# GENERATING FUNCTIONS FOR A PROBLEM OF RIORDAN

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**Abstract.** We derive a first order recurrence for $a_n(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{1+tk}\binom{n}{k}$ ($t$ fixed, $t \neq -\frac{1}{m}$, $m \in \mathrm{N}$). The first order recurrence yields an alternative proof for Riordan's theorem: $a_n(t) = \binom{1/t+n}{n}^{-1}(-1)^n$ and also yields the ordinary generating function $\sum_{n=0}^{\infty} a_n(t)x^n$ for $t \in \mathrm{N}$. From the latter, one easily computes $\sum_{n=0}^{\infty} a_n(t)$ which turns out to be the well-known $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2$ for $t = 1$. For $t = 2$, we get $\sum_{n=0}^{\infty}(-1)^n \frac{(2n)}{(2n+1)} = \frac{\ln(\sqrt{2}+1)}{\sqrt{2}}$.

In the present paper we investigate the sums $a_n(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{1+tk}\binom{n}{k}$ ($t$ fixed, $t \neq -\frac{1}{m}$, $m \in \mathrm{N}$). Riordan proved $a_n(t) = \binom{1/t+n}{n}^{-1}(-1)^n$ by the first order two-variable recurrence $(-1)^n a_n(t) = a_{n-1}(t) - \frac{1}{t+1}a_{n-1}\left(\frac{t}{t+1}\right)$ in [2], Chapter 1, Problems 4–5. Our aim is to derive a first order one-variable recurrence for $a_n(t)$, which will give an alternative proof for Riordan's theorem and make it possible to compute the ordinary generating function of $a_n(t)$. The ordinary generating function yields a number of numerical identities including the well-known

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2.$$

The exponential generating function for $a_n(t)$ is also of some importance. As we learned from Raji Sinha, in plasma physics the solution $E(z)$ of $E'(z) = 1 - 2zE(z)$ is the Conte function, which is closely related to the plasma dispersion function [1], p. 79. The explicit solution of the differential equation is

$$E(z) = ce^{-z^2} + \sum_{\ell=0}^{\infty} \frac{z^{2\ell+1}}{\ell!} a_\ell(2).$$

**Lemma 1.** *The recurrence formula* $(tn+1)a_n(t) = -tna_{n-1}(t)$ *holds.*

Proof: We have $a_n(t) = \int_0^1 (x^t - 1)^n\,dx = \int_0^1 (x^t - 1)^{n-1}(x^t - 1)\,dx$. Letting $u' = x^t - 1$ and integrating by parts yields $a_n(t) = -\frac{(n-1)t}{t+1}\int_0^1 (x^t - 1)^{n-1} x^{t-1} x\,dx$ $+\frac{(n-1)t^2}{t+1}\int_0^1 (x^t - 1)^{n-2} x^{t-1} x\,dx$. Letting $u' = (x^t-1)^{n-1}x^{t-1}$, resp. $(x^t-1)^{n-2}x^{t-1}$ in the above integrals and again integrating by parts yields $a_n(t) = \frac{n-1}{n(t+1)}a_n(t)$ $-\frac{t}{t+1}a_{n-1}(t)$. Hence $a_n(t) = \frac{-nt}{nt+1}a_{n-1}(t)$. ∎

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**Theorem 1 (Riordan).**

$$a_n(t) = \binom{1/t+n}{n}^{-1}(-1)^n.$$

Proof: We have from Lemma 1 $a_n(t) = \frac{-nt}{nt+1}a_{n-1}(t) = (-1)^n\left(\prod_{k=1}^{n}\frac{k}{k+1/t}\right)a_0(t)$ $= (-1)^n\binom{1/t+n}{n}^{-1}$. ∎

For $y(x) = \sum_{n=0}^{\infty} a_n(t)x^n$ the recurrence implies the differential equation

$$y\frac{1+tx}{x(t+tx)} + y' = \frac{1}{x(t+tx)}$$

with $y(0) = a_0(t) = 1$. As is well-known,

$$y(x) = e^{-\int\frac{(1+tx)dx}{x(t+tx)}}\left(\int\frac{1}{x(t+tx)}e^{\int\frac{(1+tx)dx}{x(t+tx)}}\,dx + c(t)\right).$$

It is easy to check that $e^{\int\frac{1+tx}{x(t+tx)}dx} = x^{1/t}(x+1)^{(t-1)/t}$, and

$$\int\frac{1}{x(t+tx)}x^{1/t}(x+1)^{(t-1)/t}\,dx = \int\frac{du}{(u^t+1)^{1/t}}$$

with the substitution $x = u^t$. The latter is equal to $-\int\frac{dz}{z^t-1}$ with the substitution $u^t = \frac{z^t}{1-z^t}$, and from now on we need $t \in \mathbf{N}$ to split the integrand into partial fractions. For even $t$ we have from [3], p. 61,

$$\int\frac{dz}{z^t-1} = \frac{1}{t}\ln\left|\frac{z-1}{z+1}\right| + \frac{1}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\cos\frac{2k\pi}{t}\ln\left(z^2-2z\cos\frac{2k\pi}{t}+1\right)$$
$$-\frac{2}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\sin\frac{2k\pi}{t}\arctan\frac{z-\cos\frac{2k\pi}{t}}{\sin\frac{2k\pi}{t}};$$

and analogously for odd $t$

$$\int\frac{dz}{z^t-1} = \frac{1}{t}\ln|z-1| + \frac{1}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\cos\frac{2k\pi}{t}\ln\left(z^2-2z\cos\frac{2k\pi}{t}+1\right)$$
$$-\frac{2}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\sin\frac{2k\pi}{t}\arctan\frac{z-\cos\frac{2k\pi}{t}}{\sin\frac{2k\pi}{t}}.$$

After this point it is easy to get the general solution of the differential equation:

**Theorem 2.** *For $t \in \mathrm{N}$ we have*

$$y(x) = -x^{-1/t}(x+1)^{(1-t)/t}\Bigg[A(x) + \frac{1}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\cos\frac{2k\pi}{t}\ln\left(\frac{x^{2/t}}{(x+1)^{2/t}} - \frac{2x^{1/t}}{(x+1)^{1/t}}\cos\frac{2k\pi}{t}+1\right) - \frac{2}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\sin\frac{2k\pi}{t}\arctan\left(\frac{\frac{x^{1/t}}{(x+1)^{1/t}} - \cos\frac{2k\pi}{t}}{\sin\frac{2k\pi}{t}}\right) + c(t)\Bigg],$$

*where* $A(x) = \frac{1}{t}\ln\left|\frac{x^{1/t}}{(x+1)^{1/t}} - 1\right|$ *for odd $t$ and* $A(x) = \frac{1}{t}\ln\left|\frac{\frac{x^{1/t}}{(x+1)^{1/t}}-1}{\frac{x^{1/t}}{(x+1)^{1/t}}+1}\right|$ *for even $t$.*

With $y(0) = 1$, we have $c(t) = \lim_{x\to 0}[-y(x)x^{1/t}(x+1)^{(t-1)/t} - G(x)]$, where $G(x)$ is the function in the brackets in Theorem 2 without $c(t)$. Hence,

$$c(t) = \frac{2}{t}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}\sin\frac{2k\pi}{t}\arctan\left(-\cot\frac{2k\pi}{t}\right),$$

noting with arctan an odd function, the above sum is zero for $t$ even; consequently

$$c(t) = \begin{cases} 0, & t \text{ even} \\ \frac{\pi}{t^2}\sum_{k=1}^{\lfloor\frac{t-1}{2}\rfloor}(4k-t)\sin\frac{2k\pi}{t}, & t \text{ odd.}\end{cases}$$

∎

Letting $t = 1$ and $t = 2$, evaluating $y(1)$ we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \ln 2$$

and

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!!}{(2n+1)!!} = \frac{\ln(\sqrt{2}+1)}{\sqrt{2}},$$

where

$$s!! = \begin{cases} s(s-2)(s-4)\cdots 2, & s \text{ even} \\ s(s-2)(s-4)\cdots 1, & s \text{ odd.}\end{cases}$$

We remark, that $y(-x)$ is the generating function for the sequence

$$\sum_{k=0}^{n}\frac{(-1)^k}{1+tk}\binom{n}{k} = \binom{1/t+n}{n}^{-1},$$

which was the original problem of Riordan.

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## References

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