Inequalities Involving the Rank of a Graph

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Abstract. Let $r(G)$ denote the rank, over the field of rational numbers, of the adjacency matrix of a graph G. Van Nuffelen and Ellingham have obtained several inequalities which relate $r(G)$ to other graph parameters such as chromatic number, clique number, Dilworth number, and domination number. We obtain additional results of this type. Our main theorem is that for graphs G having no isolated vertices, $OIR(G) \leq r(G)$, where $OIR(G)$ denotes the upper open irredundance number of G.

Introduction.

Throughout the paper $G = (V, E)$ is an undirected graph with no loops or multiple edges. We let $A(G)$ be an adjacency matrix of G relative to some ordering of the vertices. Since the rank of this matrix is independent of any particular ordering, we write simply $r(G)$ to denote its rank. We shall also let $N(G)$ be the closed neighborhood matrix relative to this ordering, that is, $N(G) = A(G) + I$. Given a vertex $v \in V$, $N(v)$ denotes the set of vertices adjacent to v, and $N[v]$ denotes $N(v) \cup \{v\}$. Moreover, if $S \subseteq V$ is a set of vertices, $N(S)$ and $N[S]$ denote $\cup_{v \in S} N(v)$ and $\cup_{v \in S} N[v]$, respectively.

In [6] Van Nuffelen states several inequalities relating $r(G)$ to other parameters, many without proof. For example, if G is a nontrivial graph (that is, contains edges) and $w(G)$ is its clique number, then $w(G) \leq r(G)$. (See Ellingham [4] for a proof). Without proof, Van Nuffelen states that

(i) For any nontrivial graph G, the *Dilworth number of* $G, \Delta(G)$, is at most $r(G)$. Here $\Delta(G)$ is the maximum size of a set of vertices which are incomparable under the partial order: $x \leq y$ if and only if $N(x) \subseteq N[y]$.

(ii) For any nontrivial graph G, $\gamma(G) \leq r(G)$. Here $\gamma(G)$ is the size of a smallest dominating set of vertices. A set $D \subseteq V$ is said to be *dominating* if each vertex in $V - D$ is adjacent to at least one vertex in D.

In the same paper Van Nuffelen conjectures that:

(iii) For any nontrivial graph G, $\chi(G) \leq r(G)$. Here $\chi(G)$ denotes the chromatic number of G.

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Recently, Ellingham [4] observed that (i) is actually false and provided a counterexample. Also recently, Alon and Seymour [1] exhibited a counterexample to (iii) by constructing a graph with chromatic number 32 but rank 29.

In light of these counterexamples, it seems reasonable to ask if (ii) is valid. To begin, note that (ii) is certainly false if no restriction is placed on G. For example, take G to be the graph with four vertices but only one edge. Then $\gamma(G) = 3$ and $r(G) = 2$. It seems, therefore, (ii) at least needs to be reformulated. In the next section we establish the inequality $\gamma(G) \leq r(G)$ assuming G has no isolated vertices. In fact, we prove a stronger result.

Main results: For any vertex $v \in V$ and subset $S \subseteq V$ we define

$$I[v,S] \equiv N[v] - N[S-\{v\}]$$
$$I(v,S] \equiv N(v) - N[S-\{v\}],$$
$$I(v,S) \equiv N(v) - N(S-\{v\}).$$

A set of vertices S is *irredundant* if for each $v \in S$ we have $I[v,S] \neq \emptyset$. We say S is *open irredundant* if $I(v,S] \neq \emptyset$ for each $v \in S$. And we say S is *open-open irredundant* if for each $v \in S$, $I(v,S) \neq \emptyset$.

The *upper irredundance number* $IR(G)$ and the *upper open irredundance number* $OIR(G)$ are defined to be the size of a largest irredundant and open irredundant set, respectively. Similarly, $OOIR(G)$ is the *upper open-open irredundance number* and is the size of a largest open-open irredundant set.

Note that if a graph has no edges it can have no nonempty open or open-open irredundant sets and so for such graphs we have $OIR(G) = OOIR(G) = 0$.

We next define $p(G)$ to be the maximum k for which $A(G)$ contains a $k \times k$ permutation submatrix. Again, note if G has no edges then $p(G) = 0$.

Lemma 1. *For any graph G, $OOIR(G) = p(G)$.*

Proof: Let S be an open-open irredundant set of vertices, say $S = \{v_{i_1}, \ldots, v_{i_m}\}$. For each $v_{i_j} \in S$ there exists a vertex $u_{i_j} \in I(v_{i_j}, S)$. Let $S' = \{u_{i_1}, \ldots, u_{i_m}\}$. Now consider the submatrix M of $A(G)$ formed by taking the rows corresponding to S and the columns corresponding to S'. The definition of open-open irredundance guarantees the u_{i_j} are distinct, and so M is an $m \times m$ submatrix. Also, by open-open irredundance, we get for each j, $1 \leq j \leq m$, $N(v_{i_j}) \cap S' = \{u_{i_j}\}$ and $N(u_{i_j}) \cap S = \{v_{i_j}\}$. Hence M is a permutation matrix and $OOIR(G) \leq p(G)$.

Conversely, let M be an $m \times m$ permutation submatrix of $A(G)$. Let $S = \{w_1, \ldots, w_m\}$ be the vertices of G corresponding to the rows of M. We claim S is open-open irredundant. For let $w_i \in S$, and we must show $I(w_i, S) \neq \emptyset$. In the matrix M let u_i be the vertex corresponding to the column containing a one in w_i's row. Then clearly $u_i \in N(w_i)$. However, since M is a permutation matrix this is the only one in u_i's column. That is $u_i \notin N(S - \{w_i\})$. Hence, $u_i \in I(w_i, S)$. ∎

Lemma 2. *(Bollobás and Cockayne) Let G be a graph having no isolated vertices. Then G has a minimum dominating set D such that for all $d \in D$, there exists a vertex $f(d) \in I[d, D] - D$.*

Proof: See [2, p. 247, Prop. 6]. ∎

Lemma 3. *For a set of vertices $D \subseteq V$, if $d \in D$ then $I(d, D] = I[d, D] - D$.*

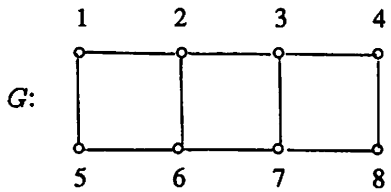
Proof: Straightforward. ∎

Lemma 4. *Let G be a graph having no isolated vertices. Then $\gamma(G) \leq OIR(G)$.*

Proof: Let D be the set described in Lemma 2 have cardinality $\gamma(G)$. By Lemma 3, for each $d \in D$, $f(d) \in I[d, D] - D = I(d, D]$ and so D is open irredundant. Hence $\gamma(G) = |D| \leq OIR(G)$. ∎

The following graph G in Figure 1 shows that there exist graphs for which $\gamma < OIR$.

Figure 1



The set consisting of vertices $\{1, 4, 7\}$ forms a minimum dominating set, but the set of vertices $\{1, 2, 3, 4\}$ is an open-irredundant set, and hence $\gamma(G) < OIR(G)$.

We now obtain a strengthening of the Van Nuffelen inequality:

Theorem 1. *For any graph G having no isolated vertices*

$$\gamma(G) \leq OIR(G) \leq OOIR(G) = p(G) \leq r(G).$$

Proof: The first inequality is Lemma 4. The next inequality holds because open irredundant sets are also open-open irredundant. The equality follows from Lemma 1. The last inequality holds because the rows of a permutation submatrix of $A(G)$ are linearly independent and therefore the rows they occur in, within $A(G)$, must be linearly independent. ∎

Recall that $\Gamma(G)$, the upper domination number, is the cardinality of a largest minimal dominating set. Since $\gamma(G) \leq \Gamma(G)$ one might ask if $\Gamma(G) \leq r(G)$. In general this is not true: Let $G = K_{n,n}$ where $n \geq 3$. Then $r(G) = 2$ and $\Gamma(G) = n$.

If we are willing to compute the rank of the closed neighborhood matrix $N(G)$ then it is true that $\Gamma(G) \leq r(N(G))$. In fact a stronger statement is possible.

Let $p'(G)$ be the largest k for which the matrix $N(G)$ contains a $k \times k$ permutation submatrix. The proof of the following lemma is very similar to the proof of Lemma 1, and is omitted.

Lemma 5. *For any graph* G, $IR(G) = p'(G)$.

We now have a closed neighborhood analog to Theorem 1.

Theorem 2. *For any graph* G,

$$\Gamma(G) \leq IR(G) = p'(G) \leq r(N(G)).$$

Proof: The first inequality can be found in [3]. The equality is Lemma 5, and the last inequality is clear by reasoning similar to that in Theorem 1. ∎

Question: S.M. Hedetniemi [5] has shown $OIR(T) = \beta_1(T)$ where T is a tree and β_1 is its matching number. It follows by Theorem 1 that for trees

$$\beta_1(T) = OIR(T) \leq r(T).$$

Does this result (that is, $\beta_1(G) \leq r(G)$) hold for other classes of graphs?

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