The Number of Repeated Blocks in Balanced Ternary Designs with Block Size Three

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Abstract. Let D denote any balanced ternary design with block size three, index two, and $\rho_2 = 1$ (that is, with each element occurring repeated in just one block). This paper shows that there exists such a design D on V elements containing exactly k pairs of repeated blocks if and only if $V \equiv 0 \pmod 3$,

$$0 \leq k \leq t_V = \frac{1}{6}V(V-3), \quad k \neq t_V - 1, \text{ and } (k,V) \neq (1,6).$$

1. Introduction.

For the purposes of this paper, a *balanced ternary design* or BTD is a pair (P, B) where P is a finite set and B is a collection of multisets of size 3 (called *blocks*) of the form $\{x,x,y\}$ or $\{x,y,z\}$, where $x \neq y \neq z \neq x$, such that each pair of distinct elements occurs exactly twice among the blocks of B, and each element of P occurs twice in exactly *one* block. (The pair $\{x,y\}$ occurs twice in the block $\{x,x,y\}$.) $V = |P|$ is the *order* of the BTD. (For the more general definition and a survey of such designs, see [1].)

A necessary and sufficient condition for the existence of such a BTD is that $V \equiv 0 \pmod 3$ (see, for instance, [2]).

Some examples which we use later:

$|P| = 3$:

112
223
331

$|P| = 6$:

112	453
223	453
331	462
441	462
552	561
663	561

or

112	245
223	345
331	246
441	346
551	256
661	356

Here is our result:

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Main Theorem. *Let k, V be non-negative integers. There is a BTD of order V with exactly k pairs of repeated blocks if and only if $V \equiv 0 \pmod 3$, $0 \le k \le t_V = \frac{1}{6}V(V-3)$, $k \ne t_V - 1$, and $(k,V) \ne (1,6)$.*

Let $R(V) = \{k \mid \text{there exists a BTD on } V \text{ elements with } k \text{ repeated blocks}\}$, and let $J_V = \{k \mid 0 \le k \le t_V,\ k \ne t_V - 1\}$. So we must prove $R(V) = J_V$ for $V \equiv 0 \pmod 3$, except $1 \notin R(6)$. (The corresponding problem for two-fold triple systems was settled in [9].)

The number of blocks in a BTD of order V is $\frac{1}{3}V^2$, and V of the blocks are of the form $\{x,x,y\}$, leaving $\frac{1}{3}V(V-3)$ of type $\{x,y,z\}$. This latter number is even. Thus, the maximum possible number of repeated blocks is $t_V = \frac{1}{6}V(V-3)$. Obviously, $t_V - 1 \notin R(V)$. The two examples of BTD's of order 6 given above are the only two; see [5]. The first has three pairs of repeated blocks, the second none. So $R(6) = \{0,3\}$, and we have shown the conditions given in the main theorem are necessary.

Before we show they are sufficient, we need one more definition.

A BTD of order W with a *hole* of size V is a triple (Q,P,B), where Q is a W-set, P is a V-subset of Q, and B is a collection of blocks of Q such that:

(1) each pair of distinct elements of Q not both in P, occurs exactly twice in the blocks of B;
(2) each element of $Q \backslash P$ occurs twice in exactly one block;
(3) each pair of elements of P, distinct or not, occurs in no blocks.

(A BTD with a hole has also been called a *frame*; see [6].)

For two examples, let $Q = \{1,2,3,4,5,6\}$, $P = \{1,2,3\}$, and delete the blocks 112, 223, 331 from either of the two BTD's of order 6 given above.

Naturally, if (Q,P,B) is a BTD of order W with a hole of size V, and if (P,B_0) is a BTD of order V, then $(Q, B \cup B_0)$ is a BTD of order W.

2. The odd case.

In this section, we confine ourselves to the case $V \equiv 3 \pmod 6$.

In [3], the following was established.

Theorem 1. *If $u \ge 5$ is odd, then there is a pair of group divisible designs on the same u groups of size 3, with block size 3 and index 1, having exactly k blocks in common, if and only if $0 \le k \le t_{3u}$, $k \ne t_{3u} - i$, $i \in \{1,2,3,5\}$.*

Corollary 2. *If $V \equiv 3 \pmod 6$ and $V \ge 15$, then $R(V) \backslash \{t_V - i \mid i = 2,3,5\} \subseteq J_V$.*

Proof: Given the pair of designs in Theorem 1, with $V = 3u$, we take their blocks as blocks of a BTD. On each group, we place a BTD of order 3. ∎

We need to treat the cases $V = 9$ and 15 separately:

Lemma 3. $R(9) = J_9$.

Proof: Elements $\{1, 2, \ldots, 9\}$.

$0 \in R(9)$: Cyclic design $[1,1,2], [1,3,6], [1,3,7] \pmod 9$.

$1 \in R(9)$: 123, 124, 134, 234, 158, 159, 168, 169,
268, 269, 278, 279, 358, 359, 378, 379,
567, 567, 117, 225, 336, 449, 554, 664, 774, 884, 998.

$2 \in R(9)$: 189, 189, 349, 349,
117, 227, 338, 442, 553, 668, 773, 882, 992
123, 125, 136, 145, 146, 236, 256, 458, 467,
478, 569, 578, 579, 679.

$3 \in R(9)$: 112, 223, 331, 445, 556, 664, 778, 889, 997,
258, 258, 159, 159, 357, 357,
147, 167, 349, 369, 247, 267, 348, 368,
148, 168, 249, 269.

$4 \in R(9)$: Repeat 269, 347, 389, 579.
119, 228, 336, 448, 556, 664, 772, 885, 994,
123, 124, 135, 145, 235, 245, 167, 168, 178, 678.

$5 \in R(9)$: Repeat 258, 269, 359, 378, 489.
118, 227, 336, 447, 554, 664, 779, 886, 991,
123, 124, 134, 234, 156, 157, 167, 567.

$6 \in R(9)$: Repeat 167, 268, 279, 369, 478, 589.
118, 225, 337, 449, 556, 664, 775, 883, 991,
123, 124, 135, 145, 234, 345.

$7 \in R(9)$: Repeat 158, 169, 268, 279, 359, 378, 567.
117, 225, 336, 449, 554, 664, 774, 884, 998,
123, 124, 134, 234.

$9 \in R(9)$: Cyclic design $[1,1,5], [1,2,4], [1,2,4] \pmod 9$.

So $R(9) = J_9$. ∎

We will need the following result of H.L. Fu [7].

Theorem 4. *If $n \geq 5$, then there are two Latin squares of order n which agree on exactly k cells if and only if $0 \leq k \leq n^2$, $k \neq n^2 - i$ for $i = 1, 2, 3, 5$.*

Lemma 5. $R(15) = J_{15}$.

Proof: $\{0, 1, \ldots, 24, 26, 28, 30\} \subseteq R(15)$:

Let $P = Z_5 \times \{1,2,3\}$. Take two Latin squares $[a_{ij}]$, $[b_{ij}]$ of order 5 which have at least one cell in common; without loss of generality let their (1,1) entries be 1. So these Latin squares agree in a *further* $0,1,\ldots,18,20$ or 24 places.

Take blocks of type $\{x,x,y\}$ as follows:

$$(A)\quad \begin{array}{l} \{(1,1),(1,1),(1,2)\},\ \{(1,2),(1,2),(1,3)\},\\ \{(1,3),(1,3),(1,1)\};\\ \{(2,i),(2,i),(1,i)\},\ \{(3,i),(3,i),(1,i)\},\\ \{(4,i),(4,i),(1,i)\},\ \{(5,i),(5,i),(1,i)\},\\ \{(2,i),(3,i),(4,i)\},\ \{(2,i),(3,i),(5,i)\},\\ \{(2,i),(4,i),(5,i)\},\ \{(3,i),(4,i),(5,i)\}; \end{array} \quad \text{or}$$

$$(B)\quad \begin{array}{l} \{(2,i),(2,i),(3,i)\},\ \{(3,i),(3,i),(4,i)\},\\ \{(4,i),(4,i),(5,i)\},\ \{(5,i),(5,i),(2,i)\},\\ \{(1,i),(2,i),(4,i)\},\ \{(1,i),(2,i),(4,i)\},\\ \{(1,i),(3,i),(5,i)\},\ \{(1,i),(3,i),(5,i)\}; \end{array}$$

for $i \in \{1,2,3\}$. (Note that (A) contains no repeated blocks while (B) contains *two* repeated blocks.)

Then take blocks

$$\{(i,1),(j,2),(a_{ij},3)\},\ \{(i,1),(j,2),(b_{ij},3)\},\ (i,j \text{ not both } 1).$$

According as (A) or (B) above is taken, for each of the three possible values of i, the resulting BTD of order 15 contains k repeated blocks where

$$k \in \{0,1,\ldots,18,20,24\}+\{0,2,4,6\}, \text{ that is,}$$
$$k \in \{0,1,\ldots,24,26,28,30\}.$$

$25,27 \in R(15)$: Let $P = \{0,1,\ldots,9,a,b,c,d,e\}$.

Blocks: (A) $00c$, $cc7$, 778, $88d$, $dd0$, $aa5$, 550, $bb6$, 669, $99e$, $ee0$; (B_1) *or* (B_2) where

(B_1) is 110, 220, 330, 440, 123, 124, 134, 234;

(B_2) is 112, 223, 334, 441, 013, 013, 024, 024;

(C) $0ab, 0ab, bcd, bce, bde, acd, ace, ade$;

(D) 067, 089, 568, 579, *twice* each;

(E) $15b, 16c, 17d, 18e, 19a,$
$25c, 26d, 27e, 28a, 29b,$
$35d, 36e, 37a, 38b, 39c,$
$45e, 46a, 47b, 48c, 49d,$ all twice each.

Then $(A)\ (B_i)\ (C)\ (D)\ (E)$ is a BTD of order 15; if $i = 1$ this has 25 repeated blocks, and if $i = 2$ this has 27 repeated blocks.

So $R(15) = J_{15}$. ∎

Theorem 6. *If $t \geq 2s+1$, then there is a BTD (Q,P,B) of order $6t+3$ with a hole of size $6s+3$, in which all blocks with three distinct elements are repeated.*

Proof: By [4], there is an idempotent commutative quasigroup $(T,\circ)$ of order $2t+1$ with a subquasigroup S of order $2s+1$.

Let $Q=T\times\{1,2,3\}$, $P=S\times\{1,2,3\}$. For each $i\in T\backslash S$, place the blocks

$$\{(i,1),(i,1),(i,2)\},\ \{(i,2),(i,2),(i,3)\},$$
$$\{(i,3),(i,3),(i,1)\} \text{ in } B.$$

For each unordered pair i,j of distinct elements of T which are not both in S, place the blocks

$$\{(i,1),(j,1),(i\circ j,2)\},\ \{(i,2),(j,2),(i\circ j,3)\},$$
$$\{(i,3),(j,3),(i\circ j,1)\} \text{ in } B \textit{ twice}.$$

We now patch the gaps left by Corollary 2.

Theorem 7. *If $V \equiv 3 \pmod 6$, then $R(V) = J_V$.*

Proof: This is true for $V \leq 15$, so we assume $V \geq 21$ and proceed by induction on V. We need only prove $t_V - i \in R(V)$ for $i \in \{2,3,5\}$.

Write $V = 12x + 6\delta + 3$, where $\delta \in \{0,1\}$. Let (Q,P,B) be the BTD of order V with a hole of size $U = 6x + 6\delta - 3$, given by Theorem 6. By induction, there is a BTD of order U with exactly $t_U - i$ repeated blocks to fill the hole. ∎

3. The even case.

We now proceed with the case $V \equiv 0 \pmod 6$. As in the odd case, we need to construct a BTD with a hole. We will use difference methods, so a few definitions are in order.

If d is a positive integer, and x is any integer, we define $|x|_d$ as follows: find the unique y with $y \equiv x \pmod d$, and $-d/2 < y \leq d/2$. Then $|x|_d = |y|$.

If K_d is the complete graph on vertices Z_d, and $e = xy$ is an edge, we say that e is an edge of *difference* $|x-y|_d$. So the differences are in the set $D = \{1,2,\ldots,\lfloor\frac{d}{2}\rfloor\}$.

Finally, the 3-set $\{a,b,c\} \subseteq D$ is said to be a *difference triple* $\pmod d$ if either $a+b=c$, or $a+b+c=d$.

Theorem 8. *Let V and W be integers, with $0 \leq 2V \leq W$, V odd, W even, and $V+W \equiv 0 \pmod 3$. Let k be an integer with $0 \leq k \leq \frac{1}{6}(W-V)(W+V-3)$ and $k \equiv 0 \pmod{W-V}$. Then there is a BTD (Q,P,B) of order W with a hole of size V having exactly k pairs of repeated blocks.*

Proof: Let $W-V=d=2e+1$, let $Q\backslash P = Z_d$, and let $P=\{\infty_j \mid 1 \leq j \leq V\}$. It is well known that $D=\{1,2,\ldots,e\}$ contains $\lfloor e/3\rfloor$ pairwise disjoint difference triples. (This is easy if $e \not\equiv 0 \pmod 3$; see, for example, [9]. If $e \equiv 0 \pmod 3$, there is a cyclic Steiner triple system of order $2e + 1$ [8].)

The conditions on V, W and k guarantee the existence of integers x, t, y satisfying

$$\begin{aligned} 0 \le x \le t \le y/3 + t \le e/3, \\ W = 4e - 6t + 2, \\ V = 2e - 6t + 1, \\ k = (2e + 1)(x + y). \end{aligned}$$

So we can partition D into sets A_0, A_1, A_2, A_3 such that:

(1) $|A_0| = 3x$;
(2) $|A_1| = 3t - 3x$;
(3) $|A_2| = y$;
(4) $|A_3| = e - 3t - y$;
(5) each of A_0 and A_1 can be partitioned into difference triples $\pmod d$.

For each difference triple $a < b < c$ in the partition of A_0, and for each $i \in Z_d$, take the block $\{i,\ i + a,\ i + a + b\}$ twice.

For each difference triple $a < b < c$ in the partition of A_1, and for each $i \in Z_d$, take the blocks $\{i,\ i + a,\ i + a + b\}$ and $\{i,\ i + b,\ i + a + b\}$ each once.

Consider the graph G on vertices Z_d and edges with difference in A_2. This is a graph which is regular of degree $2y$, and so by Vizing's theorem [10], it can be properly $2y + 1$ edge coloured with colours $1, 2, \ldots, 2y + 1$.

For each $1 \le j \le 2y + 1$, and each edge ab of G coloured j, take the block $\{\infty_j, a, b\}$ twice.

For each $a \in Z_d$, there is exactly one colour i that does not occur on an edge of G incident with a. Take the block $\{a, a, \infty_i\}$.

Partition the set $\{\infty_j | 2y + 2 \le j \le V\}$ into $e - 3t - y$ pairs $\{s_j, t_j\}$, $1 \le j \le e - 3t - y$. For each such j, let f be the jth difference in A_3, and for each edge ab of difference f, take the blocks $\{s_j, a, b\}$ and $\{t_j, a, b\}$ each one. ∎

Before we can use this theorem, we will need some more special cases.

Lemma 9. $R(12) = J_{12}$.

Proof: It is easy to find two Latin squares of order 4 which agree in k places where $k \in \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$. Take two such Latin squares $[a_{ij}]$ and $[b_{ij}]$ which agree in at least *one* place, so that $a_{11} = b_{11}$. Let $P = Q \times \{1, 2, 3\}$ where the Latin squares are based on the set $Q = \{1, 2, 3, 4\}$, and take blocks in B as follows:

(1) $\{(1,1),(1,1),(1,2)\}$, $\{(1,2),(1,2),(1,3)\}$,
$\{(1,3),(1,3),(1,1)\}$;
$\{(2,i),(2,i),(1,i)\}$, $\{(3,i),(3,i),(1,i)\}$,
$\{(4,i),(4,i),(1,i)\}$, $1 \le i \le 3$.

(2) $\{(2,i),(3,i),(4,i)\}$, twice, $1 \le i \le 3$.

(3) $\{(r,1),(s,2),(a_{rs},3)\}$, $\{(r,1),(s,2),(b_{rs},3)\}$, r, $s \in Q$; r,s not both 1.

Then if the two Latin squares agree in $k \ge 1$ places, there are $k-1$ repeated blocks of type (3).

The BTD (P,B) thus contains $(k-1)+3$ repeated blocks, so $\{3,4,5,6,8,10,11,14,18\} \subseteq R(12)$.

$0 \in R(12)$: Take base blocks

$[0,0,5]$, $[0,1,3]$, $[0,3,4]$, $[0,6,8]$ modulo 12.

$9,12,15 \in R(12)$: Take a BTD on 6 elements with 3 repeated blocks. Use a doubling construction where old elements are $\{x\}$, new elements are $\{x\}\times\{1,2\}$, and where block $\{x,y,z\}$ is replaced by four blocks

$$(A)\ \begin{matrix}\{(x,1),(y,1),(z,1)\}, & \{(x,1),(y,2),(z,2)\},\\ \{(x,2),(y,1),(z,2)\}, & \{(x,2),(y,2),(z,1)\}\end{matrix} \quad \text{or}$$

$$(B)\ \begin{matrix}\{(x,2),(y,2),(z,2)\}, & \{(x,2),(y,1),(z,1)\},\\ \{(x,1),(y,2),(z,1)\}, & \{(x,1),(y,1),(z,2)\}.\end{matrix}$$

(Here x *may* equal y.)

Using (A) for all 3 repeated blocks yields a BTD on 12 elements with 18 repeated blocks.

Using (A) twice for two, and (A) once, (B) once for one of the repeated blocks, shows $15 \in R(12)$; (A) twice for one, and (A) and (B) for two of the repeated blocks shows $12 \in R(12)$, and (A) and (B) for the 3 repeated blocks shows $9 \in R(12)$. Hence $\{9,12,15,18\} \subseteq R(12)$.

$1 \in R(12)$: $P = \{1, 2, \ldots, 9, T, E, W\}$.

$112, 223, 331, TT3, EE3, WW3;$

$$(\star)\begin{cases} 441, 551, 661, 772, 882, 992, \\ 245, 345, 246, 346, 256, 356, \\ 178, 378, 179, 379, 189, 389, \end{cases}$$

$1TE, 2TE, 1TW, 2TW, 1EW, 2EW;$

$47T, 47T;$

$58E, 68W, 69E, 59W;$

$48W, 49W, 49E, 48E;$

$75E, 76E, 76W, 75W;$

$T58, T59, T69, T68.$

$7 \in R(12)$: From the above BTD, remove blocks $(\star)$ and replace with:

$442, 553, 661, 772, 883, 991,$

$256, 256, 346, 346, 145, 145,$

$289, 289, 379, 379, 178, 178.$

$16 \in R(12)$: $P = \{0, 1, \ldots, 9, T, E\}$.

$004, 114, 224, 339, 443, 559, 66T, 774, 88T, 992, TTI, EE0;$

$012, 013, 023, 123$; then the following blocks, each repeated:

$056, 078, 09T, 157, 169, 18E, 258, 267, 2TE,$

$35E, 368, 37T, 45T, 46E, 489, 79E.$

$2 \in R(12)$: $P = \{0_L, 1_L, \ldots, 5_L, 0_R, 1_R, \ldots, 5_R\}$.

B: Take base blocks, modulo 6:

short block $[0_L, 2_L, 4_L]$, twice.

$[0_L, 0_L, 0_R]$, $[0_R, 0_R, 1_R]$; $[0_R, 2_R, 4_R]$ once (short);

$[0_L, 1_L, 2_R]$, $[0_L, 1_L, 4_R]$; $[0_L, 3_L, 4_R]$, $[3_L, 0_R, 2_R]$, $[1_L, 0_R, 3_R]$.

$13 \in R(12)$: $P = \{1, 2, \ldots, 9, T, E, W\}$.

B: $115, 559, 991, 221, 331, 441, 66T, TT1, 77E, EE1, 88W, WW1;$

repeated blocks: $25T, 26E, 27W, 289,$
$35E, 36W, 379, 38T,$
$45W, 469, 47T, 48E$, and 234;

$9TE, 9TW, 9EW, TEW;$

$167, 168, 178, 567, 568, 578.$

So $R(12) = \{0, 1, \ldots, 16, 18\}$. ∎

Lemma 10. $J_{18} \backslash \{40, 43\} \subseteq R(18)$.

Proof: Let $P = Z_6 \times \{1,2,3\}$. On each $Z_6 \times \{i\}$, $i = 1,2,3$, take a BTD which may have either 0 or 3 repeated blocks. Then let $[a_{ij}]$ and $[b_{ij}]$ be two Latin squares of order 6 which agree in k places, $k \in \{0,1,\dots,30,32,36\}$. Take blocks $\{(i,1),(j,2),(a_{ij},3)\}$ and $\{(i,1),(j,2),(b_{ij},3)\}$. Then $x+k \in R(18)$, where $x = 0,3,6$ or 9 and k is as above. This shows that $\{0,1,\dots,39,41,42,45\} \subseteq R(18)$. ▮

Lemma 11. $J_{24} \backslash \{79, 81, 82\} \subseteq R(24)$.

Proof: Take $P = \{\{1,2,\dots,8\} \times \{1,2,3\}\}$ and blocks in B as follows:

$$\{(1,1),(1,1),(1,2)\},\ \{(1,2),(1,2),(1,3)\},\ \{(1,3),(1,3),(1,1)\},$$
$$\{(i,j),(i,j),(1,j)\},\quad 2 \leq i \leq 8,\quad j = 1,2,3;$$

on $\{(j,i) \mid 2 \leq j \leq 8\}$, for each $i = 1,2,3$ take a $TTS(7)$, which may contain $0,1,3$ or 7 repeated blocks; then, using two Latin squares $[a_{ij}]$ and $[b_{ij}]$ of order 8 based on $\{1,2,\dots,8\}$ with $a_{11} = b_{11} = 1$, and agreeing in k *more* cells, where $k \in \{0,1,\dots,57,59,63\}$, take blocks

$$\{(i,1),(j,2),(a_{ij},3)\} \text{ and } \{(i,1),(j,2),(b_{ij},3)\},$$

where i,j are not both 1. The result is a BTD containing S repeated blocks where

$$S \in 3 \times \{0,1,3,7\} + k, \quad k \text{ as above.}$$

This shows that $\{0,1,\dots,78,80,84\} \subseteq R(24)$. ▮

Theorem 12. *If* $V \equiv 0 \pmod 6$, *then* $R(V) = J_V$.

Proof: By Lemma 9, we may assume $V \geq 18$. So let $n \in J_V$; we now show $n \in R(V)$. By Lemma 10, if $V = 18$, we may assume $n \in \{40,43\}$, and by Lemma 11, we may assume $n \in \{79,81,82\}$ if $V = 24$.

Let $V = 12x + 6\delta$, $\delta \in \{0,1\}$, and write n as $n = k + \ell$, where $0 \leq k \leq 3(2x+1)(3x+2\delta-1)$, $k \equiv 0 \pmod{6x+3}$, and $\ell \in R(6x+6\delta-3)$ by Theorem 7. By Theorem 8, there is a BTD of order V with a hole of size $6x+6\delta-3$, having exactly k pairs of repeated triples. Now fill the hole with a BTD of order $6x+6\delta-3$ having ℓ pairs of repeated triples. ▮

Of course, Theorem 7 and Theorem 12 prove our Main Theorem.

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