# Embeddings of Maximal Arc Type in Finite Projective Planes[1]

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**Abstract.** Let $D$ and $\overline{D}^d$ be two designs such that there is a joint embedding $D'$ and $\overline{D}'$ of $D$ and $\overline{D}$ in a finite projective plane $\pi$ of order $n$ such that the points of $D'$ and the lines of $\overline{D}'$ are mutually all of the exterior elements of each other. We show that there is a tactical decomposition $T$ of $\pi$, two of the tactical configurations of which are $D'$ and $\overline{D}'$, and determine combinatorial restrictions on $n$ and the parameters of $D$ and $\overline{D}^d$. We also determine the entries of the incidence matrices of $T$.

## 1. Introduction.

In this paper we continue the study, begun in [4], of certain types of joint embeddings of a design and a dual design in a finite projective plane. We start with a brief recapitulation of the terminology introduced in [4].

Let $\pi$ be a finite projective plane and $D$ a design (dual design). We say that $D$ is a *subdesign* (*subdualdesign*) of $\pi$ if $\pi$ has a substructure $D'$ which is isomorphic to $D$. We also say that $D$ is *embeddable* in $\pi$, that $D'$ is *embedded* in $\pi$ and that $D'$ is an *embedding* of $D$ in $\pi$. If there is an embedding $D'$ of a design $D$ and an embedding $\overline{D}'$ of a dual design $\overline{D}$ in a finite projective plane $\pi$, we say that $D$ and $\overline{D}$ are *jointly embeddable* in $\pi$ and that $D'$ and $\overline{D}'$ constitute a *joint embedding* of $D$ and $\overline{D}$ in $\pi$.

For the purposes of this paper a design will be an incidence structure with $v$ points and $b$ blocks such that each block is incident with $k > 0$ points, each pair of distinct points are jointly incident with $\lambda > 0$ blocks and $v \geq k+1$. As is well-known any design is a tactical configuration (and likewise for any dual design). Thus each point of a design will be incident with the same number, $r$ say, of blocks of the design. We shall refer to a design with parameters $v, b, r, k, \lambda$, as introduced above, as a "$(v, b, r, k, \lambda)$-design". For such a design to be embedded in a finite projective plane clearly we must have $\lambda = 1$.

Let $P$ be a set of points of a projective plane $\pi$. A line $\ell$ of $\pi$ is said to be a *P-tangent*, a *P-secant* or a *P-exterior line* if $\ell$ is incident with one, more than one or no point of $P$, respectively. Dual terms which cover the corresponding situations for a line set are defined analogously.

Now let $D$ be a design and $\overline{D}$ be a dual design. Note first that it is not necessarily assumed that $D$ and $\overline{D}^d$ (the dual of $\overline{D}$) have the same parameters. Suppose $D'$

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and $\overline{D}'$ are a joint embedding of $D$ and $\overline{D}$ in a finite projective plane $\pi$ with $P'$ the point set of $D'$ and $\overline{L}'$ the line set of $\overline{D}'$ and consider the following three cases:

Type 1: $P'$ and $\overline{L}'$ are mutually all of the tangents of each other.

Type 2: $P'$ and $\overline{L}'$ are mutually all of the secants of each other.

Type 3: $P'$ and $\overline{L}'$ are mutually all of the exterior elements of each other.

As pointed out in [4], for the case of Type 2 joint embeddings, $D' = \overline{D}'$ and so $\pi' = D' = \overline{D}'$ is a projective subplane of $\pi$, whence the sets of $P'$-tangents, $P'$-secants and $P'$-exterior lines (if there are any of the latter) and their $\overline{L}'$ counterparts form a tactical decomposition of $\pi$, the entries in the incidence matrices of which are readily determinable in terms of the orders of $\pi$ and $\pi'$. In [4] we established analogous results to these for Type 1 joint embeddings. In this paper we wish to do the same for Type 3 joint embeddings.

A *maximal arc* (*with parameter* $s$) of a finite projective plane $\pi$ of order $n$ is a set $P_3$ of points of $\pi$ such that a line of $\pi$ meets $P_3$ in no points or in a fixed number $s+1$ of points, where $1 \leq s \leq n-2$. For a maximal arc with parameter $s$ in a plane $\pi$ of order $n$, simple counting yields $|P_3| = s(n+1)+1$. Letting $L_2$ be the set of $P_3$-secant lines, $L_3$ the set of $P_3$-exterior lines and $P_2$ the set of $L_3$-secant points, we have that $P_3$ and $L_2$ form an embedded design $D'$ of $\pi$ with parameters $v = s(n+1)+1$, $b = (s+1)^{-1}(n+1)(s(n+1)+1)$, $r = n+1$, $k = s+1$, $\lambda = 1$, and $P_2$ and $L_3$ form an embedded dual design $\overline{D}'$ such that $(\overline{D}')^d$ has parameters $v = (s+1)^{-1}n(n-s)$, $b = (n-s)(n+1)$, $r = n+1$, $k = (s+1)^{-1}n$, $\lambda = 1$. These combinatorial results are contained in Wallis [5], but note that, where Wallis uses "$n$" we use "$s+1$" and where he uses "$s$" we use "$n$".

Also it is not difficult to verify that $P_2, P_3, L_2, L_3$, form a tactical decomposition of $\pi$ with incidence matrices

$$\begin{bmatrix} n-s & n+1 \\ s+1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} n+1-(s+1)^{-1}n & n+1 \\ (s+1)^{-1}n & 0 \end{bmatrix}.$$

(Here the $ij$ entry in the first matrix gives the number of $P_{i+1}$ points on an $L_{j+1}$ line, and for the second matrix, it gives the number of $L_{i+1}$ lines on a $P_{j+1}$ point.) Clearly, the existence of a maximal arc with parameter $s$ in a plane of order $n$ implies the well-known condition that $s+1$ is a divisor of $n$. Also note that the point set $P_3$ of $D'$ and the line set $L_3$ of $\overline{D}'$ are mutually all of the exterior elements of each other, and so $D'$ and $\overline{D}'$ are a joint embedding of Type 3. It would thus not be unreasonable to refer to the design $D'$ in a joint embedding of Type 3 as an "embedding of maximal arc type".

**2. Tactical decompositions.**

Basic definitions and facts concerning tactical decompositions of finite incidence structures can be found in [1] (pp. 4-8 and p. 60). Of particular importance is the fact that the number of point classes and the number of block classes in a tactical decomposition of a symmetric design, such as a finite projective plane, are equal ([1], p. 60). If $P_i$ and $L_j$ are a point class and a line class of a tactical decomposition $T$ of a finite incidence structure $\eta$, we denote the constant number of $P_i$ points incident with an $L_j$ line by $(P_iL_j)$, and the constant number of $L_j$ lines incident with a $P_i$ point by $(L_jP_i)$. We also denote the incidence matrix of $T$ with $(P_iL_j)$ in the $i$th row and $j$th column by $M(P/L)$, and the incidence matrix of $T$ with $(L_iP_j)$ in the $i$th row and $j$th column by $M(L/P)$.

A tactical decomposition $T$ of a finite projective plane $\pi$ with point classes $P_i$ and line classes $L_i$, where $i = 1,2,\dots,m$, is said to be *symmetric* if there is a permutation $\begin{pmatrix} 1 & 2 \dots & m \\ j_1 & j_2 \dots & j_m \end{pmatrix}$ such that, upon relabelling $L_i$ by $L_{j_i}$ for each $i$ from 1 to $m$, $|P_k| = |L_k|$ for $k = 1,2,\dots,m$. A tactical decomposition $T$ of a finite projective plane $\pi$ with point classes $P_i$ and line classes $L_i$, where $i = 1,2,\dots,m$, is said to be *strongly symmetric* if there is a permutation $\begin{pmatrix} 1 & 2 \dots & m \\ j_1 & j_2 \dots & j_m \end{pmatrix}$ such that, upon relabelling $L_i$ by $L_{j_i}$ for each $i$ from 1 to $m$, the incidence matrices of $T$ come out to be equal.

Now suppose that $T$ is a tactical decomposition of a finite projective plane $\pi$ of order $n$ and $P_i$, $i = 1,2,\dots,m$, and $L_j$, $j = 1,2,\dots,m$, are the point and line classes of $T$, respectively. We shall make use of the following fundamental counting equations later.

$$\sum_{i=1}^{m} |P_i| = n^2 + n + 1, \qquad (P1)$$

$$\sum_{i=1}^{m} |L_i| = n^2 + n + 1, \qquad (L1)$$

$$\sum_{i=1}^{m} (P_iL_j) = n + 1, \quad j = 1,2,\dots,m, \qquad (Aj)$$

$$\sum_{i=1}^{m} (L_iP_j) = n + 1, \quad j = 1,2,\dots,m, \qquad (Bj)$$

$$(P_iL_j)|L_j| = (L_jP_i)|P_i|, i, \quad j = 1,2,\dots,m, \qquad (Cij)$$

$$\sum_{k=1}^{m} (P_iL_k)(L_kP_j) = |P_i| + \delta_{ij}n, \ i, \quad j = 1,2,\dots,m, \qquad (Dij)$$

$$\sum_{k=1}^{m} (L_iP_k)(P_kL_j) = |L_i| + \delta_{ij}n, \ i, \quad j = 1,2,\dots,m, \qquad (Eij)$$

where $\delta_{ij} = 0$ or 1 accordingly as $i \neq j$ or $i = j$.

Equations $(Cij)$ hold since $P_i$, $L_j$ form a tactical configuration for all $i, j$. Equations $(Dij)$ are obtained by counting incidences of $P_i$ points with the lines on a $P_j$ point in two ways, and Equations $(Eij)$ by a counting argument that is the dual of this. Note that, if $T$ is a tactical decomposition with $M(P/L) = M(L/P)$, then $|P_i| = |L_i|$, for all $i = 1, 2, \ldots, m$, from Equations $(Cii)$, $i = 1, 2, \ldots, m$. Consequently, a strongly symmetric tactical decomposition is necessarily symmetric. It is not true, however, that a symmetric tactical decomposition is necessarily strongly symmetric. For example, the tactical decomposition of any finite non-self-polar projective plane with each point and line class of order one is symmetric, but not strongly symmetric.

We now show, for a Type 3 joint embedding, that the sets of $P'$-tangents, $P'$-secants and $P'$-exterior lines and their $\overline{L}'$ counterparts form a tactical decomposition. This will enable us to employ Equations $(P1)$ to $(Emm)$, with $m = 3$, to sort out the combinatorics of Type 3 joint embeddings in the next section.

**Theorem 1.** *Let $D$ and $\overline{D}^d$ be designs, and let $D'$ and $\overline{D}'$ be a joint embedding of Type 3 of $D$ and $\overline{D}$ in a finite projective plane $\pi$ of order $n$. Let $P_1$, $P_2$, $P_3$ be the sets of $\overline{L}'$-tangents, $\overline{L}'$-secants, $\overline{L}'$-exterior points, respectively, and $L_1$, $L_2$ and $L_3$ be the sets of $P'$-tangents, $P'$-secants and $P'$-exterior lines, respectively, where $P'$ is the point set of $D'$ and $\overline{L}'$ is the line set of $\overline{D}'$. Then*

(a) *$P_3 = P'$, $L_2 =$ the block set of $D'$, $L_3 = \overline{L}'$ and $P_2 =$ the point set of $\overline{D}'$,*
(b) *$P_1 = \phi$ if and only if $L_1 = \phi$,*
(c) *if $P_1 = \phi = L_1$, then $P_2, P_3, L_2, L_3$, form a tactical decomposition of $\pi$, and*
(d) *if $P_1 \neq \phi \neq L_1$, then $P_1, P_2, P_3, L_1, L_2, L_3$, form a tactical decomposition of $\pi$.*

Proof:

(a) Clearly $P_3 = P'$ and $L_3 = \overline{L}'$ since, by assumption, $P'$ is the complete set of $\overline{L}'$-exterior points and $\overline{L}'$ is the complete set of $P'$-exterior lines. Also, every line of $\pi$ which is a block of $D'$ is a $P_3$-secant, and conversely, every $P_3$-secant is easily shown to be a block of $D'$. So $L_2$ is the set of blocks of $D'$. Dually we can obtain that $P_2$ is the set of points of $\overline{D}'$.
(b) Suppose $L_1 = \phi$. Then, $P_3$ is a maximal arc with, say, parameter $s$, where $1 \leq s \leq n-2$, since $\overline{D}'$ is a design. But then each non-$P_3$ point is on $(s+1)^{-1}n > 1$ $L_3$ lines. So there are no $P_1$ points. A dual argument establishes the converse.
(c) Suppose $P_1 = \phi = L_1$. Then $P_3$ is a maximal arc and so this part follows.
(d) Suppose $P_1 \neq \phi \neq L_1$. Clearly, there are a constant number of lines of each type on a $P_3$ point, and dually for points on each $L_3$ line.

Let $P$ be a $P_1$ point, and suppose that there are $\alpha$ $L_1$ lines on $P$ and $\beta$ $L_2$ lines. Then $\alpha + \beta = n$ since $P$ is an $L_3$-tangent. Also $\alpha + k\beta = |P_3|$, where $k$ is the blocksize of $D'$. Since $k > 1$ we can solve these equations uniquely for $\alpha$ and $\beta$. Clearly, $\alpha$ and $\beta$ are independent of $P$ in $P_1$. A dual argument suffices for the $L_1$ lines.

Let $Q$ be a $P_2$ point, and suppose that there are $\gamma$ $L_1$ lines on $Q$ and $\delta$ $L_2$ lines on $Q$. Then $\gamma + \delta = n + 1 - \bar{k}$, where $\bar{k}$ is the blocksize of $(\overline{D}')^d$. Also $\gamma + k\delta = |P_3|$. By similar reasoning to that employed for the $P_1$ points we have that $\gamma$ and $\delta$ are independent of $Q$. A dual argument suffices for the $L_2$ lines.

## 3. Combinatorics of embeddings of maximal arc type.

Having established that a Type 3 joint embedding in a finite projective plane $\pi$ yields a tactical decomposition of $\pi$ in the manner described in Theorem 1 we turn to the problem of determining the entries in the incidence matrices of the tactical decomposition in terms of as few of the relevant parameters as possible, and of finding whatever combinatorial restrictions apply to these parameters. We summarize our results in the following theorem.

**Theorem 2.** *Let $D$ and $\overline{D}^d$ be designs, and let $D'$ and $\overline{D}'$ be a joint embedding of Type 3 of $D$ and $\overline{D}$ in a finite projective plane $\pi$ of order $n$. Let $P_1, P_2, P_3$ be the sets of $\overline{L}'$-tangents, $\overline{L}'$-secants and $\overline{L}'$-exterior points, respectively, and $L_1, L_2, L_3$ be the sets of $P'$-tangents, $P'$-secants, $P'$-exterior lines, respectively, where $P'$ is the point set of $D'$ and $\overline{L}'$ is the line set of $\overline{D}'$.*

(a) *If $P_1 = \phi = L_1$, then $P_3$ is a maximal arc, $D$ is an $((s+1)(s\mu+1), (s\mu+1)((s+1)\mu+1), (s+1)\mu+1, s+1, 1)$-design and $\overline{D}^d$ is an $((\bar{s}+1)(\bar{s}\bar{\mu}+1), (\bar{s}\bar{\mu}+1)((\bar{s}+1)\bar{\mu}+1), (\bar{s}+1)\bar{\mu}+1, \bar{s}+1, 1)$-design, where $\bar{\mu} = s+1$ and $\bar{s} = \mu - 1$, for some $s \geq 1$ and $\mu \geq 2$, $n = (s+1)\mu$ and the incidence matrices $M(P/L)$ and $M(L/P)$, where $(P_{i+1} L_{j+1})$ is in the $ij$ position of $M(P/L)$ and dually for $M(L/P)$, of the tactical decomposition formed by $P_2, P_3, L_2, L_3$ are given by*

$$M(P/L) = \begin{bmatrix} (s+1)\mu - s & (s+1)\mu + 1 \\ s+1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{s}\bar{\mu} + 1 & (\bar{s}+1)\bar{\mu} + 1 \\ \bar{\mu} & 0 \end{bmatrix}$$

*and*

$$M(L/P) = \begin{bmatrix} s\mu + 1 & (s+1)\mu + 1 \\ \mu & 0 \end{bmatrix} = \begin{bmatrix} (\bar{s}+1)\bar{\mu} - \bar{s} & (\bar{s}+1)\bar{\mu} + 1 \\ \bar{s}+1 & 0 \end{bmatrix}.$$

*Also, the tactical decomposition is symmetric if and only if $\mu = s+1$, in which case it is strongly symmetric.*

(b) *If* $P_1 \neq \phi \neq L_1$, *then* $D$ *and* $\overline{D}^d$ *have the same blocksize* $(s+1$, *say, where* $s \geq 1)$, $s \mid n-1$ *and the incidence matrices of the tactical decomposition formed by* $P_1, P_2, P_3, L_1, L_2$ *and* $L_3$ *are given by*

$$M(P/L) = \begin{bmatrix} n-\bar{r}+\rho & n-\bar{r}+\rho-(s+1) & n-\bar{r}+1 \\ \bar{r}-\rho & \bar{r}-\rho+1 & \bar{r} \\ 1 & s+1 & 0 \end{bmatrix} \tag{1}$$

*and*

$$M(L/P) = \begin{bmatrix} n-r+\rho & n-r+\rho-(s+1) & n-r+1 \\ r-\rho & r-\rho+1 & r \\ 1 & s+1 & 0 \end{bmatrix}, \tag{2}$$

*where* $r = (L_2 P_3)$, $\bar{r} = (P_2 L_3)$ *and* $\rho = \frac{n-1}{s}$. *Furthermore,* $n \geq r \geq s+1$ *and* $n \geq \bar{r} \geq s+1$ *and also either*

(i) $r = \bar{r}$, *whence*

$$M(P/L) = M(L/P) = \begin{bmatrix} n-r+\rho & n-r+\rho-(s+1) & n-r+1 \\ r-\rho & r-\rho+1 & r \\ 1 & s+1 & 0 \end{bmatrix},$$

$$(n-s)(s+1) \geq rs+1(=|P_3|=|L_3|) \geq n, \tag{3}$$

$$s^2r^2-((n+2)(s+1)-1)sr+(s+1)(n^2-1)=0 \tag{4}$$

*and* $|P_1| = |L_1| = (n+1)(n-r+\rho) - r(s+1)$, $|P_2| = |L_2| = (n+1)(r-\rho)+r$, *or*

(ii) $r+\bar{r} = n+\rho$, *where* $\rho = \frac{n-1}{s}$ *as before,*

$$M(P/L) = \begin{bmatrix} r & r-(s+1) & n-\bar{r}+1 \\ n-r & n-r+1 & \bar{r} \\ 1 & s+1 & 0 \end{bmatrix}, \tag{5}$$

$$M(L/P) = \begin{bmatrix} \bar{r} & \bar{r}-(s+1) & n-r+1 \\ n-\bar{r} & n-\bar{r}+1 & r \\ 1 & s+1 & 0 \end{bmatrix}, \tag{6}$$

$r$ *and* $\bar{r}$ *are the solutions of the quadratic equation*

$$s^2x^2-(n(s+1)-1)sx+(s+1)n(n-(s+1))=0 \tag{7}$$

*and* $|P_1| = r(n-s)$, $|L_1| = \bar{r}(n-s)$, $|P_2| = n(n-r+1)$ *and* $|L_2| = n(n-\bar{r}+1)$.

*The tactical decomposition in Part (b)(ii) is symmetric if and only if* $n = r = s^2 + s + 1$ *and* $\bar{r} = s + 1$ *or* $n = \bar{r} = s^2 + s + 1$ *and* $r = s + 1$. *In either of these cases the tactical decomposition is strongly symmetric.*

Proof:

(a) In this case the result follows from those concerning maximal arcs mentioned in the introductory section of this paper upon putting $\mu = (s+1)^{-1} n$.

(b) We have $(P_3 L_1) = (L_3 P_1) = 1$ and $(P_3 L_3) = (L_3 P_3) = 0$, by definition of the $P_i$ and $L_j$. Also, let $(P_3 L_2) = s+1$, $(L_2 P_3) = r$, $(P_2 L_3) = \bar{r}$ and $(L_3 P_2) = \nu$.

From Equation (A3) listed earlier we have $(P_1 L_3) = n - \bar{r} + 1$ and from $(B3)$ $(L_1 P_3) = n - r + 1$. Next $(B1)$ and $(D31)$ yield $(L_2 P_1) = \frac{|P_3| - n}{s}$, $(L_1 P_1) = n - \frac{|P_3| - n}{s}$, and $(B2)$ and $(D32)$ give $(L_2 P_2) = \frac{|P_3| - n + (\nu - 1)}{s}$ and $(L_1 P_2) = n + 1 - \nu - \frac{|P_3| - n + (\nu - 1)}{s}$. Then, from $(A1)$ and $(E31)$, we have $(P_2 L_1) = \frac{|L_3| - n}{\nu - 1}$ and $(P_1 L_1) = n - \frac{|L_3| - n}{\nu - 1}$, and from $(A2)$ and $(E32)$ follows $(P_2 L_2) = \frac{|L_3| - n + s}{\nu - 1}$ and $(P_1 L_2) = n - s - \frac{|L_3| - n + s}{\nu - 1}$.

Now, from the expression for $(L_2 P_1)$, we have $s \mid |P_3| - n$, and then, from that for $(L_2 P_2)$ we have $s \mid \nu - 1$. Also, from the expression for $(P_2 L_1)$, we have $\nu - 1 \mid |L_3| - n$ and then, from that for $(P_2 L_2)$ we have $\nu - 1 \mid s$. Clearly, we must have $\nu - 1 = s$ and so $(L_3 P_2) = s + 1$. We note that $(L_3 P_2)$ is the blocksize of $\overline{D}^d$.

From $(D33)$ we readily have $|P_3| = rs + 1$, and from $(E33)$, that $|L_3| = \bar{r}s + 1$. Now, since $s \mid |P_3| - 1$ we have from the expression for $(L_2 P_1)$ that $s \mid n - 1$. Also, replacing $|P_3|$ by $rs + 1$ and $|L_3|$ by $\bar{r}s + 1$ in the expressions for $(P_1 L_1)$, $(P_1 L_2)$, $(P_2 L_1)$ and $(P_2 L_2)$, and also in their $(L_i P_j)$ counterparts, yields the incidence matrices given by (1) and (2).

Clearly, $n$ must be greater than or equal to $r$ and $\bar{r}$, and $r \geq s + 1$ and $\bar{r} \geq s + 1$ are simply Fisher's Inequality for $D$ and $\overline{D}^d$.

Next, let $z = |P_3|$ and $\bar{z} = |L_3|$. Then, from $(C11)$, $(C13)$ and $(C31)$ we have

$$(\bar{z} - n + s - (n+1)s)(z - 1 - (n+1)s)z = (z - n + s - (n+1)s)(\bar{z} - 1 - (n+1)s)\bar{z} \quad (8)$$

and, from $(C22)$, $(C23)$ and $(C32)$, we have

$$(\bar{z} - n + s)(z - 1)z = (z - n + s)(\bar{z} - 1)\bar{z}. \quad (9)$$

Using (9) we can simplify (8) down to the equation

$$(z - \bar{z})(z + \bar{z} - (n(s + 1) + 1)) = 0,$$

whence $z = \bar{z}$ or $z + \bar{z} = n(s+1) + 1$. But then, using $z = rs + 1$ and $\bar{z} = \bar{r}s + 1$, we have $r = \bar{r}$ or $r + \bar{r} = n + \rho$, where $\rho = \frac{n-1}{s}$.

Obviously, when $r = \bar{r}$ we have $M(P/L)$ and $M(L/P)$ equal to the matrix on the right side of Equation (2). For $i = 1, 2$, $(Di1)$ to $(Di3)$ each yield $|P_i|$ as given in the statement of Part (b)(i). Similarly $(E11)$ to $(E23)$ yield $|L_1|$ and $|L_2|$ as given. Equations $(Cii)$ are identities and Equations $(Cij)$, where $i \neq j$, all reduce (after some algebra) to Equation (4). Equations $(P1)$ and $(L1)$ are clearly satisfied and, since the entries in the incidence matrices must be non-negative, we have that $r \geq \rho$ and $n - r + \rho \geq s + 1$ which readily yield (3).

Turning to the case where $r + \bar{r} = n + \rho$ we clearly have $M(P/L)$ and $M(L/P)$ given by (5) and (6). For $i = 1, 2$, $(Di1)$ to $(Di3)$ each yield $|P_i|$ as given in Part (b)(ii). Dually Equations $(E11)$ to $(E23)$ yield $|L_1|$ and $|L_2|$ as given. Equations $(Cii)$ are identities and Equations $(Cij)$, where $i \neq j$, each reduce to Equation (7) with $x = r$ upon eliminating $\bar{r}$, or with $x = \bar{r}$ upon eliminating $r$. $(P1)$ and $(L1)$ are obviously satisfied and the inequalities concerning $n$, $r$, $\bar{r}$ and $s$ we have already mentioned ensure that the entries of the incidence matrices are non-negative.

Finally, we examine the possibility of the tactical decomposition being symmetric in the case where $r + \bar{r} = n + \rho$.

We first note that $r = \bar{r}$ is impossible since then $r = \bar{r} = \frac{sn+n-1}{2s}$ which is readily shown to be impossible using (7). So $|P_i| \neq |L_i|$ for $i = 1, 2, 3$. So, for a symmetric tactical decomposition, we must have $|P_1| = |L_2|$, $|P_2| = |L_3|$ and $|P_3| = |L_1|$ or $|P_1| = |L_3|$, $|P_2| = |L_1|$ and $|P_3| = |L_2|$. Using the expressions for $|P_i|$ and $|L_j|$ in terms of $n$, $r$, $\bar{r}$ and $s$ and $r + \bar{r} = n + \frac{n-1}{s}$ readily yields $n = r = s^2 + s + 1$, $\bar{r} = s + 1$ in the first case, and $n = \bar{r} = s^2 + s + 1$, $r = s + 1$ in the second. That these cases yield strongly symmetric tactical decompositions is then easily verified.

Remarks:

(a) Note that an $((s+1)(s\mu+1), (s\mu+1)((s+1)\mu+1), (s+1)\mu+1, s+1, 1)$-design is simply a $(v, b, r, k, \lambda)$-design with $k$ a divisor of $v$ and $\lambda = 1$, and conversely. We have stated Part (a) of Theorem 2 as above in order to avoid quotients in the expressions for the parameters of $D$ and $\overline{D}^d$, and in order to bring out the symmetry of the situation.

(b) For a Type 3 joint embedding as described in Part (b)(ii) of Theorem 2 there is a Type 3 joint embedding in the dual plane which can be described by interchanging the roles of $r$ and $\bar{r}$.

## 4. Concluding remarks.

Equations (4) and (7) in the statement of Theorem 2 give some further information about the various parameters beyond that explicitly given there. In Part (b)(i) it is straightforward to show, using (4), that $s$ is a divisor of $2(r - \rho)$. In Part (b)(ii) we can readily show, using (7), that $s$ is a divisor of $\rho - 1$ and so $n = \tau s^2 + s + 1$ for some $\tau \geq 1$.

For $s \geq 2$ the case $n = r = s^2 + s + 1$ and $\bar{r} = s + 1$ corresponds to $D'$ an $((s+1)(s^2+1), (s^2+1)(s^2+s+1), s^2+s+1, s+1, 1)$-design (that is, a generalized unital with $\sigma = 1$ — see [4]), and $\overline{D}'$ a projective subplane of order $s$, in a plane of order $s^2 + s + 1$. Also for $s \geq 2$ the case $n = \bar{r} = s^2 + s + 1$ and $r = s + 1$ corresponds to $D'$ a projective subplane of order $s$ and $\overline{D}'$ a dual of a generalized unital with $\sigma = 1$, in a plane of order $s^2 + s + 1$. A projective subplane $\pi'$ of order $s$ in a plane of order $s^2 + s + 1$ will always give rise to Type 3 joint embeddings of both these sorts since the exterior points of the line set of $\pi'$ and the tangent lines of the point set of $\pi'$ form a generalized unital with $\sigma = 1$, and the tangent points of the line set of $\pi'$ and the exterior lines of the point set of $\pi'$ form the dual of a generalized unital with $\sigma = 1$, and these structures are embedded relative to $\pi'$ in the requisite ways. (This generalized unital/dual generalized unital pair will also form a Type 1 joint embedding.) However, there are no examples of such subplanes known, and indeed the $s = 2$ case has been shown to be impossible in [3]. Furthermore, the Bruck-Ryser Theorem eliminates some instances (for example, $s = 6, n = 43$ and $s = 7, n = 57$).

When $n = r = s^2 + s + 1, \bar{r} = s + 1$ and $s = 1$ in the $r + \bar{r} = n + \rho$ case we have $D'$ is an affine subplane of order two in a plane of order three and $\overline{D}'$ is a $(3, 3, 2, 2, 1)$-design. That a quadrangle and its diagonal triangle are embedded in $PG_1(2,3)$ in this way is easily verified. This example and the corresponding one where $n = \bar{r} = s^2 + s + 1$ and $r = s + 1$ with $s = 1$, in $PG_1(2,3)$, are the only examples known to the author of Type 3 joint embeddings as described in Theorem 2 (b)(ii).

The equations $n = 3s^3 - 3s^2 - 2s + 1, r = 3s^2 - 2$, where $s \geq 3$, yield an infinite class of values of $n$ and $r$ satisfying the restrictions which apply in Theorem 2 (b)(i). Some of these values can be eliminated by applying the Bruck-Ryser Theorem (for example, $s = 7, n = 869$ and $s = 10, n = 2681$).

The only example known to the author of a Type 3 joint embedding as described in Theorem 2 (b)(i) occurs in $PG_1(2,7)$. $PG_1(2,7)$ possesses affine subplanes of order three (see [2]). The exterior line set of the point set of such an affine subplane $\alpha$ of $PG_1(2,7)$, and the set of secant points of this line set, is an embedded dual of an affine plane of order three which, along with $\alpha$, constitutes a joint embedding of Type 3 in $PG_1(2,7)$.

The author has a list of all values of $n$, $s$ and $r$ satisfying the restrictions in each of the two cases in Theorem 2(b) for $n$ up to $10^4$. For Part (b)(i) there are 102 solutions for $n$, $s$ and $r$ of which 19 can be eliminated by the Bruck-Ryser Theorem. (For example, $n = 94, r = 76, s = 3$ can be so eliminated.) For Part (b)(ii) there are 145 pairs of solutions of which 67 pairs can be so eliminated. (For example, $n = 213$, $r = 195$, $s = 4$ and $n = 213$, $r = 71, s = 4$ is such a pair.) Of course, the solutions for Part (b)(ii) always occur in pairs since, if $n$, $s$, $r$ is a solution, then $n$, $s$, $\bar{r}$ is also a solution with $\bar{r} = n + \rho - r \neq r$. The first eighteen sets of values of $n$, $s$ and $r$ in each case (ignoring values excluded by the Bruck-Ryser Theorem) are given in the tables below.

Table 1 $(r = \bar{r})$

| $n$ | $s$ | $r$ |
|---|---|---|
| 7 | 2 | 4 |
| 9 | 2 | 6 |
| 11 | 2 | 9 |
| 11 | 2 | 10 |
| 49 | 3 | 25 |
| 61 | 3 | 32 |

| $n$ | $s$ | $r$ |
|---|---|---|
| 79 | 3 | 65 |
| 137 | 4 | 46 |
| 160 | 3 | 92 |
| 181 | 3 | 105 |
| 211 | 3 | 160 |
| 221 | 4 | 75 |

| $n$ | $s$ | $r$ |
|---|---|---|
| 235 | 3 | 177 |
| 291 | 5 | 73 |
| 364 | 3 | 220 |
| 373 | 4 | 341 |
| 439 | 3 | 320 |
| 529 | 6 | 106 |

Table 2 $(r + \bar{r} = n + \rho)$

| $n$ | $s$ | $r$ |
|---|---|---|
| 3 | 1 | 2 |
| 3 | 1 | 3 |
| 7 | 2 | 3 |
| 7 | 2 | 7 |
| 13 | 3 | 4 |
| 13 | 3 | 13 |

| $n$ | $s$ | $r$ |
|---|---|---|
| 31 | 5 | 6 |
| 31 | 5 | 31 |
| 58 | 3 | 29 |
| 58 | 3 | 48 |
| 73 | 8 | 9 |
| 73 | 8 | 73 |

| $n$ | $s$ | $r$ |
|---|---|---|
| 85 | 3 | 45 |
| 85 | 3 | 68 |
| 91 | 9 | 10 |
| 91 | 9 | 91 |
| 111 | 10 | 11 |
| 111 | 10 | 111 |

None of the sets of $n$, $s$, $r$ values listed in Table 1 can be eliminated using Result 12 on p. 62 of [1].

Using the discriminants of the quadratic expressions on the left side of (4) and (7) it is easy to show that $s = 1$ never occurs when $r = \bar{r}$ and that $s = 1$ only occurs when $n = 3$ in the $r + \bar{r} = n + \rho$ case. Also, in this way, we can show that $n \leq 11$ when $s = 2$ in the $r = \bar{r}$ case, and $n \leq 7$ when $s = 2$ in the $r + \bar{r} = n + \rho$ case. The values of $n$ and $r$ given in Table 1 and Table 2 with $s = 1$ or $s = 2$ are thus the only possible ones with $s$ taking either of these values. This sort of argument fails for larger values of $s$. Indeed $s = 3$ occurs many times in the list the author has of $n$, $s$, $r$ values, and right up to near $n = 10^4$.

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