INDEPENDENCE AND RELATED PROPERTIES FOR INTEGER-INTERVAL GRAPHS AND THEIR COMPLEMENTS

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1. Introduction.

For any positive integer n, let $I_n = [0, n]$. We define the integer-interval graph G_n on I_n as follows:

(Vertices) If $0 \leq m < p \leq n$, with m, p integers, then there is a vertex v in G corresponding to the closed interval $[m, p]$.

(Edges) For integers i, j, m, p, with $0 \leq i < j \leq n$ and $0 \leq m < p \leq n$, there is an edge joining the vertices determined by $[i, j]$ and $[m, p]$ if these closed intervals have a nonempty intersection.

Throughout this discussion we use $[m, p]$, $0 \leq m < p \leq n$, to represent either the closed interval or the vertex that corresponds to it. The context will indicate the appropriate meaning.

If $m, n \in \mathbf{Z}$, with $n > 0$, the integer-interval graph for $[m, m+n]$ is isomorphic to the graph determined by $[0, n]$. Consequently, we restrict our attention to closed intervals with left endpoint 0.

For any undefined terms the reader should see [1] or [5].

2. Independence in G_n.

In an undirected graph $G = (V, E)$, a subset W of V is called *independent* if for any $x, y \in W$ there is no edge $\{x, y\}$ in E. Given $a, b, c, d \in \mathbf{Z}$ with $0 \leq a < b < c < d \leq n$, $\{[a, b], [c, d]\}$ is an independent set of size 2 for the graph G_n.

In the graph G_n there are $\binom{n+1}{2} = v$ vertices. Let e denote the number of edges in G_n. Our first result determines e in terms of n.

Theorem 1. *In the integer-interval graph G_n, the number of edges, e, is given by* $e = (1/12)(n-1)(n)(n+1)(n+4)$.

Proof: In any undirected graph $G = (V, E)$ with $|V| = v$ and $|E| = e$, the number of independent subsets of V of size 2 is $\binom{v}{2} - e$. Applying this idea to G_n, there are $\binom{n+1}{4}$ independent subsets of two vertices, so the number of edges in G_n satisfies $\binom{v}{2} - e = \binom{n+1}{4}$, where $v = \binom{n+1}{2}$. It follows that $e = (\frac{1}{12})(n-1)(n)(n+1)(n+4)$.

[This formula for the number of edges in G_n was derived by recursion in Theorem 1 of [6].]

Consequently, in $\overline{G}_n$, the complement of G_n, there are $\binom{n+1}{2}$ vertices and $\binom{n+1}{4}$ edges.

We now use the result of Theorem 1 to establish a certain property for G_n. In [6] it was observed that for all $n \geq 1$, G_n is a Hamiltonian graph. The following extends this property.

An undirected graph $G = (V, E)$ is called *pancyclic* if G contains a cycle of length ℓ for all $3 \leq \ell \leq |V|$. In addition, we have the following result due to J.A. Bondy [3]: Let $G = (V, E)$ be an undirected Hamiltonian graph where $|V| = v$ and $|E| = e$, and $e \geq (1/4)v^2$. Then either G is pancyclic, or v is even and G is isomorphic to the bipartite graph $K(v/2, v/2)$.

These concepts lead to the following result.

Theorem 2. *For $n \geq 1$, the integer-interval graph G_n is pancyclic.*

Proof: For $n = 1$, G_1 is an isolated vertex and the result is immediate, so let $n \geq 2$.

Since G_2 is isomorphic to K_3, an odd cycle, and G_2 is a subgraph of G_n for all $n \geq 3$, G_n is not bipartite for $n \geq 2$. Hence G_n will be pancyclic for $n \geq 2$ if $e \geq (1/4)v^2$, and this follows easily from Theorem 1.

Returning now to the property of independence in G_n we make the following observations:

(1) Let $k, n \in \mathbf{Z}$, $n > 0$, and $0 \leq 2k \leq n+1$. If $0 \leq a_1 < a_2 < \ldots < a_{2k} \leq n+1$, with $a_i \in \mathbf{Z}$, $1 \leq i \leq 2k$, then $\{[a_1, a_2], [a_3, a_4], \ldots, [a_{2k-1}, a_{2k}]\}$ is an independent set of k vertices in G_n. There are $\binom{n+1}{2k}$ such independent subsets.

(2) For $n \in \mathbf{Z}^+$, n odd, there is a unique maximal independent set $\{[0,1], [2,3], [4,5], \ldots, [n-1, n]\}$ in G_n and the independence number $\beta(G_n)$ is $(\frac{1}{2})(n+1)$.

When n is even there are $\binom{n+1}{n} = n+1$ maximal independent sets of size $n/2 = \beta(G_n)$. These $n+1$ sets are determined as follows:

(i) There are $(n/2)+1$ sets containing $(n/2)$ vertices corresponding to intervals of length 1. (ii) The remaining $(n/2)$ sets each contain exactly one vertex determined by an interval of length 2; the other $(n/2) - 1$ vertices correspond to intervals of unit length. [For each odd number m, $1 \leq m \leq n-1$, the subset $\{0, 1, 2, \ldots, n-1, n\} - \{m\}$ results in the independent set $\{[0,1], [2,3], \ldots, [m-1, m+1], [m+2, m+3], \ldots, [n-1, n]\}$.]

In general, $\beta(G_n) = \lceil \frac{n}{2} \rceil$, $n \in \mathbf{Z}^+$.

(3) For an undirected graph $G = (V, E)$, the *Fibonacci number* of G counts the total number of independent subsets of V, including the empty set. Here the total number of independent subsets is $\binom{n+1}{0} + \binom{n+1}{2} + \binom{n+1}{4} + \ldots + \binom{n+1}{n+1} = 2^n$, for G_n, when n is odd.

When n is even the total number of independent sets in G_n is $\binom{n+1}{0} + \binom{n+1}{2} + \binom{n+1}{4} + \ldots + \binom{n+1}{n} = 2^n$.

So the Fibonacci number of G_n is 2^n for all $n \in \mathbf{Z}^+$.

(4) Given an undirected graph $G = (V, E)$, the *vertex covering number* of G, denoted $\alpha(G)$, is the size of a smallest subset S of V where each edge of G is incident with at least one vertex of S. The following result, due to T. Gallai [4], now determines $\alpha(G_n)$ for the integer-interval graph G_n.

For any undirected graph $G = (V, E)$ with no isolated vertices,

$$\alpha(G) + \beta(G) = |V|.$$

From the result in observation (2) it follows that

$$\alpha(G_n) = \binom{n+1}{2} - \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n^2-1}{2} \right\rceil = \begin{cases} n^2/2, & n \text{ even} \\ (n^2-1)/2, & n \text{ odd.} \end{cases}$$

(5) Related to the invariants in (4) one finds the following for an undirected graph $G = (V, E)$. A set F of edges in G is called *(edge) independent* if for any $e_1, e_2 \in F$ there is no common vertex. The size of a largest such set F is called the *edge independence number* of G and is denoted $\beta_1(G)$. When G has no isolated vertices we define an *edge cover* of G as a subset H of E such that for all $v \in V$, v is a vertex on at least one edge in H. The size of a smallest edge cover is the *edge covering number* of G, denoted $\alpha_1(G)$.

A second result due to T. Gallai [4] yields $\alpha_1(G) + \beta_1(G) = |V|$.

Since G_n is Hamiltonian, $\alpha_1(G_n) = \lceil (1/2)\binom{n+1}{2} \rceil$, and consequently $\beta_1(G_n) = \binom{n+1}{2} - \lceil (1/2)\binom{n+1}{2} \rceil$.

3. Triangles in G_n.

In this section we derive a formula for t_n, the number of triangles (subgraphs isomorphic to K_3) in G_n. We first observe that a triangle in G_n is determined by three vertices (intervals) of the form $[a_i, b_i]$ where $i = 1, 2, 3$, and $a_i, b_i \in \{0, 1, 2, \ldots, n\}$. Let $k = b_1$ and assume that $b_1 \leq b_2 \leq b_3$, so that $k \in \{1, 2, \ldots, n\}$. These intervals are pairwise distinct but intersecting, and we find that there are

(i) $\binom{k}{3}$ triples of intervals where $b_1 = b_2 = b_3$;
(ii) $\binom{k}{2}(k+1)(n-k)$ triples when $b_1 = b_2 < b_3$; and,
(iii) $k[2\binom{k+1}{2}\binom{n-k}{2} + \binom{k+1}{2}(n-k) + (k+1)\binom{n-k}{2}]$ such triples where $b_1 < b_2 \leq b_3$.

Of the terms in brackets in (iii), the first arises from the subcase $a_2 \neq a_3, b_2 < b_3$; the second accounts for the subcase where $a_2 \neq a_3$, but $b_2 = b_3$; and the third arises for $a_2 = a_3, b_2 < b_3$. In all three subcases, a_1 may be chosen in k ways.

Consequently

$$t_n = \sum_{k=1}^{n} \left\{ \binom{k}{3} + \binom{k}{2}(k+1)(n-k) + k\left[2\binom{k+1}{2}\binom{n-k}{2} + \binom{k+1}{2}(n-k) + (k+1)\binom{n-k}{2}\right]\right\}.$$

Theorem 3. *For $n \geq 1$, the number of triangles t_n that occur in G_n is given by*

$$t_n = \binom{n+1}{3} + 12\binom{n+1}{4} + 16\binom{n+1}{5} + 6\binom{n+1}{6}.$$

Proof: Both this sum and the preceding summation are polynomials in n of degree 6. Using the summation formulas for $\sum_{k=1}^{n} k^i$, where $i = 1,2,3,4,5$, one may show (in a tedious way) that these two formulas for t_n are identical.

The formula for t_n in the statement of Theorem 3 was originally derived by considering the number of distinct values, namely, 3, 4, 5, or 6, among the a_i's and b_i's in the intervals $[a_i, b_i]$, $i = 1,2,3$.

4. The Complement of G_n.

In this final section we shall make several observations involving $\overline{G}_n$, the complement of G_n.

(1) For $n = 1$, $G_n = \overline{G}_n$ and each graph consists of an isolated vertex. When $n = 2$, G_n is isomorphic to K_3 and $\overline{G}_n$ contains three isolated vertices.

For $n \geq 3$, we claim that $\overline{G}_n$ consists of five components: The four isolated vertices $[0, n-1]$, $[0, n]$, $[1, n-1]$, and $[1, n]$, and a component C_n that contains the other $\binom{n+1}{2} - 4$ vertices.

To show that C_n is connected, let $[a, b]$, $[c, d]$ be any two vertices in $\overline{G}_n$, excluding $[0, n-1]$, $[0, n]$, $[1, n-1]$, and $[1, n]$. If $[a, b] \cap [c, d] = \emptyset$, then $\{[a, b], [c, d]\}$ is an edge in $\overline{G}_n$.

If $[a, b] \cap [c, d] \neq \emptyset$, let a be minimal, $a \leq c$. When $a > 1$, then $[a, b] \cap [0, 1] = \emptyset = [c, d] \cap [0, 1]$ and the edges $\{[a, b], [0, 1]\}$, $\{[0, 1], [c, d]\}$ in $\overline{G}_n$ provide a path (in $\overline{G}_n$) of length two. If $a = 0$ or 1, then $b \leq n-2$. For $d \leq n-2$, we have $[a, b] \cap [n-1, n] = \emptyset = [c, d] \cap [n-1, n]$, and the edges $\{[a, b], [n-1, n]\}$, $\{[n-1, n], [c, d]\}$ connect the vertices $[a, b]$ and $[c, d]$. Otherwise, $d = n-1$ or n, and $2 \leq c$. Here $[a, b] \cap [n-1, n] = \emptyset = [n-1, n] \cap [0, 1] = [0, 1] \cap [c, d]$, and the edges $\{[a, b], [n-1, n]\}$, $\{[n-1, n], [0, 1]\}$, and $\{[0, 1], [c, d]\}$ provide a path (in $\overline{G}_n$) of length 3 connecting $[a, b]$ and $[c, d]$.

(2) Following M. Golumbic [5], an undirected graph G is called *chordal* if every cycle in G of length greater than three possesses a chord, that is, an edge

that joins two nonconsecutive vertices of the cycle. The graphs G_n, $n \geq 1$, are chordal graphs. Since $\overline{G}_1$ and $\overline{G}_2$ each consist of just isolated vertices, they are chordal. The graph $\overline{G}_3$ consists of four isolated vertices and one edge; there is no cycle found among the five edges of $\overline{G}_4$. So $\overline{G}_n$ is chordal for $1 \leq n \leq 4$.

When $n \geq 5$, the four vertices $[0,1]$, $[3,5]$, $[0,2]$, and $[3,4]$ form a cycle in $\overline{G}_n$ with no chord joining $[3,4]$ and $[3,5]$, or $[0,1]$ and $[0,2]$. Hence $\overline{G}_n$ is not chordal for $n \geq 5$.

(3) From [5] an undirected graph G is called *perfect* if $\omega(H) = \mathcal{X}(H)$, for all induced subgraphs H of G. Here $\omega(H)$ = the *clique number* of H, $\mathcal{X}(H)$ = the *chromatic number* of H.

From the results of C. Berge [2] and A. Hajnal and J. Surányi [7], every chordal graph is perfect. Hence G_n is perfect for all $n \geq 1$. In addition, in [8] L. Lovász shows that an undirected graph G is perfect if and only if $\overline{G}$ is perfect. Hence each of G_n and $\overline{G}_n$ is perfect for any $n \geq 1$. From observation (2) above we find that the family of graphs, $\overline{G}_n$, $n \geq 5$, is such that each graph is perfect, though not chordal.

(4) For any undirected graph G, $\omega(G) = \beta(\overline{G})$ and $\beta(G) = \omega(\overline{G})$. For the graphs G_n, $\overline{G}_n$, we also have $\omega(G_n) = \mathcal{X}(G_n)$ and $\omega(\overline{G}_n) = \mathcal{X}(\overline{G}_n)$, so $\mathcal{X}(G_n) = \beta(\overline{G}_n)$ and $\beta(G_n) = \mathcal{X}(\overline{G}_n)$.

From the second observation in Section 2 it follows that $\omega(\overline{G}_n) = \mathcal{X}(\overline{G}_n) = \lceil \frac{n}{2} \rceil = \beta(G_n)$. Theorem 2 of [6] leads to $\beta(\overline{G}_n) = \omega(G_n) = (1/4)n^2 + n + (1/8)(-1)^n - (1/8)$.

Also, the number of triangles in $\overline{G}_n$ is the number of independent subsets of three vertices in G_n, namely $\binom{n+1}{6}$. And from Theorem 3 it follows that $\overline{G}_n$ contains $\binom{n+1}{3} + 12\binom{n+1}{4} + 16\binom{n+1}{5} + 6\binom{n+1}{6}$ independent subsets of size 3.

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