Support Sizes of Triple Systems with Small Index

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Abstract. For each integer $v \geq 0$ and each $\lambda \in \{4,5,7,8\}$, the possible numbers of distinct blocks in a triple system of order v and index λ is determined. This essentially completes the determination of possible support sizes for triple systems with $\lambda \leq 8$.

1. The support size conjecture.

A triple system (V, B) of **order** v and **index** λ, denoted $TS(v,\lambda)$, is a v-set V and a collection B of 3-element subsets of V called **triples**; each 2-subset of V appears in precisely λ triples. The **support** of a triple system (V, B) is the set B^* of distinct triples; its **support size** is the number of distinct triples.

We define $SS(v,\lambda) = \{b^*: \exists\, TS(v,\lambda)(V,B) \text{ with } b^* = |B^*|\}$. The determination of $SS(v,\lambda)$ is of interest in statistical applications [6,15]. We recall the support size conjecture from [3].

Let $m_v = [v(v-1)/6]$, $s_v = [v(v+2)/6]$ and $M_{v,\lambda} = [\min(\lambda, v-2)\cdot v(v-1)]/6$. Now define $PS(v,\lambda)$ to be empty if no $TS(v,\lambda)$ exists, otherwise according to the following table:

v (mod 12)	$PS(v,\lambda)$
0,4	$s_v, s_v+2,\ s_v+3, \ldots, M_{v,\lambda}$
1,3,7,9	$m_v, m_v+4,\ m_v+6, m_v+7, \ldots, M_{v,\lambda}$
2	$s_v+8, \ldots, M_{v,\lambda}$
5,11	$m_v+7, m_v+10, m_v+11, \ldots, M_{v,\lambda}$
6,10	$s_v+1, s_v+2, \ldots, M_{v,\lambda}$
8	$s_v+7, s_v+9, s_v+10, \ldots, M_{v,\lambda}$.

The Support Size Conjecture [3]:
For $v \notin \{6,7,8,9,10,12,14\}$,

1. if $\lambda \neq v-2$, $SS(v,\lambda) = PS(v,\lambda)$;
2. if $\lambda = v-2$, $SS(v,\lambda) = PS(v,\lambda) \setminus \{M_{v,\lambda}-5,\ M_{v,\lambda}-3,\ M_{v,\lambda}-2,\ M_{v,\lambda}-1\}$.

In [3], the necessity of these conditions is proved with one possible omission; the case $s_v + 8$ for $v \equiv 8 \pmod{12}$ is not eliminated by the necessary conditions in [3].

Rosa and Hoffman [13] determined $SS(v,2)$; Colbourn and Mahmoodian [2,3] determined $SS(v,3)$ and $SS(v,6)$ (with the possible exception noted above for the latter). In this paper, we determine $SS(v,\lambda)$ for $\lambda \in \{4,5,7,8\}$.

2. Small orders.

For each $v \leq 13$, and every λ, a precise determination of $SS(v,\lambda)$ is available. In the following table, we summarize the results for $v \leq 13$ and $\lambda \leq 8$; we list the values in $PS(v,\lambda) \backslash SS(v,\lambda)$.

v	2	3	4	5	6	7	8
6	9		9,11,12,13,15,17,18,19		9,11,12,13,15,18		9,11,12,13,15
7		16	16,24,27	16,30,32,33,34	16,34	16	16
8					20,51,53,54,55		
9	16,19	16,19	16,19	16,19	16,19	16,19,79,81,82,83	16,19
10	22		22		22		22,115,117,118,119
12	30		30		30		30

This table is a consequence of a number of computational investigations for $v = 6$ [7], $v = 7$ [8], $v = 8$ [6], $v = 9$ [9], $v = 10$ and 11 [10], $v = 12$ [4] and $v = 13$ [11].

3. Recursive constructions.

In general, we apply the $v \to 2v + s (1 \leq s \leq 7)$ constructions from [2, 3], and minor variants of them; many such variants have appeared in the literature, and those required here are easily obtained. Together with the observation that $SS(v,\lambda) \subseteq SS(v,\lambda\, l)$ for $l \geq 1$, and the determination of $SS(v,\lambda)$ for $\lambda \in \{2,3\}$, these establish the following for $\lambda \leq 8$:

Proposition 3.1. *If* $\lambda < v - 2$, $SS(v,\lambda) = PS(v,\lambda) \neq \phi$ *and* $1 \leq s \leq 7$, $SS(2v+s,\lambda) = PS(2v+s,\lambda)$.

Using this proposition, we obtain a complete solution for $SS(v,\lambda)$, $\lambda \leq 8$, given a solution for orders $v \leq 25$. Our task is reduced to the determination of $SS(v,\lambda)$ for $14 \leq v \leq 25, \lambda \in \{4,5,7,8\}$. In the $v \to 2v+1$ construction, the complete determination of support sizes of λ-factorizations of λK_n in [5] affords much flexibility.

We employ two other main classes of constructions to handle the intermediate values. The first class consists of tripling constructions.

Lemma 3.2. *Let $s \in \{0,1,3\}$. Let x be the support size of a λ-factorization of $\lambda K_{v,v}$, $y \in SS(v+s,\lambda)$ and z_1, z_2 be support sizes of triple systems of order v and index λ missing a subdesign on s elements.*

Then
$$x + y + z_1 + z_2 \in SS(3v+s,\lambda).$$

Proof: Form three disjoint sets of size v, X, Y and Z; suppose that $Z = \{1,2,\dots,v\}$. Form a λ-factorization on $\lambda K_{v,v}$ having bipartition (X,Y); let the λ-factors be $F_1,\dots,F_v$. Now form triples by taking, for each $1 \le i \le v$, and each edge $\{x,y\}$ in F_i, the triple $\{i,x,y\}$. The support size of this collection of triples is precisely the support size of the λ-factorization.

Then add s new points S. Place a $TS(v+s,\lambda)$ on $X \cup S$, and place triple systems of order v and index λ having a hole of size s on $Y \cup S$ and $Z \cup S$, so that the holes correspond to S in each case. It is evident that no two ingredients share any triples, and hence the support size is as required. ∎

To use this effectively, observe that $x = tv$ for $t = v$ and $v+2 \le t \le \lambda v$ are possible support sizes of λ-factorizations of $\lambda K_{v,v}$ for $\lambda \le v$ (this can be improved upon using results of [1] and [5]).

The last main construction used is novel. We define an **enclosing** $E(v:w,\lambda:\mu)$ to be a pair (V,B), with the following properties. V is a v-set, and $W \subset V$ is a w-set. B consists of triples so that each pair of elements not both in W appears in λ triples, while each pair in W appears in μ triples ($\mu \le \lambda$). An enclosing is **faithful** if no triple of B appears on three elements of W, and it is **simple** if it has no repeated blocks. A simple faithful $E(v:w, \lambda:\mu)$ is denoted $SFE(v:w, \lambda:\mu)$. We use such enclosings as follows.

Lemma 3.3. *Suppose there is an $SFE(v:w, \lambda:\lambda-\mu)$ and that $t \in SS(w,\mu)$. Then $\lambda v(v-1)/6 - \mu w(w-1)/6 + t \in SS(v,\lambda)$.*

Proof: Since the $SFE(v:w,\lambda:\mu)$ is faithful, no triple in a $TS(w,\mu)$ on W repeats a triple in the SFE. The number of distinct triples in the SFE is $\lambda v(v-1)/6 - \mu w(w-1)/6$, since the SFE is simple. Hence, the union of the SFE and the $TS(w,\mu)$ is a $TS(v,\lambda)$ with the specified number of distinct blocks. ∎

Very little is known about the existence of SFE's. Even when $\lambda = \mu$, while enclosings are known by Stern's theorem [14], and these are automatically faithful, no general result for SFE's is known here (this would be the "simple" analogue of Stern's theorem). One general result is known, however [12]:

Lemma 3.4. *Let $v \equiv 1,3 \pmod 6$, $0 \le \mu \le v-2$ and $0 \le \lambda - \mu \le v+1$. Then there exists an $SFE(2v+1:v, \lambda:\mu)$.*

In the next section, we establish the existence of some further SFE's which are useful in determining support sizes. We also have occasion to use faithful enclosings which are *not* simple; proceeding as in Lemma 3.3 enables us to show that no repeated blocks appear, other than those repeated in the enclosing itself.

4. Intermediate orders.

In this section, we employ doubling, tripling and SFE constructions to settle the required intermediate cases for $\lambda \in \{4,5,7,8\}$.

4.1 The case $\lambda = 4$.

We must determine $SS(v,4)$ for $v \in \{15,16,18,19,21,22,24,25\}$. For $v = 15$, the $v \to 2v+1$ construction suffices. For $v = 16$, we use a $v \to 2v+2$ construction to embed a $TS(7,4)$ on $X = \{x_1,\dots,x_7\}$ using nine further points $Y = \{y_1,\dots,y_9\}$. Let Q_1,Q_2,Q_3,Q_4 be the four parallel classes of a Kirkman triple system of order nine on Y. Now each x_i appears in triples with the pairs of a 4-factor on Y. One 4-factor on Y will remain. Taking this last 4-factor to be $Q_1 \cup Q_2$, we are left with edges of Q_1 and Q_2 three times, and of Q_3 and Q_4 four times. In this way we obtain that if $t \in SS(7,4), 8 \le x \le 14$, $t+6+9x \in SS(16,4)$. Instead, taking Q_1 twice establishes that if $t \in SS(7,4)$, $x = 7$ or $9 \le x \le 14$, $t+3+9x \in SS(16,4)$. Together with $SS(16,2) \subseteq SS(16,4)$, this leaves only $\{82,90,156,159\}$ in doubt. One can instead form four of the 4-factors by taking a 4-factorization of $2K_9$ in either case. If there is a 4-factorization of $2K_9$ with support size s and $t \in SS(7,4)$, we have $t+s+u \in SS(16,4)$ for $u \in \{30,42,48,51,57,60\}$. For the remaining values, we take

t	s	u	Support size
22	30	30	82
18	30	42	90
28	68	60	156
28	71	60	159

The required values for s are immediate consequences of [5]. For $v = 18$, $\{61,\dots,102\} = SS(18,2) \subseteq SS(18,4)$. The tripling construction $3 \cdot 6 = 18$ can be used by selecting any of $\{10,14,16,20\}$ on each of the three $TS(6,4)$'s. For the "cross" triples, we use any 4-factorization of $4K_{6,6}$; hence we can obtain support sizes $\{36,40,42-108\}$ for this ingredient by extending a 3-factorization of $3K_6$ [1], showing that $\{61,\dots,168\} \subseteq SS(18,4)$. Moreover, since the support sizes of 2-factorizations of $2K_{3,3}$ are $\{9,15,17,18\}$ [1], the support sizes of 4-factorizations of $4K_{6,6}$ include $\{109,\dots,144\}$ as well yielding $SS(18,4) = PS(18,4)$. For $v = 19$, $v = 21$ and $v = 25$, the $v \to 2v+1$ construction suffices. For $v = 22$ and $v = 24$, the $v \to 2v+4$ construction suffices.

4.2 The case $\lambda = 4$.

We must determine $SS(v,5)$ for $v \in \{15,19,21,25\}$. For $v = 15$, the $v \to 2v+1$ construction produces all support sizes except $\{170,172,173,174\}$. Using the $SFE(15{:}\,7,5{:}\,4)$ from Lemma 3.4 handles 170, 172 and 173. Using the $SFE(15{:}\,7,5{:}\,3)$ from Lemma 3.4 handles 174.

For $v = 19$, the $v \to 2v + 1$ construction suffices. For $v = 21$, we apply the $3v + 3$ construction. The "cross" triples can have support sizes $6x$ for $x = 6$ and $8 \le x \le 30$. Two $TS(9,5)$'s missing a subdesign of size 3 can each have support sizes 11 or 55 (all repeated or all distinct) while the last has support size $t \in SS(9,5) = \{12,18,20,21,\ldots,60\}$. Since also $SS(21,2) = \{70,74,76,\ldots,140\}$ for decomposable $TS(21,2)$'s [13], we have $SS(21,5) = PS(21,5)$.

For $v = 25$, we adopt a similar strategy using the $3v + 1$ construction; the details are easy.

4.3 The case $\lambda = 7$.

We need the values of $SS(v,7)$ for $v \in \{15,19,21,25\}$. For $v = 15$, the $2v+1$ construction gives all values up to 231 (using 7-factorizations of $7K_8$ from [5]). Now an $SFE(15{:}\,7,7{:}\,5)$ handles all values (up to 245, the maximum) except $\{240,242,243,244\}$. An $SFE(15{:}\,7,7{:}\,4)$ handles 240, 242 and 243, and an $SFE(15{:}\,7,7{:}\,3)$ handles 244.

The $2v + 1$ construction handles $v = 19$ completely. For $v = 21$, we employ a $3v$ construction, to obtain $t_1 + t_2 + t_3 + 7_x \in SS(21,7)$ for $t_i \in SS(7,7)$ and $x = 7$ or $9 \le x \le 49$. This handles all support sizes up to 448.

To handle the remaining cases, we construct an $SFE(21{:}\,15,7{:}\,5)$ on 21 elements V with 15 distinguished elements X. First form a partial triple system P on V having a leave which precisely is a 10-regular graph G on X. Form a simple 4-factorization of $K_{15} \cup X$ and associate each factor with an element of $V \backslash X$, forming triples in the usual way. Finally, take any simple 2-factorization of $6K_6$ and associate each factor with an element of X. This is the required SFE, and it yields $\{327,333,335,336,\ldots,490\} \subseteq SS(21,7)$.

For $v = 25$, the $3v + 1$ construction is sufficient, as follows. Given that t_i, $i = 1,2,3,4$ is the support size of a factorization of $3K_{4,4}$, and that s_i, $i = 1,2,3,4$ is the support size of a 4-factorization of $4K_{4,4}$, we have $\sum_{i=1}^{4} t_i + s_i$ as the support size of a 7-factorization of $7K_{8,8}$. Using results on $3K_{4,4}$ in [1], the construction is complete.

4.4 The case $\lambda = 8$.

We need to determine $SS(v,8)$ for $v \in \{15,16,18,19,21,22,24,25\}$. For $v = 15$, the standard $v \to 2v+1$ construction provides all support sizes up to 231. Rosa's construction [12] can be used to produce $FE(15{:}\,7,8{:}\,5)$ with support size 210 or 217; together with $SS(7,5)$ this gives support sizes $\{231,\ldots,246,248,252\}$. An $FE(15{:}\,7,8{:}\,4)$ with support size 224 gives support sizes $\{247,249,250\}$.

Next we form an $SFE(15{:}\,9,8{:}\,5)$ as follows. On the six additional points, place four blocks so that each point is in two blocks. Associate each of the 9 points with a 4-factor of a simple 4-factorization of the remaining edges on the six points. Finally, associate each of the six points with a 4-factor of a 4-factorization of $3K_9$. This provides $\{251,\ldots,280\}$, and completes the determination of $SS(15,8)$.

For $v = 16$, $\{48,50,51,\ldots,160\} = SS(16,4) \subseteq SS(16,8)$. Next we apply a $2v+2$ construction. Let X be the elements of the $TS(7,8)$ and let Y be nine additional elements. Form a Hamiltonian decomposition Q_1,Q_2,Q_3,Q_4 of K_9 on Y. Now form 7 8-factors, each obtained by taking the union of four of the $\{Q_i\}$, so that the resulting 8-factors partition $7K_9$.

The resulting 8-factorization can have support size $9x$ for $10 \le x \le 28$. Each 8-factor is associated with an element of X; finally a $TS(9,1)$ is placed on Y. Thus, we have that

$$t + 12 + 9x \in SS(16,8)$$

for $t \in SS(7,8)$ and $10 \le x \le 28$. What remains is $\{300\text{-}320\}$.

For $v = 18$, $\{61,\ldots,204\} = SS(18,4) \subseteq SS(18,8)$. The standard $2v+4$ construction handles all values up to 387. To handle the remaining values, we produce an $SFE(18{:}\,9,8{:}\,6)$ on elements $\{x_1,\ldots,x_9\} \cup \{y_1,\ldots,y_9\}$. Let F_i, $i = 1,\ldots,9$ be a near 2-factorization on $\{x_i\}$ so that F_i misses x_i. Let G_i, $i = 1,\ldots,9$ be a near 2-factorization on $\{y_i\}$ so that G_i misses y_i. Associate edges of F_i with y_i in triples, and associate edges of $K_9 \backslash G_i$ with x_i in triples. Finally place a $TS(9,1)$ on $\{y_i\}$. This SFE establishes that $\{388\text{-}408\}$ are support sizes of $TS(18,8)$.

For $v = 19$, the standard $v \to 2v+1$ construction together with the $SFE(19{:}\,9,8{:}\,6)$ are sufficient. For $v = 21$, the $2v+1$ construction suffices.

For $v = 22$, $\{89,\ldots,308\} = SS(22,4) \subseteq SS(22,8)$. The $2v+2$ construction settles all values except four: $\{607,609,610,611\}$. An $SFE(22{:}\,15,8{:}\,5)$ is easily produced to handle these last four values.

For $v = 24$, we employ the $2v+4$ construction. To handle the four remaining values, we produce an $SFE(24{:}\,12,8{:}\,6)$ by the same method used for $v = 18$. For $v = 25$, the $2v+1$ construction suffices.

5. Concluding remarks.

Applying Proposition 3.1 with the values produced in Section 4 establishes that the support size conjecture is true for all v and $\lambda \le 8$, with the possible exception of $s_v + 8$ for $v \equiv 8 \pmod{12}$. Naturally, we do not expect that handling further small values of λ would be overly difficult by the methods developed here. However, at this point, the next main step would be to establish the support size conjecture for all λ. Suitable general results on existence of the faithful enclosings introduced here are needed to address this problem.

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References

1. C.J. Colbourn, *Concerning 3-factorizations of* $3K_{nn}$. (to appear).
2. C.J. Colbourn and E.S. Mahmoodian, *The spectrum of support sizes for threefold triple systems*, Discrete Math. (to appear).
3. C.J. Colbourn and E.S. Mahmoodian, *Support sizes of sixfold triple systems*. (to appear).
4. C.J. Colbourn and E.S. Mahmoodian, *Support sizes of* $(12,3,\lambda)$ *designs*, Technical Report #1-1367, Sharif University of Technology, Tehran, Iran.
5. C.J. Colbourn and A. Rosa, *Support sizes of* λ*-factorizations*. (to appear).
6. W. Foody and A. Hedayat, *On theory and applications of BIB designs with repeated blocks*, Ann. Statist. 5 (1977), 932-945.
7. A. Hedayat and G.B. Khosrovshahi, *An algebraic study of BIB designs: a complete solution for* $v = 6$ *and* $k = 3$, J. Comb. Theory A30 (1981), 43-52.
8. A. Hedayat and S.Y.R. Li, *The trade-off method in the construction of BIB designs with variable support sizes*, Ann. Statist. 7 (1979), 1277-1287.
9. G.B. Khosrovshahi and E.S. Mahmoodian, *On BIB designs with various support sizes for* $v = 9$ *and* $k = 3$, Commun. Stat. (Simul. Comp.) (to appear).
10. G.B. Khosrovshahi and E.S. Mahmoodian, *A table of BIB* $(10, 30t, 9t, 3, 2t)$ *and BIB* $(11, 55t, 15t, 3, 3t)$ *designs*, Technical Report #2-1365 (1987), Sharif University of Technology, Tehran.
11. E.S. Mahmoodian. (Private communications, 1988).
12. A. Rosa, *A theorem on the maximum number of disjoint Steiner triple systems*, J. Comb. Theory A18 (1975), 305-312.
13. A. Rosa and D.G. Hoffman, *The number of repeated blocks in twofold triple systems*, J. Comb. Theory A41 (1986), 61-88.
14. G. Stern, *Tripelsysteme mit Untersysteme*, Arch. Math. 33 (1979), 204-208.
15. H.P. Wynn, *Convex sets of finite population plans*, Ann. Statist. 5 (1977), 414-418.