On The Existence of Some Balanced Arrays

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Abstract. In this paper, we obtain a polynomial inequality of degree three in m (the number of constraints), with coefficients involving the parameters μ_i's, on the existence of balanced arrays of strength four and with two symbols. Applications of the inequality to specific balanced arrays for obtaining an upper bound on the number of constraints are also discussed.

1. Introduction and Preliminaries

A matrix T of size $(m \times N)$ with elements from a set S containing s symbols (say, 0, 1, 2, .., $s-1$) is called an array T with m rows (constraints), N columns (runs or treatment - combinations), and with elements $0, 1, 2, \ldots, s-1$. The existence and construction of these arrays under some combinatorial constraint are very important to the statistical design of experiments and combinatorics. The next definition of a balanced array (B-array) is given under one such combinatorial structure:

Definition 1.1. *An array T of size $(m \times N)$ and with s elements (say; 0, 1, ..., $s-1$) is said to be of strength t $(t \leq m)$ if in every $(t \times N)$ submatrix T^* of T, we have the following condition satisfied:*

$$\lambda(\underline{\alpha};\ T^*) = \lambda(p(\underline{\alpha});\ T^*)$$

where $\underline{\alpha}$ is any $(t \times 1)$ column vector of T^*, $p(\underline{\alpha})$ is a vector obtained by permuting the elements of $\underline{\alpha}$, and $\lambda(\underline{\alpha};\ T^*)$ stands for the frequency of the column vector $\underline{\alpha}$ in T^*.

In this paper we restrict ourselve to B-arrays with $s = 2$ (i.e. arrays with elements 0 and 1) and $t = 4$. The condition of the above definition then reduces to the fact that every vector $\underline{\alpha}$ of weight i ($i = 0, 1, 2, 3, 4$; the weight of $\underline{\alpha}$ is the number of 1's in it) appears with the same frequency μ_i (say). The vector $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the index set of the array T, and the B-array is sometimes denoted by $(m,\ N,\ t = 4,\ s = 2;\ \underline{\mu}')$. It is quite obvious that

$$N = \sum_{i=0}^{4} \binom{4}{i} \mu_i.$$

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If $\mu_i = \mu$ for each i, then the B-array is reduced to an orthogonal array (O-array). Thus B-arrays include O-arrays as special cases. Also the incidence matrices of incomplete block designs, BIB designs and doubly balanced designs are certain kind of B-arrays. Furthermore, B-arrays have been very useful in the construction of symmetrical and asymmetrical fractional factorial designs of different resolutions. To gain further insight into the importance of B-arrays to statistical design of experiments and combinatorics, the interested reader may consult the bibliography given at the end, and the further references therein.

2. Existence Conditions For Balanced Arrays

The following results are easy to establish.

Lemma 2.1. *A B-array T with index set $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$, and with $m = T = 4$ always exists.*

Remark: It is quite obvious that the construction of a B-array T with $t = 4$, index set $\underline{\mu}'$, and with $m > 4$ is non-trivial.

Lemma 2.2. *A B-array T of strength four with index set $\underline{\mu}'$ is also of strength t' with $0 < t' \leq 4$. Considered as an array of strength t', its index set $\underline{\mu}^{*'} = (\mu_0^*, \mu_1^*, ..., \mu_{t'}^*)$ is given by*

$$\mu_j^* = \sum_{i=0}^{t-t'} \binom{t-t'}{i} \mu_{i+j}$$

where $j = 0, 1, 2, ..., t'$, and with the convention that $\binom{t-t'}{i} = 1$ if $t - t' = i = 0$.

Remark: In view of the above lemma it is quite clear that the index sets $\underline{\mu}^{*'}$ for $t' = 3, 2,$ and 1 are respectively given by $(\mu_0 + \mu_1, \mu_1 + \mu_2, \mu_2 + \mu_3, \mu_3 + \mu_4)$, $(\mu_0 + 2\mu_1 + \mu_2, \mu_1 + 2\mu_2 + \mu_3, \mu_2 + 2\mu_3 + \mu_4)$, and $(\mu_0 + 3\mu_1 + 3\mu_2 + \mu_3, \mu_1 + 3\mu_2 + 3\mu_3 + \mu_4)$.

Lemma 2.3. *Let x_j $(0 \leq j \leq m)$ denote the number of columns of weight j in a B-array T of strength four and index set $\underline{\mu}'$. Then the following results hold:*

$$\sum_{j=0}^{m} \binom{j}{4} x_j = \binom{m}{4} \mu_4,$$

$$\sum \binom{j}{3} x_j = \binom{m}{3} A, \quad A \text{ being } = \sum_{i=0}^{1} \binom{1}{i} \mu_{i+3},$$

$$\sum \binom{j}{2} x_j = \binom{m}{2} B, \quad B \text{ being } = \sum_{i=0}^{2} \binom{2}{i} \mu_{i+2},$$

$$\sum \binom{j}{1} x_j = \binom{m}{1} C, \quad C \text{ being } = \sum_{i=0}^{3} \binom{3}{i} \mu_{i+1},$$

$$\sum x_j = N, \quad N \text{ being } = \sum_{i=0}^{4} \binom{4}{i} \mu_i.$$

The above are obviously reduced to the following:

$$\sum_{j=0}^{m} x_j = N \tag{2.1}$$

$$\sum_{j=0}^{m} j x_j = mC \tag{2.2}$$

$$\sum j^2 x_j = m(m-1)B + mC \tag{2.3}$$

$$\sum j^3 x_j = m(m-1)(m-2)A + 3m(m-1)B + mC \tag{2.4}$$

$$\sum j^4 x_j = m(m-1)(m-2)(m-3)\mu_4 + 6m(m-1)(m-2)A + 7m(m-1)B + mC \tag{2.5}$$

Theorem 2.1. *Consider a B-array T with index set $\underline{\mu}'$ and $m \geq t = 4$. Then we have*

$$am^3 + bm^2 + cm + d \geq 0 \tag{2.6}$$

where a, b, c, and d are polynomials of degree four in μ_i's ($i = 0, 1, 2, 3$, and 4).

Proof: The following is quite obvious $\sum x_j (j - \bar{j})^4 \geq 0$, where $\bar{j}$ is the mean of the number of 1's in the columns of T.

Expanding the above, we obtain

$$\sum j^4 x_j - 4\bar{j} \sum j^3 x_j + 6\bar{j}^2 \sum j^2 x_j - 3N\bar{j}^4 \geq 0$$

Substituting from (2.1) - (2.5) and simplifying it further, we obtain $am^3 + bm^2 + cm + d \geq 0$, where

$$\begin{aligned}
a &= \mu_4 N^3 - 4N^2(\mu_3 + \mu_4)(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4) \\
&\quad + 6N(\mu_2 + 2\mu_3 + \mu_4)(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4)^2 \\
&\quad - 3(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4)^4, \\
b &= 6\mu_3 N^3 - 12N^2(\mu_2 + \mu_3)(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4) \\
&\quad + 6N(\mu_1 + 2\mu_2 + \mu_3)(\mu_1 + 3\mu_2 + 3\mu_3 + \mu_4)^2, \\
c &= N^3(7\mu_2 - 4\mu_3) + 4N^2(\mu_3 - \mu_1)(\mu_1 + 3\mu_2 + 3\mu_2 + \mu_4), \\
d &= N^3(\mu_1 - 4\mu_2 + \mu_3)
\end{aligned}$$

The above result is quite useful in obtaining an upper bound on the number of constraints m for a B-array T with a given index set $\underline{\mu}'$ and/or discussing the existence of B-arrays. For the polynomial inequality to satisfy, it is quite obvious that at least one of a, b, c, and d must be such that it is ≥ 0. The following result is a direct consequence of the above theorem.

Corollary 2.1. *Let T be a B-array of size $(m \times N)$ with index set $\underline{\mu}' = (\mu_0, \ldots, \mu_4)$ such that $\mu_1 + \mu_3 = 4\mu_2$, then the following result holds:*

$$[N\mu_4 - (\mu_2 + 2\mu_3 + \mu_4)^2]\, m^2 + 6[N\mu_3 - 2\mu_2(\mu_2 + 2\mu_3 + \mu_4)]\, m + [N(7\mu_2 - 4\mu_3) - 36\mu_2^2] \geq 0 \qquad (2.7)$$

Next, we give some examples illustrating the applications of the above results.

Example 1: Consider an array with $\underline{\mu}' = (3, 3, 2, 5, 3)$, $N = 50$. The parameters $\underline{\mu}'$'s satisfy the condition in Corollary 2.1, therefore using (2.7) above, we obtain

$$-25m^2 + 380m - 148 \geq 0$$

It can be easily checked that m must lie between $\frac{2}{5}$ and $14\frac{4}{5}$. Thus $m = 14$ is an upper bound for the above array.

Remark: It is interesting to note that results given by Chopra (1982) do not work for the above example.

A computer program was prepared to obtain the zeros of the polynomial $f(m) = am^3 + bm^2 + cm + d$ and to determine the intervals over which $f(m) > 0$ with $m \geq 4$.

Example 2: Consider an array with $\underline{\mu}' = (4, 4, 4, 4, 3), N = 63$. That such an array exists with $m = 8$ is shown in Chopra (1975). Here $a = -16{,}644$, $b = 1{,}512$, $c = 3000564$, and $d = -2000376$. It was observed that $f(13) > 0, f(14) < 0$. Thus one zero is in the interval (13, 14). The other two zeros were found in the intervals (-14, -13), and (0, 1) which are of no consequence in obtaining an upper bound for m. Thus the largest value of m for this array = 13. It can be easily checked that $m \leq 63, m \leq 32$ by using the results given in Chopra (1982), while those of Chopra (1985) do not give us any upper bound. Hence, we have a significant improvement over the results available in literature.

Example 3: Consider $\underline{\mu}' = (3, 4, 5, 4, 1), N = 66$. We find $a = 30984$, $b = -855360$, $c = 5462424$, and $d = -3449952$. The zeros are in the intervals (0, 1), (8, 9), and (18, 19). For this array $m \leq 8$ since $f(m) < 0$ in (9, 18), and $f(19) > 0$. The results of Chopra (1982, 1985) give us $m > 8$.

Finally we give a table of some selected values of $\underline{\mu}'$ along with an upper bound for m in each case.

$\underline{\mu}'$	N	upper bound
4,3,2,3,4	44	12
3,4,4,4,3	62	09
4,4,5,4,0	66	06
1,5,5,5,0	71	04
1,5,5,5,1	72	05
2,5,5,5,2	74	05
3,5,5,5,3	76	07
4,5,5,5,4	78	10
1,6,6,6,0	85	04
1,6,6,6,1	86	04
2,6,6,6,2	88	05
3,6,6,6,3	90	06
4,6,6,6,4	92	08
4,6,6,6,5	93	09
5,6,6,6,5	94	11
6,6,6,6,5	95	16

Remark: The above results can be generalized to B-arrays of strength $2l$, but the notation may become cumbersome and messy.

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