the same number of parallel classes in the $IKPD_3(\{4, s^*\}, g_i + w, w)$ which contain a block of size s; match these up arbitrarily, placing the g_i points of the $IKPD_3(\{4, s^*\}, g_i + w, w)$ on the i-th group of the frame and the w points in its hole on w new points.

Next, each $IKPD_3(\{4, s^*\}, g_i + w, w)$ contains $w - a$ parallel classes of quadruple, where $a = 3$ if $(s, w) = (5, 5)$, $a = 5$ if $(s, w) = (6, 6)$, and $a = w$ if $(s, w) = (5, 3)$ and $(6, 5)$. From union of this with $w - a$ holey parallel classes of the $KPD_3(\{4, s^*\}, g_m + w)$, to form $w - a$ additional parallel classes. There remain g_m parallel classes of the $KPD_3(\{4, s^*\}, g_m + w)$, which can be match arbitrarily with the g_m frame parallel classes of the m-th group to complete the construction.

It is easy to check that this construction gives a three-fold Kirkman packing design with $w - a + \sum_{1\le i\le m} g_i$ resolution classes. □

Lemma 3.2 *If there exists an idempotent $TD[6;t]$, then there exists a $KPD_3(\{4,5^*\}, 20t + 4k + 5)$ for $5 \le k \le t$.*

Proof: We start with the idempotent $TD[6;t]$ and give the $t - k$ points in one group weight 0 and the remaining points weight 1 to obtain a $\{5,6,k,t\}-GDD$ of type 6^k5^{t-k} by Theorem 2.2. And then give the points of the resulting GDD weight 4 to obtain a (4,3)-frame of type 24^k20^{t-k} by Theorem 2.4. The result then follows from Theorem 3.1 with $w = 5$, the input designs we need $IKPD_3(\{4,5^*\}, 25, 5)$ and $IKPD_3(\{4,5^*\}, 29, 5)$ come from Lemma 2.5 and Lemma 2.6 respectively. □

Lemma 3.3 *If there exists an idempotent $TD[7;t]$, then there exists a $KPD_3(\{4,6^*\}, 24t + 4k + 6)$ for $5 \le k \le t$.*

Proof: The proof is similar to Lemma 3.2, the input designs we need $IKPD_3(\{4,6^*\}, 30, 6)$ and $IKPD_3(\{4,6^*\}, 34, 6)$ come from Lemma 2.9 and Lemma 2.10 respectively. □

Lemma 3.4 *If there exists an idempotent $TD[9;t]$, then there exists a $KPD_3(\{4,6^*\}, 32t + 4k + 6)$ for $5 \le k \le t$.*

Proof: The proof is similar to Lemma 3.2, the input designs we need $IKPD_3(\{4,6^*\}, 38, 6)$ and $IKPD_3(\{4,6^*\}, 42, 6)$ come from Lemma 2.9 and Lemma 2.10 respectively. □

We also need the following result which follows from Theorem 3.1.

Lemma 3.5 *If there exist a (4,3)-frame of type g^5m^1, an $IKPD_3(\{4,s^*\}, g + w, w)$ and a $KPD_3(\{4,s^*\}, m+w)$, then there exists a $KPD_3(\{4,s^*\}, 5g + m + w)$.*

3.1 $KPD_3(\{4,5^*\}, v)$

Lemma 3.6 *There is a $KPD_3(\{4,5^*\}, v)$ containing $v - 3$ parallel classes for $v \in \{v \equiv 1 \pmod 4) : 17 \le v \le 97\} \cup \{109\}$.*

Proof: For the cases $v \in \{v \equiv 1 \ (mod\ 4) : 17 \le v \le 45\} \cup \{53\}$, the result comes from Lemmas 2.5 to 2.8 and the fact that the existence of an $IKPD_3(\{4,5^*\}, v, w)$ implies the existence of a $KPD_3(\{4,5^*\}, v)$. For the cases $v = 73$ and 93, the result comes from Lemma 3.5 with $w =$

3, $m = 0$ and $g = 14$ and 18. The input designs we need (4,3)-frames of type g^5 and IKPD$_3(\{4,5^*\}, g+3, 3)$s come from Theorem 2.3 and Lemmas 2.8 respectively. For the other values of v, take the point set $X = Z_t \cup \{\infty_1, \infty_2, \infty_3\}$, $t = v - 3$. The $v - 3$ parallel classes will be generated modulo t from an initial parallel class. The blocks of the initial parallel class for each X are listed in Appendix I. □

Lemma 3.7 *There is a KPD$_3(\{4,5^*\}, v)$ containing $v - 3$ parallel classes for $v \in \{v \equiv 1 \ (mod\ 4) : 101 \leq v \leq 257\}$.*

Proof: For the case $v = 109$, see Lemma 3.6. For the case $v = 101$, we start with the (4,3)-frame of type 14^7 and apply Theorem 3.1 with $w = 3$ to obtain the desired result. The input designs we need (4,3)-frame and IKPD$_3(\{4,5^*\}, 17, 3)$ come from Theorem 2.3 and Lemmas 2.8 respectively. For the cases $v = 113$ and 133, the result comes from Lemma 3.5 with $w = 3$, $m = 0$ and $g = 22$ and 26. The input designs we need (4,3)-frames of type g^5 and IKPD$_3(\{4,5^*\}, g+3, 3)$s come from Theorem 2.3 and Lemmas 2.8 respectively. For the other values of v, the result comes from Lemma 3.5 with $w = 5$ and $g = 4s$, $5 \leq s \leq 10$. The input designs we need (4,3)-frames of type g^5m^1 and IKPD$_3(\{4,5^*\}, g+5, 5)$s come from Theorem 2.3 and Lemmas 2.5 to 2.7 respectively. □

Lemma 3.8 *There is a KPD$_3(\{4,5^*\}, 321)$ containing $v-3$ parallel classes.*

Proof: We start with the $TD(9,9)$ and give the 2 points in one group weight 0 and the remaining points weight 1 to obtain a $\{8,9\} - GDD$ of type $9^8 7^1$ by Theorem 2.2. And then give the points of the resulting GDD weight 4 to obtain a (4,3)-frame of type $36^8 28^1$ by Theorem 2.4. The result then follows from Theorem 3.1 with $w = 5$, the input designs we need IKPD$_3(\{4,5^*\}, 41, 5)$ and KPD$_3(\{4,5^*\}, 33)$ come from Lemma 2.7 and Lemma 3.6 respectively. □

The proof of Theorem 1.2: From Lemma 3.2 with $t \geq 11$ and $t \notin E_1$, we know that the result is true for $v \geq 261$ and $v \neq 321$. For the cases $17 \leq v < 261$ and $v = 321$, we know that the result is true from Lemmas 3.6 to 3.8. □

3.2 KPD$_3(\{4,6^*\}, v)$

Lemma 3.9 *There is a KPD$_3(\{4,6^*\}, v)$ containing $v - 5$ parallel classes for $v \in \{v \equiv 2 \ (mod\ 4) : 26 \leq v \leq 102\}$.*

Proof: For the cases $v \in \{v \equiv 2 \ (mod\ 4) : 26 \leq v \leq 46\} \cup \{54\}$, the result comes from Lemmas 2.9 to 2.11 and the fact that the existence of an IKPD$_3(\{4,6^*\}, v, w)$ implies the existence of a KPD$_3(\{4,6^*\}, v)$. For the other values of v, take the point set $X = Z_t \cup \{\infty_1, \cdots, \infty_5\}$, $t = v-5$. The $v-5$ parallel classes will be generated modulo t from an initial parallel class. The blocks of the initial parallel class for each X are listed in Appendix II. □

Lemma 3.10 *There is a $KPD_3(\{4, 6^*\}, v)$ containing $v-5$ parallel classes for $v \in \{v \equiv 2 \pmod 4) : 106 \leq v \leq 310\}$.*

Proof: For the cases $v \in \{110, 130\}$, the result comes from Lemma 3.5 with $w = 5$, $m = 0$ and $g \in \{21, 25\}$. The input designs we need (4,3)-frames of type g^5 and $\text{IKPD}_3(\{4, 6^*\}, g+5, 5)$s come from Theorem 2.3 and Lemma 2.11 respectively. For the cases $v \in \{106, 114, 118, 122, 134, 138, 142, 162\}$, take the point set $X = Z_t \cup \{\infty_1, \cdots, \infty_5\}$, $t = v - 5$. The $v - 5$ parallel classes will be generated modulo t from an initial parallel class. The blocks of the initial parallel class for each X are listed in Appendix II. For the case $v = 262$, we start with the (4,3)-frame of type 8^8 and apply Theorem 3.1 with $w = 6$ to obtain the desired result. For the other values of v, the result comes from Lemma 3.5 with $w = 6$ and $g = 4s$, $6 \leq s \leq 10$ and $s = 12$. The input designs we need (4,3)-frames of type g^5m^1 and $\text{IKPD}_3(\{4, 6^*\}, g+6, 6)$s come from Lemma 2.3 and Lemmas 2.9 and 2.10 respectively. □

Lemma 3.11 *There is a $KPD_3(\{4, 6^*\}, v)$ containing $v-5$ parallel classes for every $v \in E_3$, where $E_3 = \{v \equiv 2 \pmod 4) : 318 \leq v \leq 334,\ 374 \leq v \leq 406,\ 542 \leq v \leq 574\}$.*

Proof: For the cases $v = 334, 374$ and 406, we start with the $TD(9, 13)$ and give the $13 - k_1, 13 - k_2$ and $13 - k_3$ points in each of the last three groups and the one point in each of the other groups weight 0, respectively, and the remaining points weight 1 to obtain a $\{5, 6, 7, 8, 9\} - GDD$ of type $12^6(k_1)^1(k_2)^1(k_3)^1$ by Theorem 2.2. And then give the points of the resulting GDD weight 4 to obtain a (4,3)-frame of type $48^6(4k_1)^1(4k_2)^1(4k_3)^1$ by Theorem 2.4. The result then follows from Theorem 3.1 with $w = 6$ and $(k_1, k_2, k_3) \in \{(10, 0, 0), (12, 8, 0), (12, 8, 8)\}$, the input designs we need $\text{IKPD}_3(\{4, 6^*\}, 54, 6)$ and $\text{IKPD}_3(\{4, 6^*\}, 38, 6)$ come from Lemma 2.8. For the other values of v, the result comes from Lemma 3.4 with $t \in \{9, 11, 16\}$. □

The proof of Theorem 1.3: From Lemma 3.3 with $t \geq 11$ and $t \notin E_1 \cup E_2$, we know that the result is true for $v \geq 314$ and $v \notin E_3$. For the cases $26 \leq v < 314$ and $v \in E_3$, we know that the result is true from Lemmas 3.9 to 3.11. □

4 Concluding remarks

In this paper, we have proved that there exist $KPD_3(\{4, s^*\}, v)$s for all positive integers $v \equiv s - 4 \pmod 4$ with $v \geq 17$ if $s = 5$, and $v \geq 26$ if $s = 6$. For small order of v, we know that $m(9) = m(10) = 5$ by a computer exhaustive search. It is easy to see that $m(v) = 3$ for $v \in \{5, 6\}$. Therefore, there are 4 values of $v \in \{13, 14, 18, 22\}$ for which the existence of a $KPD_3(\{4, s^*\}, v)$ remains undecided.

References

[1] R.J.R. Abel, personal communication.

[2] R.J.R. Abel, C.J. Colbourn and M. Wojtas, Concerning seven and eight mutually orthogonal latin squares, *J. Combin. Designs* **12** (2004), 123-131.

[3] R.J.R. Abel, G. Ge, M. Greig and A.C.H. Ling, Further results on $(v, 5, w^*, 1)$-PBDs, Preprint.

[4] H. Cao and B. Du, Kirkman packing designs $KPD(\{w, s^*\}, v)$ and related threshold schemes, *Discrete Math.* **281** (2004), 83-95.

[5] H. Cao and L. Zhu, Kirkman packing designs $KPD(\{3, 5^*\}, v)$, *Disigns, Codes and Cryptography* **26** (2002), 127-138.

[6] A. Černý, P. Horák and W.D. Wallis, Kirkman's school projects, *Discrete Math.* **167/168** (1997), 189-196.

[7] C.J. Colbourn and J.H. Dinitz (eds.), *CRC Handbook of Combinatorial Designs*, CRC Press Inc., Boca Raton, 1996.

[8] C.J. Colbourn and A.C.H. Ling, Kirkman school project designs, *Discrete Math.* **203** (1999), 49-60.

[9] B. Du, Two-fold Kirkman packing designs, *Bulletin of the ICA*, to appear.

[10] S.C. Furino, S. Kageyama, A.C.H. Ling, Y. Miao, J. Yin, Frames with block size four and index three, *J. Stat. Plann. Infer.* **106** (2002), 117-124.

[11] G. Ge, Uniform frames with block size four and index one or three, *J. Combin. Designs* **9** (2001), 28-39.

[12] G. Ge, C.W.H. Lam, A.C.H. Ling, Some new uniform frames with block size four and index one or three, *J. Combin. Designs* **12** (2004), 112-122.

[13] G. Ge and A.C.H. Ling, some more 5-GDDs and optimal $(v, 5, 1)$ packings, *J. Combin. Designs* **12** (2004), 132-141.

[14] N.C.K. Phillips, W.D. Wallis and R.S. Rees, Kirkman packing and covering designs, *JCMCC* **28** (1998), 299-325.

[15] R.S. Rees, Group-divisible designs with block size k having $k+1$ groups for $k = 4, 5$, *J. Combin. Designs* **8** (2000), 363-386.

[16] D.R. Stinson, Frames for Kirkman triple systems, *Discrete Math.* **65** (1987), 289-300.

[17] J. Wang, Embeddings of resolvable designs, Ph.D. thesis, Shanghai Jiao Tong University, 2004.

[18] R.M. Wilson, Constructions and uses of pairwise balanced designs, *Math. Centre Tracts* **55** (1974), 18-41.

[19] M. Wojtas, Three new constructions of mutually orthogonal latin squares, *J. Combin. Designs* **8** (2000), 218-220.

[20] Y. Zhang and B. Du, The spectrum of directed Kirkman packing designs, *Applied Math.-JCU* **18** (2003), 325-331.

Appendix I Some $KPD_3(\{4, 5^*\}, v)$s

$IKPD_3(\{4, 5^*\}, v, 3)$ for $v \in \{17, 25, 29\}$.

$v = 17:$	0 1 2 3 7	4 8 12 ∞_1	5 10 13 ∞_2	6 9 11 ∞_3
$v = 25:$	0 1 2 3 7	4 9 13 19	5 11 15 18	
	6 14 17 ∞_1	8 16 21 ∞_2	10 12 20 ∞_3	
$v = 29:$	0 1 2 3 6	4 9 16 20	5 12 14 22	
	7 13 19 24	8 18 21 ∞_1	10 17 25 ∞_2	
	11 15 23 ∞_3			

$KPD_3(\{4, 5^*\}, v)$ for $v \in \{49, 57, 61, 65, 69, 77, 81, 85, 89, 97, 109\}$.

$v = 49:$	0 8 17 27 38	1 2 3 4	5 7 10 13
	6 11 15 29	9 22 34 39	12 26 30 41
	14 25 35 45	16 23 36 42	18 24 31 40
	19 33 37 ∞_1	20 32 44 ∞_2	21 28 43 ∞_3
$v = 57:$	0 10 21 33 46	1 5 11 17	2 6 12 19
	3 7 14 22	4 9 16 24	8 13 38 39
	15 35 37 51	18 31 40 49	20 23 47 48
	25 34 36 53	26 29 45 50	27 41 44 ∞_1
	28 42 43 ∞_2	30 32 52 ∞_3	

$v = 61$:	0 8 17 27 38	1 6 12 19	2 7 13 20
	3 9 16 24	4 14 22 31	5 10 25 34
	11 15 43 45	18 37 41 53	21 44 46 47
	23 39 42 56	26 36 48 50	28 40 54 55
	29 33 49 ∞_1	30 51 52 ∞_2	32 35 57 ∞_3
$v = 69$:	0 11 23 36 50	1 7 14 22	2 8 15 24
	3 9 17 25	4 13 27 37	5 12 29 39
	6 16 28 40	10 19 38 55	18 21 41 59
	20 49 51 54	26 30 56 61	31 42 57 60
	32 33 53 58	34 35 52 63	43 47 62 ∞_1
	44 46 48 ∞_2	45 64 65 ∞_3	
$v = 77$:	0 16 33 51 70	1 13 26 40	2 14 27 41
	3 15 28 43	4 18 34 49	5 20 36 53
	6 23 42 69	7 29 47 71	8 31 52 73
	9 30 63 72	10 32 39 61	11 35 59 66
	12 48 58 68	17 21 57 65	19 22 24 25
	37 38 45 46	44 50 55 ∞_1	54 56 60 ∞_2
	62 64 67 ∞_3		
$v = 81$:	0 14 29 45 62	1 10 20 31	2 11 21 32
	3 12 22 34	4 15 27 39	5 18 33 46
	6 19 35 50	7 24 41 59	8 13 36 42
	9 16 23 48	17 54 68 74	25 43 61 67
	26 49 52 53	28 55 60 63	30 38 70 71
	37 73 75 77	40 44 64 66	47 69 72 ∞_1
	51 56 76 ∞_2	57 58 65 ∞_3	
$v = 85$:	0 16 33 51 70	1 12 24 37	2 13 25 38
	3 14 27 41	4 18 32 47	5 20 35 52
	6 22 39 57	7 23 43 61	8 11 15 34
	9 17 36 42	10 31 55 63	19 28 62 67
	21 29 73 79	26 48 68 74	30 71 72 81
	40 50 77 80	44 45 65 66	46 49 53 75
	54 59 64 ∞_1	56 58 60 ∞_2	69 76 78 ∞_3
$v = 89$:	0 16 33 51 70	1 12 24 37	2 13 25 38
	3 14 26 39	4 18 32 46	5 20 35 50
	6 22 40 59	7 27 44 61	8 11 15 41
	9 17 36 43	10 19 28 57	21 52 60 81
	23 29 64 69	30 34 72 77	31 55 76 79
	42 74 82 84	45 65 66 67	47 48 53 75
	49 56 78 80	54 58 85 ∞_1	62 68 71 ∞_2
	63 73 83 ∞_3		

$v = 97$:	0 17 35 54 74	1 12 24 37	2 13 25 38
	3 14 26 39	4 18 32 46	5 20 36 51
	6 21 40 56	7 23 41 58	8 27 45 69
	9 15 42 48	10 16 44 61	11 19 43 63
	22 29 67 75	28 49 59 81	30 52 57 78
	31 33 71 80	34 55 84 85	47 50 79 88
	53 83 92 93	60 64 68 90	62 65 89 91
	66 73 76 ∞_1	70 72 77 ∞_2	82 86 87 ∞_3

$v = 109$:	0 24 49 75 102	1 17 34 52	2 18 35 53
	3 19 36 55	4 22 41 60	5 25 45 65
	6 27 48 69	7 29 51 73	8 31 54 77
	9 33 57 83	10 37 62 87	11 38 68 103
	12 20 26 56	13 21 28 58	14 23 59 66
	15 44 72 82	16 50 61 63	30 94 98 104
	32 90 95 99	39 70 71 80	40 43 46 74
	42 89 100 101	47 81 86 88	64 67 79 93
	76 78 91 ∞_1	84 85 96 ∞_2	92 97 105 ∞_3

Appendix II Some $KPD_3(\{4, 6^*\}, v)$s

$IKPD_3(\{4, 6^*\}, v, 5)$ for $v \in \{26, 30\}$.

$v = 26$:	0 1 2 3 5 10	4 11 17 ∞_1	6 14 18 ∞_2
	7 13 16 ∞_3	8 12 19 ∞_4	9 15 20 ∞_5

$v = 30$:	0 1 2 3 5 10	7 11 18 23	4 14 20 ∞_1
	6 13 19 ∞_2	8 16 21 ∞_3	9 17 23 ∞_4
	12 15 24 ∞_5		

$KPD_3(\{4, 6^*\}, v)$ for $v \in \{50, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98, 102, 106, 114, 118, 122, 134, 138, 142, 162\}$.

$v = 50$:	0 6 13 21 30 40	1 4 8 14	2 5 9 15
	3 7 25 36	10 19 35 43	11 28 37 42
	12 20 23 39	16 18 38 ∞_1	17 29 34 ∞_2
	22 24 44 ∞_3	26 27 41 ∞_4	31 32 33 ∞_5

$v = 58$:	0 8 17 27 38 50	1 6 12 19	2 7 14 21
	3 9 15 23	4 13 22 32	5 18 34 44
	10 35 39 43	11 16 37 41	20 33 36 51
	24 26 48 ∞_1	25 40 42 ∞_2	28 29 45 ∞_3
	30 31 52 ∞_4	46 47 49 ∞_5	

$v = 62$:	0 9 19 30 42 55	1 7 14 22	2 8 15 24
	3 10 18 26	4 13 33 51	5 17 35 45
	6 12 29 43	11 25 36 56	16 21 38 54
	20 23 47 52	27 31 53 ∞_1	28 32 46 ∞_2
	34 37 39 ∞_3	40 41 44 ∞_4	48 49 50 ∞_5
$v = 66$:	0 10 21 33 46 60	1 7 14 22	2 8 15 23
	3 9 17 26	4 11 20 29	5 28 40 56
	6 18 36 47	12 30 49 59	13 37 41 57
	16 19 38 43	24 27 53 58	25 51 55 ∞_1
	31 48 50 ∞_2	32 52 54 ∞_3	34 35 39 ∞_4
	42 44 45 ∞_5		
$v = 70$:	0 8 17 27 38 50	1 9 18 28	2 10 19 30
	3 13 24 36	4 16 29 44	5 20 33 46
	6 22 41 55	7 25 31 51	11 15 56 62
	12 37 43 59	14 45 48 49	21 35 57 64
	23 26 52 ∞_1	32 34 39 ∞_2	40 42 47 ∞_3
	53 54 58 ∞_4	60 61 63 ∞_5	
$v = 74$:	0 8 17 27 38 50	1 9 18 28	2 10 19 30
	3 13 24 36	4 16 29 42	5 20 33 48
	6 21 35 51	7 11 14 32	12 15 37 44
	22 40 46 62	23 43 45 59	25 26 60 64
	31 63 66 68	34 52 57 ∞_1	39 41 65 ∞_2
	47 61 67 ∞_3	49 55 56 ∞_4	53 54 58 ∞_5
$v = 78$:	0 12 25 39 54 70	1 10 20 31	2 11 21 32
	3 13 24 36	4 16 29 42	5 14 28 43
	6 22 38 52	7 27 44 62	8 30 37 61
	9 26 34 60	15 18 51 68	17 35 41 63
	19 23 47 55	33 40 66 72	45 49 50 ∞_1
	46 48 53 ∞_2	56 57 58 ∞_3	59 64 67 ∞_4
	65 69 71 ∞_5		
$v = 82$:	0 13 27 42 58 75	1 11 22 34	2 12 23 35
	3 14 26 39	4 17 31 45	5 15 30 46
	6 24 40 59	7 25 44 62	8 16 33 53
	9 18 38 60	10 19 36 57	20 65 66 73
	21 68 70 74	28 48 54 55	29 67 72 76
	32 37 71 ∞_1	41 63 69 ∞_2	43 47 50 ∞_3
	49 51 52 ∞_4	56 61 64 ∞_5	
$v = 86$:	0 14 29 45 62 80	1 11 22 34	2 12 23 36
	3 13 24 37	4 16 28 41	5 19 33 48
	6 25 42 59	7 26 44 66	8 15 31 38
	9 17 35 55	10 18 40 49	20 47 52 72
	21 57 61 63	27 43 69 70	30 39 76 79
	32 58 60 64	46 50 77 ∞_1	51 54 56 ∞_2
	53 73 78 ∞_3	65 71 74 ∞_4	67 68 75 ∞_5

$v = 90:$	0 14 29 45 62 80	1 11 22 34	2 12 23 35
	3 13 24 37	4 16 30 43	5 18 32 47
	6 21 38 55	7 25 44 60	8 15 33 39
	9 17 36 52	10 19 41 48	20 40 64 68
	26 70 71 73	27 63 69 72	28 31 67 75
	42 49 51 81	46 54 74 76	50 56 78 ∞_1
	53 79 83 ∞_2	57 77 82 ∞_3	58 59 84 ∞_4
	61 65 66 ∞_5		
$v = 94:$	0 15 31 48 66 85	1 12 24 37	2 13 25 38
	3 14 27 41	4 16 30 44	5 20 35 51
	6 22 39 56	7 26 46 64	8 17 42 49
	9 18 36 45	10 32 40 60	11 19 43 53
	21 23 67 68	28 80 86 88	29 69 70 75
	33 54 82 87	34 71 78 81	47 73 76 77
	50 57 79 ∞_1	52 72 74 ∞_2	55 61 65 ∞_3
	58 63 84 ∞_4	59 62 83 ∞_5	
$v = 98:$	0 16 33 51 70 90	1 12 24 37	2 13 25 38
	3 14 27 41	4 18 30 44	5 20 35 50
	6 22 39 55	7 26 46 63	8 28 49 71
	9 31 52 73	10 15 42 60	11 17 43 72
	19 23 47 81	21 61 67 92	29 56 64 74
	32 40 84 91	34 75 78 80	36 45 76 83
	48 53 54 82	57 59 66 ∞_1	58 62 86 ∞_2
	65 68 69 ∞_3	77 85 87 ∞_4	79 88 89 ∞_5
$v = 102:$	0 16 33 51 70 90	1 13 26 40	2 14 27 41
	3 15 28 43	4 18 34 49	5 20 36 53
	6 23 42 60	7 25 44 66	8 29 50 71
	9 31 54 74	10 17 21 45	11 19 39 61
	12 22 48 56	24 32 73 83	30 59 82 91
	35 38 64 88	37 63 87 93	46 81 86 92
	47 52 85 96	55 57 58 89	62 68 72 ∞_1
	65 94 95 ∞_2	67 69 76 ∞_3	75 79 84 ∞_4
	77 78 80 ∞_5		
$v = 106:$	0 16 33 51 70 90	1 13 26 40	2 14 27 41
	3 15 28 43	4 18 34 49	5 20 36 53
	6 23 42 60	7 25 44 65	8 29 50 73
	9 31 54 74	10 17 21 45	11 19 39 61
	12 22 46 52	24 32 87 95	30 75 79 85
	35 57 83 92	37 63 86 97	38 64 88 93
	47 56 94 99	48 55 89 96	58 67 68 100
	59 62 91 ∞_1	66 69 98 ∞_2	71 72 77 ∞_3
	76 78 80 ∞_4	81 82 84 ∞_5	

$v = 114$:	0 16 33 51 70 90	1 14 28 43	2 15 29 44
	3 17 32 48	4 20 37 54	5 18 36 55
	6 24 45 65	7 27 49 71	8 30 53 74
	9 34 57 78	10 35 58 82	11 38 62 86
	12 19 23 59	13 21 25 56	22 26 73 99
	31 39 91 102	40 46 95 103	41 69 94 101
	42 68 98 108	47 81 88 93	50 52 80 106
	60 61 63 72	64 67 100 105	66 76 104 ∞_1
	75 77 107 ∞_2	79 84 85 ∞_3	83 89 92 ∞_4
	87 96 97 ∞_5		
$v = 118$:	0 18 37 57 78 100	1 16 32 49	2 17 33 50
	3 19 36 54	4 22 41 60	5 20 40 61
	6 26 47 69	7 29 52 75	8 31 55 79
	9 34 58 83	10 35 62 89	11 38 64 90
	12 42 70 96	13 21 27 59	14 23 51 63
	15 24 53 66	25 30 77 88	28 72 82 112
	39 71 99 107	43 87 94 101	44 46 56 86
	45 48 84 92	65 97 108 111	67 68 73 80
	74 76 110 ∞_1	81 85 91 ∞_2	93 95 104 ∞_3
	98 102 103 ∞_4	105 106 109 ∞_5	
$v = 122$:	0 18 37 57 78 100	1 16 32 49	2 17 33 50
	3 19 38 56	4 22 41 61	5 20 40 63
	6 27 48 70	7 29 52 75	8 34 58 83
	9 35 59 84	10 36 60 87	11 39 64 94
	12 42 71 98	13 21 51 65	14 23 55 68
	15 24 53 62	25 30 66 76	26 31 82 96
	28 73 101 113	43 45 72 77	44 47 80 91
	46 107 108 109	54 106 112 116	67 69 81 115
	74 102 110 114	79 85 92 ∞_1	86 89 97 ∞_2
	88 95 99 ∞_3	90 93 103 ∞_4	104 105 111 ∞_5
$v = 134$:	0 18 37 57 78 100	1 16 32 49	2 17 33 50
	3 19 36 54	4 22 41 60	5 20 40 61
	6 26 47 69	7 29 52 75	8 31 55 79
	9 34 58 83	10 35 62 88	11 38 64 90
	12 39 67 95	13 42 70 99	14 23 53 63
	15 24 59 89	21 30 68 82	25 65 96 109
	27 72 86 122	28 81 93 125	43 48 102 113
	44 76 106 112	45 51 56 114	46 108 115 121
	66 74 116 128	71 73 107 110	77 85 119 123
	80 87 91 92	84 124 126 ∞_1	94 97 98 ∞_2
	101 103 111 ∞_3	104 105 118 ∞_4	117 120 127 ∞_5

$v = 138:$	0 22 45 69 94 120	1 20 40 61	2 21 41 62
	3 23 44 66	4 26 49 72	5 24 48 73
	6 30 55 81	7 33 60 87	8 35 63 91
	9 37 67 96	10 39 70 100	11 42 74 104
	12 43 75 107	13 29 46 64	14 31 68 86
	15 32 98 132	16 56 93 111	17 52 88 121
	18 34 71 115	19 27 95 106	25 65 113 124
	28 38 105 119	36 78 112 125	47 57 114 128
	50 59 97 109	51 53 54 58	76 79 84 85
	77 80 90 92	82 118 126 130	83 89 131 ∞_1
	99 103 110 ∞_2	101 102 116 ∞_3	108 117 123 ∞_4
	122 127 129 ∞_5		
$v = 142:$	0 22 45 69 94 120	1 20 40 61	2 21 41 62
	3 23 44 66	4 26 49 72	5 24 48 73
	6 30 55 81	7 33 60 87	8 35 63 91
	9 37 67 96	10 39 68 99	11 42 74 104
	12 43 75 107	13 27 57 92	14 28 64 79
	15 29 65 82	16 31 71 86	17 34 50 90
	18 36 54 70	19 52 89 123	25 38 78 116
	32 76 121 132	46 56 103 110	47 51 127 136
	53 58 119 129	59 105 111 118	77 112 115 124
	80 84 88 122	83 85 95 101	93 98 100 135
	97 102 108 ∞_1	106 109 117 ∞_2	113 114 126 ∞_3
	125 133 134 ∞_4	128 130 131 ∞_5	
$v = 162:$	0 28 57 87 118 150	1 31 62 94	2 7 13 20
	3 39 76 114	4 29 54 80	5 30 56 82
	6 33 60 88	8 32 59 89	9 37 66 95
	10 34 65 97	11 35 69 102	12 45 78 113
	14 48 83 117	15 36 55 74	16 38 58 77
	17 40 61 81	18 41 63 84	19 42 64 100
	21 67 103 123	22 68 109 127	23 70 85 124
	24 71 86 131	25 43 90 132	26 75 79 91
	27 44 104 116	46 53 115 125	47 52 96 156
	49 122 126 139	50 128 134 142	51 93 99 136
	72 143 146 155	73 144 147 152	92 130 140 141
	98 107 111 151	101 110 112 153	105 106 120 ∞_1
	108 145 148 ∞_2	119 121 129 ∞_3	133 135 149 ∞_4
	137 138 154 ∞_5		