# The forcing semi-H-cordial numbers of certain graphs

R. Khoeilar and S.M. Sheikholeslami
Department of Mathematics
Azarbaijan University of Tarbiat Moallem, Tabriz, Iran

Abdollah Khodkar
Department of Mathematics, University of West Georgia
Carrollton, GA 30118

**Abstract**

A labeling $f$ of a graph $G$ is called semi-H-cordial if for each vertex $v$, $|f(v)| \leq 1$, $|e_f(1) - e_f(-1)| \leq 1$ and $|v_f(1) - v_f(-1)| \leq 1$. In this paper we study the forcing semi-$H$-cordial numbers of paths, cycles, stars, trees, Dutch-windmill graphs, wheels, grids and cylinders.

## 1 Introduction

Let $G$ a be graph with vertex set $V(G)$ and edge set $E(G)$. By a *labeling* of a graph $G$ we mean a map $f$ which assigns to each edge of $G$ an element of $\{-1, 1\}$. Let $f$ be a labeling for a graph $G$. For each vertex $v \in V(G)$ we define $f(v)$ to be the sum of the labels of all edges having $v$ as an endpoint. In other words, $f(v) = \sum_{e \in I(v)} f(e)$, where $I(v)$ is the set of all edges incident with $v$. For each integer $k$, $e_f(k)$ is the number of edges having label $k$ and $v_f(k)$ is the number of vertices having label $k$.

A labeling $f$ of a graph $G$ is called *semi-H-cordial* (see [3]) if for each vertex $v$, $|f(v)| \leq 1$, $|e_f(1) - e_f(-1)| \leq 1$ and $|v_f(1) - v_f(-1)| \leq 1$. A graph $G$ is called to be *semi-H-cordial* if it admits a semi-H-cordial labeling.

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The proof of the following lemma is straightforward and is left to the reader.

**Lemma 1** If $f$ is an assignment of integer numbers to the edges and vertices of a given graph $G$ such that $f(v) = \sum_{e \in I(v)} f(e)$ for each vertex $v$, then

$$\sum_{v \in V} f(v) = 2 \sum_{e \in E(G)} f(e).$$

**Lemma 2** Let $G$ be a semi-H-cordial graph with $m$ edges. Then

1. each vertex of even degree has label zero;
2. $\sum_{v \in V_o(G)} f(v) = 2 \sum_{e \in E(G)} f(e)$, where $V_o(G)$ is the set of vertices with odd degree;
3. $m$ is even.

**Proof.** The proofs of Parts 1 and 2 are clear. Here we only prove Part 3. Since $|v_f(1) - v_f(-1)| \leq 1$ and $|V_o(G)|$ is even we must have $v_f(1) = v_f(-1)$. This implies $2 \sum_{e \in E(G)} f(e) = \sum_{v \in V_o(G)} f(v) = 0$ by Part 2. Therefore $m$ is even. ■

A semi-$H$-cordial labeling $f$ for $G$ can also be represented by a set of ordered pairs $S_f = \{(e, f(e)) \mid e \in E(G)\}$. A subset $T$ of $S_f$ is called a *forcing subset* of $S_f$ if $S_f$ is the unique extension of $T$ to a semi-$H$-cordial labeling for $G$. The *forcing semi-$H$-cordial number of $S_f$*, $F_\gamma(S_f)$, is defined by $F_\gamma(S_f) = \min\{|T| : \text{T is a forcing subset of } S_f\}$. The *forcing semi-$H$-cordial number of $G$*, $F_\gamma(G)$, is defined by $F_\gamma(G) = \min\{F_\gamma(S_f) : S_f \text{ is a semi-}H\text{-cordial labeling for } G\}$.

The concept of forcing numbers has been studied in different areas of graph theory, including the chromatic number of a graph [2] and the domination numbers of a graph [1, 5]. For a survey of forcing parameters in graph theory see [4]. In this paper we study the forcing semi-$H$-cordial numbers of certain graphs. In Section 2 we find the forcing semi-$H$-cordial numbers of paths, cycles and stars. We also study the forcing semi-$H$-cordial numbers of trees. In Section 3 we find the forcing semi-$H$-cordial numbers of Dutch-windmill graphs and give an upper bound for the semi-$H$-cordial number of a wheel. In Section 4 we give upper bounds for the forcing semi-$H$-cordial numbers of grids and cylinders.

## 2 Paths, cycles and stars

In this section first we find the forcing semi-$H$-cordial numbers of paths, cycles and stars. Then we find sharp lower and upper bounds for the forcing semi-$H$-cordial number of a graph $G$. Finally, we prove that if $T$ is a tree then $F_\gamma(T) = 1$ if and only if $T$ is a path of odd order. Note that by Part 3 of Lemma 2 a path of even order, an odd cycle and the star $k_{1,n}$ for $n$ odd are not semi-$H$-cordial graphs.

**Lemma 3** For each odd positive integer $n$, $F_\gamma(P_n) = 1$.

**Proof.** Let $P$ be a path of odd order $n$, say $P : v_1, v_2, \ldots, v_n$. Define a labeling $f$ of $G$ by:

$$f(v_i v_{i+1}) = (-1)^{i+1} \text{ for } i = 1, 2, \ldots, n-1.$$

It is easy to see that $f$ is a semi-$H$-cordial labeling for $P_n$. Now let $T = \{(v_1v_2, 1)\}$. Obviously $T$ is a forcing subset of $S_f$. So $F_\gamma(P_n) \le F_\gamma(S_f) = 1$. This implies $F_\gamma(P_n) = 1$, since for every graph $G$ we have $F_\gamma(G) \ge 1$. ■

The proof of the following lemma is similar to that described in Lemma 3.

**Lemma 4** For each even positive integer $n$, $F_\gamma(C_n) = 1$.

**Lemma 5** For each even positive integer $n$, $F_\gamma(K_{1,n}) = \frac{n}{2}$.

**Proof.** Let $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$ and $deg(v_0) = n$. Define the labeling $f$ of $K_{1,n}$ as follows:

$$f(v_0 v_i) = (-1)^i \text{ for } 1 \le i \le n.$$

It is easy to see that $f$ is a semi-$H$-cordial labeling for $K_{1,n}$. Now let $T = \{(v_0v_i, 1) \mid i \text{ is even}\}$. Since $deg(v_0)$ is even we must have $f(v_0) = 0$ by Lemma 2. This implies that the labels of the remaining edges must be $-1$. So $T$ is a forcing subset of $S_f$. Thus $F_\gamma(K_{1,n}) \le F_\gamma(S_f) \le \frac{n}{2}$. Now we show that $F_\gamma(K_{1,n}) \ge \frac{n}{2}$. Let $T$ be a subset of $E(K_{1,n})$ such that each edge in $T$ has label 1 or $-1$ and $|T| < \frac{n}{2}$. Let $n_1$ be the number of edges with label 1 in $T$. Obviously $n_1 < \frac{n}{2}$. Now we can give label 1 to $\frac{n}{2} - n_1$ arbitrary edges of $E(K_{1,n}) \setminus T$ and give label $-1$ to the remaining edges of $E(K_{1,n}) \setminus T$. This is an extension of $T$ to a semi-$H$-cordial labeling of $K_{1,n}$. By construction this extension is not unique. So $F_\gamma(K_{1,n}) \geq \frac{n}{2}$. This completes the proof. ■

**Theorem 6** For a graph $G$, $1 \leq F_\gamma(G) \leq \frac{|E(G)|}{2}$. Furthermore these bounds are sharp.

**Proof.** Obviously $1 \leq F_\gamma(G)$. Now we prove $F_\gamma(G) \leq \frac{|E(G)|}{2}$. Let $f$ be a semi-$H$-cordial labeling for $G$. We have $|e_f(1) - e_f(-1)| \leq 1$. This implies $e_f(1) = e_f(-1)$, since $|E(G)|$ is even by Lemma 2. Now it is clear that $T = \{(e, f(e)) \mid f(e) = 1\}$ is a forcing subset of $S_f$. So $F_\gamma(S_f) \leq \frac{|E(G)|}{2}$. This implies $F_\gamma(G) \leq \frac{|E(G)|}{2}$.

The graphs $C_n$ and $K_{1,n}$, for even $n$, serve to show that these bounds are sharp. ■

**Theorem 7** Let $n$ be even and $1 \leq k \leq \frac{n}{2}$. Then there exists a graph $G_k$ with $n$ edges such that $F_\gamma(G_k) = k$.

**Proof.** For each $1 \leq k \leq \frac{n}{2}$, let $G_k = (V_k, E_k)$, where $V_k = \{v_0, v_1, \ldots, v_n\}$ and $E_k = \{v_0v_{n-2k+2}, v_0v_{n-2k+3}, \ldots, v_0v_n\} \cup \{v_iv_{i+1} \mid 0 \leq i \leq n-2k\}$. Define a labeling $f_k$ of $G_k$ as follows:

$$\begin{cases} f_k(v_0v_j) = (-1)^j & \text{if } \ j = n-2k+2, \ldots, n; \\ f_k(v_jv_{j+1}) = (-1)^{j+1} & \text{if } \ 0 \leq j \leq n-2k. \end{cases}$$

Obviously $f_k$ is a semi-$H$-cordial labeling for $G_k$.

Now let $T_k = \{(v_0v_j, 1) \mid n-2k+2 \leq j \leq n \text{ and } j \text{ is even}\}$. It is clear that $T_k$ is a forcing subset for $S_{f_k}$. So $F_\gamma(G_k) \leq k$. Now an argument similar to that described in the proof of Lemma 5 shows that $F_\gamma(G_k) \geq k$. This completes the proof. ■

The following result shows that if the forcing semi-$H$-cordial number of a tree is one then the tree must be a path of odd order.

**Theorem 8** For every tree $T$, $F_\gamma(T) = 1$ if and only if $T$ is a path of odd order.

**Proof.** Let $T$ a be tree and $F_\gamma(T) = 1$. By Lemma 2 Part 3 we see that $|E(T)|$ is even. Without loss of generality we can assume that for some $e \in E(T)$, $\{(e, 1)\}$ is a forcing subset for a semi-$H$-cordial labeling of $T$. We consider two cases.

**Case 1.** There exists a maximal even path, say $P : u_1u_2 \ldots u_n$, which contains $e$.
If $T = P$ then the proof is complete. Now let $T \neq P$. Let $e = u_iu_{i+1}$ for some $1 \leq i \leq n-1$. We define

$$g(u_su_{s+1}) = \begin{cases} (-1)^s & \text{if } \ 1 \leq s \leq n-1 \ \text{ and } i \text{ is even} \\ (-1)^{s+1} & \text{if } \ 1 \leq s \leq n-1 \ \text{ and } i \text{ is odd.} \end{cases}$$

Note that $g(e) = 1$ for both $i$ even and odd. Now we extend $g$ to a semi-$H$-cordial labeling $f$ for T using the following algorithm.

**Algorithm** Let $f(e') = g(e')$ for $e' \in E(P)$. Define the three variables $S$, $A$ and $a$ by $S = A = E(T) \setminus E(P)$ and $a = 1$. We update $S$, $a$ and $f$ using the following loop.

*while* $(S \neq \emptyset)$
1. Suppose that $e_1, e_2, \ldots, e_p$ is the longest path in S. For each $1 \leq i \leq p$, define $f(e_i)$ to be $(-1)^i a$ and then delete $e_i$ from $S$.
2. Let $b = \sum_{e \in E(T)\setminus S} f(e)$. If $b \neq 0$ then set $a = b$, otherwise set $a = 1$.
*end while.*

We claim that $f$ is a semi-$H$-cordial labeling for T. First note that after each iteration we have $a \in \{-1, 1\}$, because in the $k$-th iteration if $p$ is even, then $f(e_1)+\ldots+f(e_p) = 0$ and $\sum_{e \in A\setminus S} f(e)$ does not change. Otherwise we have $f(e_1) + \ldots + f(e_p) = -a$ and $\sum_{e \in A\setminus S} f(e)$ changes to 0 or $-a$. So for each

$e \in E(T)$ we have $f(e) \in \{-1, 1\}$. Now if the edges incident with a vertex $v$ are completely removed from $S$ in the $k$-th iteration, then we have $f(v) = 0$ before the $k$-th iteration and $|f(v)| \le 1$ after the $k$-th iteration until $S = \emptyset$. On the other hand we see that $\sum_{v \in V(T)} f(v) = 2\sum_{e \in E(T)} f(e) = 0$. So $v_f(-1) = v_f(1)$.

If initially we start with $S = A = E(T) \setminus E(P)$ and $a = -1$ then we obtain another extension of $g$ for $T$ which is a contradiction.

**Case 2.** Each maximal path of $T$ containing $e$ is odd.
Since $T$ is semi-$H$-cordial it follows that $T$ has at least two maximal odd paths containing $e$. Let $P : u_1u_2 \dots u_m$ be a maximal odd path of $T$ containing $e$ and $e = u_iu_{i+1}$ for some $1 \le i \le m-1$. Obviously $P - e$ has even edges. We consider two subcases.

**Subcase 2.1.** $P - e$ is connected. Then one of the endpoints of $e$ has degree 1. Now it is clear that for each maximal odd path $Q$ containing $e$, $Q - e$ is connected.

**Subcase 2.2.** $P - e$ is disconnected. Then both components of $P - e$ have the same parity, since $P$ is an odd path. Now it is clear that for each maximal odd path $Q \neq P$ containing $e$, $Q - e$ is disconnected by Subcase 2.1. Moreover, the components of $P - e$ and $Q - e$ have the same parity, otherwise we have a maximal even path of $T$ containing $e$, which is a contradiction.

Now let $Q : w_1w_2 \dots w_r$ be another maximal odd path of $T$ containing $e$ and $e = w_iw_{i+1}$ for some $1 \le i \le r-1$. Let $m_1$ and $m_2$ be the smallest and the largest subscripts such that $w_{m_1}, w_{m_2} \in V(P)$. Define a labeling $g_1$ on $E(P \cup Q)$ as follows.

$$g_1(u_su_{s+1}) = \begin{cases} (-1)^s & \text{if } 1 \le s \le m-1 \text{ and } i \text{ is even} \\ (-1)^{s+1} & \text{if } 1 \le s \le m-1 \text{ and } i \text{ is odd.} \end{cases}$$

and

$$g_1(w_tw_{t+1}) = \begin{cases} (-1)^{t+1} & \text{if } 1 \le t \le m_1 - 1,\ m_2 \le t \le r-1, i \text{ is even} \\ (-1)^t & \text{if } 1 \le t \le m_1 - 1,\ m_2 \le t \le r-1, i \text{ is odd.} \end{cases}$$

If $|E(P) \cap E(Q)|$ is even then $\sum_{e \in P \cup Q} g_1(e) = 0$ otherwise $\sum_{e \in P \cup Q} g_1(e) \in \{-1, 1\}$. Using the algorithm described in Case 1, with $S = A = E(T) \setminus E(P \cup Q)$ and $a = \sum_{e \in P \cup Q} g_1(e)$ if $\sum_{e \in P \cup Q} g_1(e) \neq 0$, otherwise $a = 1$, we can find an extension of $g_1$ to a semi-$H$-cordial labeling for $T$.

Again define a labeling $g_2$ on $E(P \cup Q)$ as follows.

$$g_2(e) = \begin{cases} g_1(e) & \text{if } e \in P \cap Q \\ -g_1(e) & \text{if } e \notin P \cap Q. \end{cases}$$

As above we can extend $g_2$ to a semi-$H$-cordial labeling for $T$. Thus we have two semi-$H$-cordial labelings of $T$ containing $\{(e,1)\}$ which is a contradiction.

Conversely, if $T$ is a path of odd order then the result follows by Lemma 3.
■

# 3 The Dutch-windmill graphs and wheels

The Dutch-windmill graph, $K_3^{(m)}$, is a graph which consists of $m$ copies of $k_3$ with a vertex in common. The wheel, $W_n$, is a graph with $n+1$ vertices $\{v_0, v_1, \ldots, v_n\}$ and edges $\{v_0v_i \mid 1 \leq i \leq n\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. In this section we find the forcing semi-$H$-cordial number of $K_3^{(m)}$ and give an upper bound for the forcing semi-$H$-cordial number of $W_n$.

**Lemma 9** $K_3^{(m)}$ is semi-$H$-cordial if and only if $m$ is even.

**Proof.** Let $K_3^{(m)}$ be a semi-$H$-cordial graph and let $f$ be a semi-$H$-cordial labeling for $K_3^{(m)}$. Since $\deg(v)$ is even for each $v \in V(G)$, we must have $f(v) = 0$ by Theorem 2. Now we have $2\sum_{e \in E(K_3^{(m)})} f(e) = \sum_{v \in V(K_3^{(m)})} f(v) = 0$. This forces $e_f(1) = e_f(-1)$. Therefore $|E(K_3^{(m)})|$ is even. Hence $m$ is even.

Conversely, let $m$ be even. Assume $v, u_i, w_i$ are the vertices of the $i$th copy of $K_3$ in $K_3^{(m)}$ ($v$ is the common vertex). Define a labeling $f$ of $K_3^{(m)}$ as follows:

$$f(e) = \begin{cases} 1 & \text{if} \quad e = vu_i, vw_i \text{ and } 1 \le i \le \frac{m}{2}; \\ 1 & \text{if} \quad e = u_iw_i \text{ and } \frac{m}{2}+1 \le i \le m; \\ -1 & \text{if} \quad e = vu_i, vw_i \text{ and } \frac{m}{2}+1 \le i \le m; \\ -1 & \text{if} \quad e = u_iw_i \text{ and } 1 \le i \le \frac{m}{2}. \end{cases}$$

It is easy to see that $f$ is a semi-$H$-cordial labeling for $K_3^{(m)}$. This completes the proof. ■

**Theorem 10** For every even positive integer $m$, $F_\gamma(K_3^{(m)}) = \frac{m}{2}$.

**Proof.** If $m = 2$ the result is trivial. Let $m \ge 4$ and let $f$ be the semi-$H$-cordial labeling for $K_3^{(m)}$ as described in Lemma 9. It is easy to see that $T = \{(u_iw_i, 1) \mid \frac{m}{2} + 1 \le i \le m\}$ is a forcing subset for $S_f$. So we have $F_\gamma(K_3^{(m)}) \le \frac{m}{2}$. Now we show that $F_\gamma(K_3^{(m)}) \ge \frac{m}{2}$. Let $E$ be a subset of edges with $|E| < \frac{m}{2}$. Without loss of generality we can assume $E$ does not intersect the first and the second copies of $K_3$. Let $a_e \in \{1, -1\}$ be the label of $e \in E$. Define $S = \{(e, a_e) \mid e \in E\}$, $S_1 = S \cup \{(u_1w_1, 1), (u_2w_2, -1)\}$ and $S_2 = S \cup \{(u_1w_1, -1), (u_2w_2, 1)\}$. Since $|S| < \frac{m}{2}$ it is easy to extend $S_1$ (and $S_2$) to a semi-$H$-cordial labeling of $K_3^{(m)}$. This implies that $S$ is not a forcing subset for any semi-$H$-cordial labeling of $K_3^{(m)}$. Therefore, $F_\gamma(K_3^{(m)}) \ge \frac{m}{2}$. This completes the proof. ■

**Lemma 11** For each $n \ge 3$, $W_n$ is a semi-$H$-cordial graph.

**Proof.** If $n$ is even define a labeling $f$ of $W_n$ as follows:

$$f(e) = \begin{cases} 1 & \text{if} \quad e = v_0v_i, v_iv_{i+1},\ 1 \le i \le n \text{ and } i \text{ is odd}; \\ -1 & \text{otherwise}. \end{cases}$$

Obviously, $f$ is a semi-$H$-cordial labeling of $W_n$. If $n$ is odd define a labeling $g$ of $W_n$ as follows:

$$g(e) = \begin{cases} 1 & \text{if} \quad e = v_0v_1, v_0v_i, v_iv_{i+1},\ 1 \le i \le n \text{ and } i \text{ is even}; \\ -1 & \text{otherwise}. \end{cases}$$

Obviously $g$ is a semi-$H$-cordial labeling of $W_n$. So the result follows. ■

**Theorem 12** For each $n \geq 3$, $F_\gamma(W_n) \leq \lfloor \frac{n+3}{2} \rfloor$.

**Proof.** If $n$ is even we define $T_1 = \{(v_1v_2, 1), (v_0v_{2i-1}, 1) \mid 1 \leq i \leq \frac{n}{2}\}$. Obviously $T_1$ has a unique extension to a semi-$H$-cordial labeling for $W_n$. If $n$ is odd we define $T_2 = \{(v_0v_1, 1), (v_{n-1}v_n, 1), (v_0v_{2i}, 1) \mid 1 \leq i \leq \frac{n-1}{2}\}$. Obviously $T_2$ has a unique extension to a semi-$H$-cordial labeling for $W_n$. Since $|T_1| = |T_2| = \lfloor \frac{n+3}{2} \rfloor$ it follows that $F_\gamma(W_n) \leq \lfloor \frac{n+3}{2} \rfloor$. ■

# 4 Grids and cylinders

In this section we find upper bounds for the forcing semi-$H$-cordial numbers of grids and cylinders. Throughout this section we assume $m, n \geq 2$ and the vertices of the $i$-th copy of $P_n$ in grid $P_n \times P_m$ (cylinder $P_n \times C_m$) are $u_1^i, u_2^i, u_3^i, \ldots, u_n^i$ for $i = 1, 2, \ldots, m$ (see Figure 1).

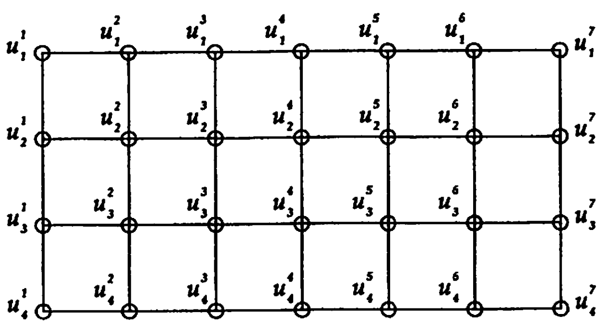

Figure 1: $P_4 \times P_7$

**Theorem 13** $P_n \times P_m$ is a semi-H-cordial graph if and only if $m + n$ is even.

**Proof.** Let $P_n \times P_m$ be a semi-H-cordial graph. By lemma 2 we see that $[2mn - (m + n)]$, the number of edges of $P_n \times P_m$, is even. This implies that $m + n$ is even.

Conversely, let $m+n$ be even. We show that $P_n \times P_m$ admits a semi-H-cordial labeling. Define $f : E(P_n \times P_m) \to \{-1,1\}$ by:

$$\begin{array}{lll} f(u_i^j u_i^{j+1}) = (-1)^{j+1} & \text{if} & i=1; \\ f(u_i^j u_i^{j+1}) = (-1)^{i+j+1} & \text{if} & i \geq 2; \\ f(u_i^j u_{i+1}^j) = (-1)^i & \text{if} & j=1; \\ f(u_i^j u_{i+1}^j) = (-1)^{i+j} & \text{if} & j \geq 2. \end{array}$$

It is easy to see that $f$ is a semi-H-cordial labeling for $P_n \times P_m$. ∎

**Lemma 14** $F_\gamma(P_2 \times P_m) \leq m-1$ for each even positive integer $m$.

**Proof.** Let $f$ be the semi-H-cordial labeling for $P_2 \times P_m$ as in Theorem 13. Define

$$T = \{(u_1^j u_2^j, (-1)^{j+1}) \mid 2 \leq j \leq m-1\} \cup \{(u_2^{m-1} u_2^m, 1)\}.$$

We show that $T$ is a forcing semi-H-cordial for $S_f$. Let $g : E(P_2 \times P_m) \to \{-1,1\}$ be a function such that $S_g$ is an extension of $T$ to a semi-H-cordial labeling for $P_2 \times P_m$. We prove $g = f$. Note that, by Lemma 2, $g(v) = 0$ if $\deg(v)$ is even and, by definition, $g(v) = \pm 1$ if $\deg(v)$ is odd. This forces $g(u_1^j u_1^{j-1}) = g(u_2^j u_2^{j-1}) = (-1)^j$ for $j = m, m-1, \ldots, 2$ and $g(u_1^1, u_2^1) = -1$. Thus $g = f$. Therefore $F_\gamma(P_2 \times P_m) \leq F_\gamma(S_f) \leq m-1$. ∎

**Lemma 15** Let $m, n \geq 3$ and let $m+n$ be even. Then $F_\gamma(P_n \times P_m) \leq n+m-3$.

**Proof.** Let $n \leq m$ (the case $m \leq n$ is similar). Let $f$ be the semi-H-cordial labeling for $P_n \times P_m$ as in Theorem 13. Define

$$\begin{array}{rcl} T & = & \{(u_2^1 u_2^2, 1)\} \cup \{(u_i^{m-n+i-1} u_i^{m-n+i}, 1) \mid 3 \leq i \leq n\} \cup \\ & & \{(u_2^j u_3^j, (-1)^j) \mid 2 \leq j \leq m-n+2\} \cup \\ & & \{(u_i^j u_{i+1}^j, 1) \mid 3 \leq i \leq n-1 \text{ and } j = m-n+i\}. \end{array}$$

Now let $g : E(P_n \times P_m) \to \{-1,1\}$ be a function such that $S_g$ is an extension of $T$ to a semi-H-cordial labeling for $P_n \times P_m$. By Lemma 2, $g(v) = 0$ if $\deg(v)$ is even. This forces a unique labeling for all edges which are incident with at least one vertex of degree four. On the other hand, $g(v) = \pm 1$ if $\deg(v)$ is odd. This implies $g(u_i^m u_{i-1}^m) = (-1)^{m+i+1}$ for $i = n, n-1, \ldots, 2$ and $g(u_1^j u_1^{j-1}) = (-1)^j$ for $j = m, m-1, \ldots, 2$, respectively. Now we must have $g(u_n^j u_n^{j-1}) = (-1)^{n+j}$ for $j = m-1, \ldots, 2$ and $g(u_i^1 u_{i-1}^1) = (-1)^{i-1}$ for $i = n, n-1, \ldots, 2$, respectively. Thus $g = f$. Therefore $F_\gamma(P_n \times P_m) \leq F_\gamma(f) \leq n+m-3$. ■

By Lemmas 14 and 15 we obtain the following result.

**Theorem 16** Let $m, n \geq 2$ and let $m+n$ be even. Then $F_\gamma(P_n \times P_m) \leq n+m-3$.

Now we find an upper bound for $F_\gamma(P_r \times P_s \times P_2)$. We assume that the vertices of the $k$-th copy of $P_r \times P_s$ in $P_r \times P_s \times P_2$ are

$$u_{1,1}^k, u_{1,2}^k, \ldots, u_{1,s}^k, u_{2,1}^k, u_{2,2}^k, \ldots, u_{2,s}^k, \ldots, u_{r,1}^k, u_{r,2}^k, \ldots, u_{r,s}^k$$

for $k = 1, 2$.

**Lemma 17** Let $G = P_r \times P_s \times P_2$. Then $G$ is a semi-$H$-cordial graph if and only if either $r$ or $s$ is even.

**Proof.** Let $G$ be a semi-$H$-cordial graph. Then $5rs - 2r - 2s$, the number of edges of $G$, must be even by Theorem 2. This forces either $r$ or $s$ to be even.

Conversely, let $s$ be even (the case $r$ even is similar). Define the mapping $f : E(G) \longrightarrow \{-1, 1\}$ by:

$$\begin{array}{rcll}
f(u_{i,j}^k u_{i+1,j}^k) & = & (-1)^{i+j} & \text{if } 1 \leq i \leq r-1,\ 1 \leq j \leq s,\ k = 1,2; \\
f(u_{i,j}^k u_{i,j+1}^k) & = & (-1)^{i+j} & \text{if } 1 \leq i \leq r,\ 1 \leq j \leq s-1,\ k = 1,2; \\
f(u_{i,j}^1 u_{i,j}^2) & = & (-1)^{i+j+1} & \text{if } 1 \leq i \leq r-1,\ 1 \leq j \leq s-1; \\
f(u_{r,j}^1 u_{r,j}^2) & = & (-1)^{r+j} & \text{if } 2 \leq j \leq s; \\
f(u_{i,s}^1 u_{i,s}^2) & = & (-1)^{i+s} & \text{if } 1 \leq i \leq r;
\end{array}$$

and $f(u_{r,1}^1 u_{r,1}^2) = (-1)^r$. It is straightforward to see that $f$ is a semi-$H$-cordial labeling for $G$. ■

**Theorem 18** Let $r$ or $s$ be even. Then $F_\gamma(P_r \times P_s \times P_2) \leq (r-1)(s-1)+7$.

**Proof.** Without loss of generality we can assume $s$ is even. Consider the semi-$H$-cordial labeling $f$ as described in Lemma 17. Define

$$\begin{aligned} T \;=\; & \{(u^k_{1,1}u^k_{1,2}, f(u^k_{1,1}u^k_{1,2})), (u^k_{1,1}u^k_{2,1}, f(u^k_{1,1}u^k_{2,1})),\\ & (u^k_{1,s}u^k_{2,s}, f(u^k_{1,s}u^k_{2,s})) \mid k = 1,2\} \cup \\ & \{(u^1_{1,s}u^2_{1,s}, f(u^1_{1,s}u^2_{1,s})), (u^1_{i,1}u^2_{i,1}, f(u^1_{i,1}u^2_{i,1})) \mid 2 \le i \le r\} \cup \\ & \{(u^1_{i,j}u^2_{i,j}, f(u^1_{i,j}u^2_{i,j})) \mid 1 \le i \le r-1,\ \ 2 \le j \le s-1\}. \end{aligned}$$

Obviously $|T| = (r-1)(s-1)+7$. It is easy to see that $S_f$ is the unique extension of $T$ to a semi-$H$-cordial labeling for $G$. This completes the proof. ■

Now we find an upper bound for $F_\gamma(P_n \times C_m)$. In what follows we assume $u_i^{m+1}$ is the same as $u_i^1$ for all $i$.

**Theorem 19** Let $n \ge 2$ and $m \ge 3$. Then $P_n \times C_m$ is a semi-H-cordial graph if and only if $m$ is even.

**Proof.** Let $P_n \times C_m$ be a semi-H-cordial graph. By lemma 2 we see that $[(n-1)m + mn]$, the number of edges of $P_n \times C_m$, is even. This implies that $m$ is even.

Conversely, let $m$ be even. We show that $P_n \times C_m$ admits a semi-H-cordial labeling. Define $f : E(P_n \times C_m) \to \{-1, 1\}$ by:

$$\begin{aligned} & f(u_i^j u_i^{j+1}) = (-1)^{i+j} && \text{if} \quad 1 \le i \le n, \quad 1 \le j \le m; \\ & f(u_i^j u_{i+1}^j) = (-1)^{i+j+1} && \text{if} \quad 1 \le i \le n-1, \quad 1 \le j \le m. \end{aligned}$$

It is easy to see that $f$ is a semi-H-cordial labeling for $P_n \times C_m$. ■

**Theorem 20** Let $n \ge 2$, $m \ge 3$ and let $m$ be even. Then

$$F_\gamma(P_n \times C_m) \le \begin{cases} n+m & \text{if} \quad n < m; \\ 2n & \text{if} \quad n = m; \\ 2n-1 & \text{if} \quad n > m. \end{cases}$$

**Proof.** Let $f$ be the semi-H-cordial labeling for $P_n \times C_m$ as described in Theorem 19. We consider three cases.

**Case 1.** $n < m$. Define

$$\begin{aligned} T &= \{(u_1^1 u_2^1, -1)\} \cup \{(u_i^i u_i^{i+1}, 1) \mid 1 \le i \le n\} \cup \\ &\quad \{(u_i^{i+1} u_{i+1}^{i+1}, 1) \mid 1 \le i \le n-1\} \cup \\ &\quad \{(u_{n-1}^j u_n^j, (-1)^{n+j}) \mid n+1 \le j \le m\}. \end{aligned}$$

Note that $|T| = n + m$.

**Case 2.** $n = m$. Define

$$\begin{aligned} T &= \{(u_1^1 u_2^1, -1), (u_n^n u_n^1, 1)\} \cup \{(u_i^i u_i^{i+1}, 1) \mid 1 \le i \le n-1\} \cup \\ &\quad \{(u_i^{i+1} u_{i+1}^{i+1}, 1) \mid 1 \le i \le n-1\}. \end{aligned}$$

Note that $|T| = 2n$.

**Case 3.** $n > m$. Let $n = km + r$, where $k \ge 0$ and $0 \le r < m$. Define

$$\begin{aligned} T &= \{(u_{sm+i}^i u_{sm+i}^{i+1}, 1) \mid 0 \le s \le k-1,\ 1 \le i \le m-1\} \cup \\ &\quad \{(u_{sm+i}^{i+1} u_{sm+i+1}^{i+1}, 1) \mid 0 \le s \le k-1,\ 1 \le i \le m-1\} \cup \\ &\quad \{(u_{km+i}^{i+1} u_{km+i+1}^{i+1}, 1) \mid 1 \le i \le r-1\} \cup \\ &\quad \{(u_{km+i}^i u_{km+i}^{i+1}, 1) \mid 1 \le i \le r\} \cup \\ &\quad \{(u_{ms}^m u_{ms}^1, 1) \mid 1 \le s \le k\} \cup \{(u_{ms}^1 u_{ms+1}^1, 1) \mid 1 \le s \le k\}. \end{aligned}$$

Note that $|T| = 2n - 1$.

Now let $g : E(P_n \times C_m) \to \{-1, 1\}$ be a function such that $S_g$ is an extension of $T$ to a semi-H-cordial labeling for $P_n \times C_m$. By Lemma 2, $g(v) = 0$ if $\deg(v)$ is even. This forces a unique labeling for all edges which are incident with at least one vertex of degree four. On the other hand, $g(v) = \pm 1$ if $\deg(v)$ is odd. This implies $g(u_i^j u_i^{j+1}) = (-1)^{i+j}$ for $i = 1, n$ and $1 \le j \le m$. Thus $g = f$ and the result follows. ■

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