ON MINIMAL TRIANGLE-FREE 5-CHROMATIC GRAPHS

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Abstract. Avis has shown that the number of vertices of a minimal triangle-free 5-chromatic graph is no fewer than 19. Mycielski has shown that this number is no more than 23. In this paper, we improve these bounds to 21 and 22, respectively.

Let $f(k)$ denote the number of vertices in the smallest k-chromatic triangle-free graph. Chvatal [2] has demonstrated that

$$f(k) \geq \binom{k+2}{2} - 4, \quad \text{for } k \geq 4. \tag{1}$$

Mycielski [5] has constructed a sequence of graphs which demonstrates that

$$f(k) \leq 2^k - 2^{k-2} - 1, \quad k = 2, 3, 4, \ldots, \tag{2}$$

and Erdős [3] has shown that

$$f(k) < c(k \cdot \log k)^2. \tag{3}$$

In addition, Erdős [4] has constructed a sequence of graphs, G_n, which are triangle-free and for which $\alpha(G_n) < |G_n|^\beta$, for some $\beta < 1$. Let $\alpha(G)$ be the size of the largest maximum independent subset of a graph G. Since for any graph G, $\chi(G) \geq |G|/\alpha(G)$, we have $\chi(G_n) \geq |G_n|/|G_n|^\beta = |G_n|^{1-\beta}$. Thus, if we let $k = |G_n|^{1-\beta}$, the sequence G_n demonstrates constructively that

$$f(k) \leq k^{1/(1-\beta)}. \tag{4}$$

It is easy to check that $f(2) = 2$ and $f(3) = 5$. These values, together with (1), show that Mycielski's construction gives the smallest triangle-free k-chromatic graphs for $k = 2, 3$, and 4.

We are interested in the value of $f(5)$. Avis [1] has shown that $f(5) \geq 19$, and Mycielski's construction gives $f(5) \leq 23$. Using a computer algorithm, we have shown that $21 \leq f(5) \leq 22$.

Following Avis' proof, we will show that it is impossible to construct any edge-maximal, vertex-minimal, triangle-free, 5-chromatic graphs with 19 or 20 vertices.

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Let $\mathcal{G}$ be the collection of all such graphs, and let $G \in \mathcal{G}$. Let H_1 be an independent subset of G of size $\alpha(G)$, and let H_2 be the induced subgraph of G formed from the vertices of G that are not in H_1. If H_2 were 3-colorable, then we could color all the vertices in H_1 with the same fourth color and G would be 4-colorable, which it is not. Therefore H_2 must be 4-chromatic. Since H_2 is also triangle-free, it must have at least 11 vertices, since the Mycielski graph on 11 vertices is the smallest such graph. Since the Ramsey number $R(3,6)$ equals 18, any graph with 18 or more vertices either contains a triangle or an independent set of size at least 6, so H_1 must have at least 6 vertices and H_2 can have no more than 14 vertices. Thus, we know that H_2 must be a triangle-free, 4-chromatic graph with between 11 and 14 vertices.

For each vertex v_i in H_1, define S_i to be the set of neighbors of v_i in H_2. To construct all graphs in $\mathcal{G}$, one could look at every possible collection of subsets of every possible H_2 and construct a graph by adding vertices whose neighborhoods are the subsets. Then one could check whether the resulting graph has the desired properties. However, the number of 14-vertex triangle-free 4-chromatic graphs is too large for this method to be feasible. We will show that only some of these graphs need to be used. Also, we need not examine every collection of subsets.

If H_2 has 14 vertices then G has 20 vertices and $\alpha(G) = 6$. We break this into smaller cases depending on $\Delta(G)$, the largest degree in G. Brooks' theorem says that $\Delta(G) \geq \chi(G)$, and we know that $\Delta(G) \leq \alpha(G)$ since a neighborhood of a vertex is an independent set in a triangle-free graph. Therefore, $\Delta(G)$ is either 5 or 6.

If $\Delta(G) = 6$, we can choose H_1 to be the set of neighbors of a vertex of degree 6. This vertex is an element of H_2, but it has no neighbors in H_2, so without it, H_2 is still 4-chromatic. Thus, H_2 is the disjoint union of a 13-vertex, triangle-free, 4-chromatic graph and a vertex. Furthermore, since H_1 is a maximum independent set, each vertex in H_2 must be adjacent to a vertex in H_1, so $\Delta(H_2) < \Delta(G) = 6$.

If $\Delta(G) = 5$, we can choose a vertex in H_2 that is adjacent to the most vertices in H_1 (this vertex is not necessarily unique). Let x be that vertex and let β be the number of vertices to which x is adjacent in H_1. Since $\Delta(G)$ is 4, $\beta \leq 5$. Since G is edge-maximal, every pair of vertices of G either is adjacent or shares a neighbor. Since no two vertices in H_1 are adjacent, each pair shares a neighbor. If $\beta = 2$, then no vertex in H_2 is adjacent to more than 2 vertices in H_1. However, there are 15 pairs of vertices in H_1 and only 13 vertices in H_2. Therefore, $\beta > 2$. Consequently, x is adjacent to 2 or fewer vertices in H_2, which means that without x, H_2 is still 4-chromatic. So H_2 can be constructed from a 13-vertex triangle-free 4-chromatic graph by adding a vertex and 1 or 2 edges from that vertex (the case involving no additional edges is covered in the $\Delta(G) = 6$ case). Also $\Delta(H_2) < \Delta(G) = 5$.

Let $\mathcal{H}$ be the collection of all 11-, 12-, and 13-vertex triangle-free 4-chromatic graphs, together with all graphs satisfying the conditions on H_2 in the preceding cases. We now describe how the graphs in $\mathcal{H}$ can be generated by computer. Since the 14-vertex graphs in $\mathcal{H}$ arise from the 13-vertex graphs in $\mathcal{H}$ as described above, we need only explain how the 11-, 12-, and 13-vertex graphs in $\mathcal{H}$ are obtained. Let H_2 be such a graph. Then $\alpha(H_2) \geq 4$, since $R(3,4) = 9$. Furthermore, if T is a maximum independent set of H_2, then $H_2 - T$ is 3-chromatic, with at most 9 vertices. Thus, to find all such graphs H_2, it is enough to create a list of all 3-chromatic, triangle-free graphs with at most 4 vertices. For each graph U in this list, add an independent set T with at least 4 vertices, and add edges in all possible ways between T and U.

We now give some definitions which are used in the theorem which follows. Given a graph H_2, let $C_1, C_2, \ldots, C_m$ be the set of all 4-colorings of H_2. A subset S of H_2 *color-dominates* a coloring C_j if and only if S is colored with 4 colors in the coloring C_j. A collection of subsets $B = S_1, S_2, \ldots, S_k$ *color-dominates* a collection of colorings $C = C_1, C_2, \ldots, C_r$ if and only if for every coloring in C there exists a set in B that color-dominates that coloring. A collection B of subsets of H_2 is called a *color-dominating set* if the collection color-dominates all 4-colorings of H_2.

Given a graph H_2 and a collection $B = \{S_1, S_2, \ldots, S_n\}$ of subsets of H_2, the graph *generated* by these objects is the graph whose vertex set is the set of vertices in H_2 together with a vertex v_i for $i = 1, 2, \ldots, n$, and whose edge set is the set of edges in H_2 together with edges from v_i to each vertex in S_i, for $i = 1, 2, \ldots, n$.

Theorem. *If H_2 is a triangle-free, 4-chromatic graph, and $B = \{S_1, S_2, \ldots, S_n\}$ is a collection of subsets of H_2, then the graph G generated by H_2 and B is 5-chromatic if and only if B color-dominates every 4-coloring of H_2.*

Proof: If the set B does not color-dominate all 4-colorings of H_2, then there is some coloring C of H_2 which is color-dominated by no subset S_i. Under that coloring of H_2, each S_i is colored with 3 or fewer colors, so it is possible to color each v_i in H_1 with a color not found in S_i, thereby 4-coloring G. Therefore, if G is 5-chromatic, the set B must necessarily color-dominate all 4-colorings of H_2.

Let us now assume that G is 4-colorable. Any 4-coloring of G gives a 4-coloring of H_2. If B color-dominates every 4-coloring of H_2, then all 4 colors appear in some S_i in B. Then v_i is colored the same as one of its neighbors. This contradiction shows that G is 5-chromatic. ∎

This theorem gives rise to an algorithm, described below, which produces all graphs in $\mathcal{G}$. For each $H_2 \in \mathcal{H}$, we want to find a certain dominating collection of subsets of H_2 from which we can construct G. Any such subset S of H_2 in this collection must satisfy the following criteria:

1) *S is an independent set of* H_2. Since G is triangle-free, no two vertices of S can be adjacent, for otherwise they would form a triangle with a vertex in H_1.
2) *S is a maximal independent set in* H_2. Since G is edge-maximal, every two vertices are either adjacent or share a neighbor. Thus, for each i, a vertex w in H_2 is either adjacent to v_i (that is, w is an element of S_i), or shares a neighbor with v_i (that is, w is adjacent to some vertex in S_i). Therefore, if w is not adjacent to any vertex in a particular subset S, it is an element of that S.
3) *S has at least 4 vertices*. Otherwise, S dominates no 4-coloring of H_2.

We create a list of all subsets of H_2 that satisfy these criteria and a list of all 4-colorings of H_2. We then perform a standard backtrack to find the smallest collection of subsets that dominates every coloring.

When this backtrack algorithm was performed on each graph in $\mathcal{H}$, no 19- or 20-vertex solutions were generated. Since the above theorem shows that all 19- and 20-vertex graphs will be found by this algorithm, it must be the case that there are no such graphs. Thus, $f(5) \geq 21$.

It is possible to let the algorithm run deeper into the backtrack tree, thereby generating 5-chromatic, triangle-free graphs with more than 20 vertices. Upon doing this, we found a few 22-vertex graphs with these properties, one of which is given below. While it can easily be checked by computer that this graph has the required properties, to show that it is 5-chromatic by hand seems to be difficult. This shows that $f(5) \leq 22$. To determine whether $f(5) = 21$ or $f(5) = 22$, one must search for 21-vertex graphs with the required properties. The above algorithm will work, but the collection $\mathcal{H}$ must be greatly expanded, thus requiring an enormous amount of computer time.

A 22-vertex triangle-free 5-chromatic graph

Vertex	Neighbors								
1	2	5	9	10	17	19	20	22	
2	1	3	11	12	14	15			
3	2	4	8	9	18	19	21	22	
4	3	5	10	11	14	17	20		
5	1	4	8	12	15	18	21		
6	11	12	14	15	17	18	19		
7	8	9	10	11	12	17	18	19	
8	3	5	7	14	16	20			
9	1	3	7	14	15	16			
10	1	4	7	15	16	21			
11	2	4	6	7	16	21	22		
12	2	5	6	7	16	20	22		
13	14	15	16	17	18	19	20	21	22
14	2	4	6	8	9	13			
15	2	5	6	9	10	13			
16	8	9	10	11	12	13			
17	1	4	6	7	13				
18	3	5	6	7	13				
19	1	3	6	7	13				
20	1	4	8	12	13				
21	3	5	10	11	13				
22	1	3	11	12	13				

References

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