Biclique Partitions of the Complement of a Directed Cycle

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Abstract. By a refinement of a rank argument used to prove a directed version of the Graham-Pollak theorem, we show that n bicliques are needed to partition the arc-set of the complement of a directed cycle.

1. Introduction.

Let $\vec{G}$ be a simple directed graph (no loops or multiple arcs) on n labelled vertices. A *directed complete bipartite subgraph* or *biclique* of $\vec{G}$ is a subgraph $\vec{K}(X_1, Y_1)$ consisting of two disjoint subsets X_1, Y_1 of the vertex set together with all arcs $x \to y$ where $x \in X$, $y \in Y$. The *biclique partition number* or *bicontent* $bcp(\vec{G})$ of $\vec{G}$ is the minimum number of bicliques whose arc-sets partition the arc-set of $\vec{G}$.

To each partition of the arc-set of $\vec{G}$ into k bicliques $\vec{K}\ (X_i, Y_i)$, $i = 1, 2, \ldots, k$ we may associate a factorization

$$A = XY^t = \sum_{i=1}^{k} X_i Y_i^t \tag{1}$$

of its adjacency matrix A into $n \times k$ $(0,1)$-matrices X and Y. Here the subsets X_i, Y_i of the vertex set are identified with the ith $(0,1)$-column vectors of X, Y, respectively, and $X_i Y_i^t$ is the adjacency matrix of $\vec{K}\ (X_i, Y_i)$. For a more detailed discussion of these notions, including variations appropriate for (undirected) graphs and bipartite graphs, see Orlin [9]. For recent work on matrix factorizations, see **[3, 5, 6, 8, 11]**.

Since the (real) rank $r(A)$ of A is a lower bound on k in (1), and since the n out-claws (or the n in-claws) at the vertices partition the arc-set, we have the following bounds [9; 6.5, 7.1]:

$$r(A) \leq bcp(\vec{G}) \leq n. \tag{2}$$

For example, let $I = I_n$ be the $n \times n$ identity matrix and $J = J_n$ the $n \times n$ all 1's matrix. Then $J - I$ is the adjacency matrix of $\vec{K}_n$, the complete directed graph which has one arc in each direction between each pair of n distinct vertices. When $n \geq 2$, $J - I$ has rank n. Thus (2) implies the following directed version of the Graham-Pollak theorem [7, 13]:

Theorem 1. *[2,10] The arc-set of $\vec{K}_n$, $n \geq 2$ can be partitioned by n bicliques, and no fewer.*

Let $\vec{D}_n$ be the complement in the complete directed graph $\vec{K}_n$ of a spanning directed cycle. Using a refinement of the rank argument above, we will prove the following theorem. This resolves a previous conjecture [4, p.141].

Theorem 2. *The arc-set of $\vec{D}_n$, $n \geq 3$, can be partitioned into n bicliques, and no fewer.*

2. Proof of Theorem 2.

If $S = S_n$ is the $n \times n$ upward-shift permutation matrix, we may label the vertices of $\vec{D}_n$ so that $D = D_n = J - I - S$ is its adjacency matrix. By (1), it is sufficient to prove the following matrix version of Theorem 2.

Theorem 3. *If $D_n = XY^t$ where X and Y are $n \times k$ $(0,1)$-matrices and $n \geq 3$, then $k \geq n$.*

The next lemma implies that D_n has full rank if n is odd, and rank $n-1$ if n is even. Consequently, Theorem 3 is immediate if n is odd. Also, $k \geq n-1$ if n is even.

Lemma 1. *If $n \geq 3$ is odd, D_n is nonsingular. If $n \geq 4$ is even, then the nullspace of D is spanned by the vector $u = [1,-1,1,-1,\dots,1,-1]^t$.*

Proof: If $Dx = o$, then $JDx = (n-2)Jx = o$ so $Jx = o$. Thus, $(I+S)x = (J-D)x = 0$. Therefore, $x_1 = -x_n$ and $x_i = -x_{i-1}$, $2 \leq i \leq n$, so $x_1 = (-1)^n x_1$. ∎

To prove Theorem 3 in the singular case, it is sufficient to show that whenever $D_n = XY^t$ where X,Y are $n \times k$ $(0,1)$-matrices, then $D_n - xy^t$ has rank at least $n-1$ for some columns $x = X_i$ and $y = Y_i$ of X and Y. For then $D - X_iY_i^t = \Sigma X_jY_j^t$ where the sum is taken over all j such that $j \neq i$, $1 \leq j \leq k$. This implies that $k-1 \geq n-1$ and so $k \geq n$. We need the following lemma and propositions.

Lemma 2. *[12] If A and B are $n \times k$ matrices, then*

$$\det(I_n + AB^t) = \det(I_k + B^tA).$$

If x and y are $n \times 1$ $(0,1)$-matrices, we let $|x|$, $|y|$ denote the number of 1's in x, y respectively. Thus, $|x|\,|y|$ is the number of 1's in xy^t.

Proposition 1. *Let x and y be $n \times 1$ $(0,1)$-matrices. Then the characteristic polynomial of $D_n - xy^t$ is*

$$\frac{(\lambda+1)^n - (-1)^n}{(\lambda+2)^2}\left(|x|\,|y| - (n-\lambda-2)(\lambda+2)\right) - \frac{(n-\lambda-2)}{(\lambda+2)} y^t S(\lambda) x$$

where

$$S(\lambda) = \sum_{j=0}^{n-1}(-1)^j(\lambda+1)^{n-1-j}S^j.$$

Proof: Let $C(\lambda) = (\lambda+1)I + S$, $1 = [1,1,\dots,1]^t$. The characteristic polynomial of $D - xy^t$ is

$$\begin{aligned}\det(\lambda I - D + xy^t) &= \det(C(\lambda) - 11^t + xy^t)\\ &= \det C(\lambda)\ \det(I_n + C(\lambda)^{-1}[-1,x][1,y]^t)\\ &= \det C(\lambda)\ \det(I_2 + [1,y]^t C(\lambda)^{-1}[-1,x]),\end{aligned}$$

by Lemma 2. Now $C(\lambda)1 = (\lambda+2)1$, so $C(\lambda)^{-1}1 = \frac{1}{(\lambda+2)}\,1$. Thus, the expression above equals

$$\begin{aligned}&\det C(\lambda)\ \det\begin{pmatrix} 1-\frac{n}{\lambda+2} & |x|/(\lambda+2)\\ -|y|/(\lambda+2) & 1+y^tC(\lambda)^{-1}x\end{pmatrix}\\ &= \frac{\det C(\lambda)}{(\lambda+2)^2}\left(|x|\,|y| - (n-\lambda-2)(\lambda+2)(1+y^tC(\lambda)^{-1}x)\right)\end{aligned}$$

Since $\det C(\lambda)$ involves only two elementary products, it equals $(\lambda+1)^n - (-1)^n$. Since $C(\lambda)S(\lambda) = ((\lambda+1)^n - (-1)^n)I$, the above expression simplifies to that given in the statement of the proposition. ∎

Proposition 2. *Let x and y be $n\times 1$ $(0,1)$-matrices. If $D_n - xy^t$ has rank less than $n-1$, then $|x|\,|y|$ is divisible by $2(n-2)$.*

Proof: By Lemma 1, n is even and $D - xy^t$ has rank $n-2$. Since the nullspace of $D - xy^t$ has dimension 2, we may choose an $n\times 1$ matrix $u \neq o$ so that $(D - xy^t)u = o$ and $y^tu = 0$. Thus, $Du = o$ and so, by Lemma 1, we may take $u = [1,-1,1,-1,\dots,1,-1]^t$. A similar lemma and argument for D^t implies that $x^tu = 0$ as well. Thus, x and y satisfy the following *balanced subscript property:*

The number of odd subscripts at which an entry is 1 is equal to the number of even subscripts at which an entry is 1.

Since $D - xy^t$ has rank $n-2$, the constant coefficient and the coefficient of λ in the characteristic polynomial must be 0. Putting $\lambda = 0$ in the characteristic polynomial gives $y^tS(0)x = 0$. (We already know this since $S(0) = uu^t$). Since the coefficient of λ is 0, taking the derivative of the characteristic polynomial and letting λ approach 0 gives:

$$\frac{n}{4}(|x|\,|y| - 2(n-2)) - \frac{(n-2)}{2}y^tS'(0)x = 0$$

where $S'(0)$ is the circulant matrix with first row equal to $[(n-1), -(n-2), (n-3), \dots, -2, 1, 0]$. Thus, $|x|\,|y| = 2(n-2)(1+\alpha/n)$ where $\alpha = y^t S'(0)x$.

To complete the proof, it is sufficient to show that $\alpha \equiv 0 \pmod{n}$. Because of the balanced property of the odd and even subscripts of the 1 entries of x and y, we need only show that for $0 \le i, j, r, s \le n/2$,

$$(e_{2i}^t + e_{2j+1}^t)S'(0)(e_{2r} + e_{2s+1}) \equiv 0 \pmod{n}.$$

Here e_i denotes the ith standard basis vector. Since the i, j entry of $S'(0)$ is $(-1)^{i-j}(i-j-1)$ modulo n, the expression above is congruent to $(2i-2r-1)$ $-(2i-2s-1-1)$ $-(2j+1-2r-1)+(2j+1-2s-1-1) \equiv 0 \pmod{n}$.
∎

Proposition 2 implies that if x and y are $n \times 1$ $(0,1)$-matrices, then $D_n - xy^t$ must have rank at least $n-1$ whenever xy^t has less than $2(n-2)$ ones. Since D_n has $n(n-2)$ ones, this must be the case for some columns $x = X_i$ and $y = Y_i$ of any matrix factorization $D_n = XY^t$. Thus, by the remark following Lemma 1, Theorem 3 is proved.

Remarks.

1. We note that $D_n - xy^t$ can have rank $n-2$: if $x = [1,1,1,1,0,0,0,0,0,0]^t$ and $y = [0,0,0,0,0,0,1,1,1,1]^t$ then $D_{10} - xy^t$ has rank 8.
2. A result of Bridges and Ryser [1, Theorem 1.2] (see also [4]) implies that if n is odd and $D = XY^t$ where X and Y are $n \times n$ $(0,1)$-matrices, then X and Y each have constant line sums. The following example shows that this need not be the case if n is even.

$$\begin{bmatrix} 0&0&1&1&1&1\\ 1&0&0&1&1&1\\ 1&1&0&0&1&1\\ 1&1&1&0&0&1\\ 1&1&1&1&0&0\\ 0&1&1&1&1&0 \end{bmatrix} = \begin{bmatrix} 0&0&1&0&1&0\\ 0&1&0&0&1&0\\ 1&0&0&0&1&0\\ 1&0&0&1&0&0\\ 1&0&1&0&0&0\\ 0&0&1&0&0&1 \end{bmatrix} \begin{bmatrix} 1&1&0&0&0&0\\ 1&0&0&1&0&0\\ 0&0&1&1&0&0\\ 0&0&1&0&0&1\\ 0&0&0&0&1&1\\ 0&1&0&0&1&0 \end{bmatrix}$$

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