MENGER PATH SYSTEMS

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Abstract. For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G, there are m openly disjoint paths of length at most d. A collection of such paths is called a *Menger path system*. Minimal conditions involving various combinations of the connectivity, minimal degree, sum of degrees, and unions of neighborhoods of pairs of nonadjacent vertices that insure the existence of Menger path systems are investigated. For example, if for fixed positive integers $d \geq 2$ and m, a graph G has order n, connectivity $k \geq m$, and minimal degree $\delta > (n-(k-m+1)(d-2))/2+m-2$, then G has property $P_{d,m}(G)$ for n. Also, if a graph G of order n satisfies $NC(G) > 5n/(d+2)+2m$, then $P_{d,m}(G)$ is satisfied. (A graph G satisfies $NC(G) \geq t$ if the union of the neighborhoods of each pair of nonadjacent vertices is at least t.) Other extremal results related to Menger path systems are considered.

1. INTRODUCTION

Consider a graph G that models a computer network with each vertex representing a processor and each edge representing a two-way communication link. To insure that the network is fault-tolerant with respect to processor failures, it is necessary that the number of openly disjoint paths between each pair of vertices of G exceed the number of possible failures. Connectivity is clearly the crucial graph concept. However, the length of time for the information to arrive is also important, so it is desirable that the openly disjoint paths be short. This requires that between each pair of the vertices of the graph G there is a specified number of paths, each with a bound on the number of vertices.

For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G there are at least m openly disjoint paths each of length at most d. The graph G representing a computer network prone to processor failures should satisfy $P_{d,m}(G)$ for appropriate values of d and m. This is one motiviation for studying graphs with property $P_{d,m}(G)$.

Menger's classical result [M] connectivity solves the problem of the existence of a system of such paths, if there is no concern for the length of the paths in the system. Although Menger's theorem gives no information about the length of the paths, the "length problem" has been studied. For example, in [BP] Bond and Peyrat studied the effect of adding or deleting edges of the diameter of a network, and Chung and Garey considered diameter bounds for altered graphs in [CG]. Menger type results for paths of bounded length were proved by Lovász, Neumann-Lara, and Plummer in [LNP] and by Pyber and Tuza in [PT], and Mengerian theorems for long paths were given by Montejanao and Neumann-Lara in

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[MN], and by Hager in [H]. In [O] property $P_{d,m}$ and its application to computer networks and distributed processing was introduced. Extremal results for $P_{d,m}$ were investigated in [FJOST].

We will extend the results in [FJOST] by investigating various combinations of connectivity, minimum degree, degree properties and neighborhood conditions of a graph G that imply $P_{d,m}(G)$. In particular, the following six results will be proved.

A graph G of order n with various minimum degree and connectivity conditions will be considered in the following theorems. These conditions will obviously place restrictions on the order n of G, but these restrictions will not be explicitly listed in the statement of the theorems. Also, all of the theorems are either sharp or have the correct order of magnitude for n sufficiently large. Examples to verify this will be described in section 2.

THEOREM 1. *Let $d \geq 2$ and m be positive integers, and let G be a k-connected graph of order n. Then, G satisfies $P_{d,m}(G)$ if the minimal degree $\delta(G)$ satisfies:*

$$\delta(G) \begin{cases} \geq \lfloor \frac{n+m}{2} \rfloor & \text{if } k < m \\ > \min \begin{cases} \frac{n-m+2}{\lfloor \frac{d+4}{3} \rfloor} + m - 2 \\ \frac{n-(k-m+1)(d-2)+m-2}{2} \end{cases} & \text{if } m \leq k \leq \frac{n-m}{d} + m - 1 \\ \geq k & \text{if } k > \frac{n-m}{d} + m - 1 . \end{cases}$$

Some times a condition on $\delta(G)$ in a theorem can be replaced by a condition on the sum of the degrees of nonadjacent vertices to obtain a stronger result. This is true in the case of Theorem 1. In the remaining theorems, $DC(G) \geq t$ means that the sum of the degrees of each pair of nonadjacent vertices of G is at least t. Different results can also be obtained by using a condition on the union of the neighborhoods of nonadjacent vertices instead of a minimal degree condition or a sum of degrees condition. In the following theorems, $NC(G) \geq t$ means that the union of the neighborhoods of each pair of nonadjacent vertices of G is at least t.

THEOREM 2. *Let $d \geq 3$ and m be positive integers, and let G be an m-connected graph of order n. Then, G satisfies $P_{d,m}(G)$ if*

$$NC(G) > (5n - m - d - 4)/(d + 2) + 2m - 2 .$$

For a graph G of order n, the neighborhood condition $NC(G) \geq t$ for $t < n-1$ is not strong enough to imply any connectivity in G. That is the reason that it was necessary to assume that G was m-connected in Theorem 2. If the connectivity of G is $k > m$, and k is sufficiently larger then a weaker neighborhood suffices to insure that $P_{d,m}(G)$ is satisfied. This follows from the next theorem, which also deals with the degree condition $DC(G)$.

THEOREM 3. *Let $d \geq 3$, m, and $k > (d-1)(m-1)$ be positive integers and let G be a k-connected graph of order n. Then, G satisfies property $P_{d,m}$ if either*

$$NC(G) > n-2-(d-3)(k-(d-1)(m-1)), \quad or$$
$$DC(G) > n-2-(d-3)(k-(d-1)(m-1)) + m.$$

The results of Theorem 2 and Theorem 3, along with some of the theorems from [FJOST], form the basis for summary results involving degree and neighborhood conditions that are similar to Theorem 1. They are stated next.

THEOREM 4. *Let $d \geq 3$ and m be positive integers, and let G be a k-connected graph of order n. Then, G satisfies $P_{d,m}(G)$ if the degree condition $DC(G)$ satisfies:*

$$DC(G) \begin{cases} \geq n+m-2 & \text{if } k < m \\ > \min \begin{cases} 2\left(\frac{n-m+2}{\lfloor \frac{d+4}{3} \rfloor} + m - 2\right) \\ n-(d-3)(k-(m-1)(d-1)) + m \end{cases} & \text{if } m \leq k \leq \frac{n-m}{d} + m - 1 \\ \geq 2k & \text{if } k > \frac{n-m}{d} + m - 1. \end{cases}$$

THEOREM 5. *Let $d \geq 3$ and m be positive integers, and let G be a k-connected graph of order n. Then, G satisfies $P_{d,m}(G)$ if the neighborhood condition $NC(G)$ satisfies:*

$$NC(G) \begin{cases} \geq n-1 & \text{if } k < m \\ > \min \begin{cases} \frac{5n-m-d-4}{d+2} + 2m - 2 \\ n-2-(d-3)(k-(m-1)(d-1)) \end{cases} & \text{if } \leq k \leq \frac{n-m}{d} + m - 1 \\ \geq k & \text{if } k > \frac{n-m}{d} + m - 1. \end{cases}$$

The $d = 2$ case is special for degree and neighborhood conditions for nonadjacent pairs of vertices, and does not fit the same pattern as the $d > 2$ cases. Thus, it was not part of either Theorem 4 or Theorem 5. The next theorem deals with this special case.

THEOREM 6. *For any positive integer m, a graph G of order n and connectivity k satisfies $P_{2,m}(G)$ if at least one of the following inequalities are satisfied:*

$$k > \frac{n+m}{2} - 1$$
$$\delta(G) > \frac{n+m}{2} - 1$$
$$DC(G) > \frac{3n+m}{2} - 3$$
$$NC(G) > n-2.$$

Also, each of the inequalities is sharp.

2. EXAMPLES AND PRELIMINARY RESULTS

Notation and standard definitions in the paper will generally follow that found in [CL]. Any special notation will be described as needed. We start with some results that will be used in the proofs of the main theorems, and we describe some examples that indicate the sharpness of these results.

The first result gives the minimal degree required to imply $P_{d,m}$, and can be found in [FJOST].

THEOREM A. *Let $d \geq 2$ and m be positive integers. If G is a graph of order n with $\delta(G) \geq \lfloor (n+m)/2 \rfloor$, then G satisfies $P_{d,m}(G)$. Further, the condition is sharp.*

The following result does not appear in [FJOST], but it can easily be derived with the same type of elementary counting proof.

THEOREM B. *Let $d \geq 3$ and m be positive integers. If G is a graph of order n with $DC(G) \geq n+m-2$ then G satisfies $P_{d,m}(G)$. Further, the condition is sharp.*

PROOF: Suppose that the result is not true, and that x and y are a pair of vertices of G that do not have m paths of length at most d between them. If x and y are nonadjacent, then then have at most $m-1$ common adjacencies, so $d(x)+d(y) \leq n-2+m-1 < n+m-2$ a contradiction. This verifies that nonadjacent pairs of vertices have m disjoint paths between then of length 2. If x and y are adjacent, then with no loss of generality we can select a vertex z that is adjacent to x but not to y. By the previous case, there exists m paths of length 2 from y to z, and one of these paths contains x. This gives immediately m paths of length at most 3 from x to y.

The graph $H = K_{m-1} + (\lfloor K_{\frac{n-m+1}{2}} \rfloor \cup \lceil K_{\frac{n-m+1}{2}} \rceil)$ has connectivity $m-1$, $DC(H) = n+m-3$, and does not satisfy $P_{d,m}(H)$. This verifies the sharpness of Theorem B, and completes the proof of Theorem B. ∎

Before stating the next preliminary results, we will describe a family of examples related to these results. This family of graphs also plays an important role in general for graphs with Menger path systems. Graphs with the same connectivity, and even the same minimal degree, can have vastly different Menger path systems. Consider the generalized wheel graph $W_{m-2,n-m+2} = K_{m-2} + C_{n-m+2}$, which has order n, and connectivity and minimal degree m. It is easy to see that this graph does not satisfy $P_{n-m,m}$, since any m internally openly disjoint paths between a pair of adjacent vertices of the rim of the wheel will have one path that contains all of the vertices of the rim of the wheel. On the other hand, the m-cube Q_m has order $n = 2^m$, connectivity and minimal degree m, and it satisfies $P_{m+1,m}$. This last assertion is easy to verify by an induction argument on the index m of the m-cube.

"Wheel type" graphs give important information on the extremal properties related to $P_{d,m}$. We start with the wheel graph $W_r = K_1 + C_r$ that has r spokes and r vertices on the rim. Replace each vertex of W_r with a complete graph, and make each vertex of the corresponding complete graph adjacent to the vertices in the neighborhood of the replaced vertex. The graphs obtained by this expansion of vertices of a wheel form a family of "generalized wheels". More precisely, order the vertices of W_r starting with the center and followed by the vertices of the rim in a natural order around the cycle. For positive integers $p(i)$ $(0 \leq i \leq r)$, the generalized wheel obtained from W_r by replacing the i^{th} vertex with a complete graph $K_{p(i)}$ will be denoted by $W(p(0), p(1), \cdots, p(r))$.

In many of the cases of interest to us, most of the $p(i)$'s in the generalized wheel will be the same, so we will adopt the more compact notation of representing the sequence $(p(j), \cdots, p(k))$ by $(k-j+1; p)$ when $p = p(j) = \cdots = p(k)$. Thus, $W(1, r; 1) = W_r$ and $W(m-2, n-m+2; 1) = K_{m-2} + C_{n-m+2}$, which is the generalized wheel considered earlier in this section. For the following families of generalized wheels, it will be assumed that $d \geq 2$ and m are fixed positive integers.

Select any integer n such that $n-m$ is divisible by d, and consider the generalized wheel

$$W(m-2, d; (n-m)/d, 1, 1).$$

Let x and y denote the two vertices of the rim of the generalized wheel that are associated with the complete graphs that are a single vertex. This graph has order n, connectivity $m-1+(n-m)/d \geq m$, and $m-1$ internally disjoint paths of length at most 2 between x and y. However, any path from x to y not using any of the $m-2$ vertices in the center of the generalized wheel or the edge xy has length at least $d+1$. Therefore, $W(m-2, d; (n-m)/d, 1, 1)$ does not satisfy $P_{d,m}$. The following theorem (Theorem C from [FJOST]) gives that any graph with connectivity exceeding $(n-m)/d + m - 1$ does satisfy $P_{d,m}$.

THEOREM C. *Let $d \geq 2$ and m be positive integers, and let G be a graph of order n. If G has connectivity exceeding $(n-m)/d + m - 1$, then $P_{d,m}(G)$ is satisfied. This result is the best possible in that there is a graph that has connectivity $(n-m)/d + m - 1$ that does not satisfy $P_{d,m}(G)$.*

Select any positive integer p, let $n = (d+4)p + m - 4$, and consider the generalized wheel graph

$$W(m-2, 3p-2, d-2; p, 3p-2, 1, 1).$$

Again, let x and y be the vertices of the rim of the generalized wheel associated with the complete graphs with a single vertex. This graph has order n, minimum degree $3p+m-3 = 3(n-m+4)/(d+4)+m-3$, and connectivity $p+m-1 \geq m$. Just as before, any path between x and y that does not contain the edge xy or any of the $m-2$ vertices in the center of the wheel has length at least $d+1$. Thus, this graph does not satisfy $P_{d,m}$, but has minimum degree of the same order of magnitude as the degree condition in the hypothesis of the following theorem (Theorem D of [FJOST]).

THEOREM D. *Let m and d be positive integers, and let G be an m-connected graph of order n. If G has minimum degree exceeding $\lfloor (n-m+2)/\lfloor (d+4)/3 \rfloor \rfloor + m - 2$, then G satisfies $P_{d,m}(G)$.*

It should be noted that the proof of Theorem D in [FJOST] can be modified in a completely straightforward way to verify that if the sum of the degrees of each pair of nonadjacent vertices is ay least $2(\lfloor (n-m+2)/\lfloor (d+4)/3 \rfloor \rfloor + m - 2$, then G has property $P_{d,m}$ for $d \geq 3$. Also, the generalized wheel described prior to Theorem D shows that the sum of degrees condition cannot be significantly lowered. We state this result for use later.

THEOREM E. *Let m and $d \geq 3$ be positive integers, and let G be an m-connected graph of order n. If G satisfies the degree condition $DC(G) > 2(\lfloor (n-m+2)/\lfloor (d+4)/3 \rfloor \rfloor m - 2)$, then G satisfies $P_{d,m}(G)$.*

The next counting result, which can be found in [FJOST], will be used in the proof of the main results.

LEMMA F. *Let P and Q be openly disjoint paths from x to y in a graph G, such that the sum of their lengths is a minimum. If A and B are subsets of vertices of P and Q respectively, such that A does not contain any pair of consecutive vertices on P, then the number of edges between A and B is at most $|A|+|B|-1$.*

3. PROOFS

We begin with the proof of Theorem 1, which depends heavily on the results in [FJOST].

THEOREM 1. *Let $d \geq 2$ and m be positive integers, and let G be a k-connected graph of order n. Then, G satisfies $P_{d,m}(G)$ if the minimal degree $\delta(G)$ satisfies:*

$$\delta(G) \begin{cases} \geq \lfloor \frac{n+m}{2} \rfloor & \text{if } k < m \\ > \min \begin{cases} \frac{n-m+2}{\lfloor \frac{d+4}{3} \rfloor} + m - 2 \\ \frac{n-(k-m+1)(d-2)+m-2}{2} \end{cases} & \text{if } m \leq k \leq \frac{n-m}{d} + m - 1 \\ \geq k & \text{if } k > \frac{n-m}{d} + m - 1. \end{cases}$$

PROOF: For $k < m$, Theorem A implies the result. Also, if $k > (n-m)/d + m - 1$, then G satisfies $P_{d,m}$ by Theorem C. Note that in general $\delta \geq k$, and Theorem 1 is true for $k > (n-m)/d + m - 1$. We are left to consider only the following range for k : $m \leq k \leq (n-m)/d + m - 1$. By Theorem D, $\delta(G) > ((n-m+2)/\lfloor (d+4)/3 \rfloor + m - 2$ implies that $P_{d,m}$ is satisfied, so to complete the proof of Theorem 1 it is sufficient to verify that $\delta(G) > (n - (k - m + 1)(d-2) + m - 2)/2$ implies $P_{d,m}(G)$.

Suppose that G is a graph of order n that does not satisfy $P_{d,m}$. Select vertices x and y for which there does not exists m openly disjoint paths between the two vertices, each of length at most d. Since G is $k \geq m$ connected, there are k openly disjoint paths between x and y. Select k such paths with the sum of the lengths a minimum. Denote the paths by $P_1, P_2, \cdots, P_k$, and let $r_1 \leq r_2 \leq \cdots \leq r_k$ by the number of interior vertices (one less than the length of the path) of each of the paths. Then by assumption, $r_j \geq 1$ for $j \geq 2$, and $r_j \geq d$ for $j \geq m$. If r_0 is the number of vertices not on any of these paths, then

$$n = 2 + \sum_{i=0}^{k} r_i.$$

Therefore,

$$r_0 + n - 2 - \sum_{i=1}^{k} r_i \leq n - 2 - (m-2) - (k - m + 1)d.$$

By assumption, x and y have no common adjacencies off the paths, and each of x and y have precisely one adjacency on each of the k paths. Therefore, if $d(x) \leq d(y)$, then

$$d(x) \leq \frac{r_0}{2} + k \leq \frac{n - m - (k - m + 1)d + 2k}{2}.$$

This contradicts the assumption on $\delta(G)$ and completes the proof of Theorem 1. ∎

Note that each of Theorem A and Theorem C is sharp, so no improvement is possible in Theorem 1 for $k < m$ or $k > (n-m)/d + m - 1$. Although the result in Theorem D is not sharp, it has the correct order of magnitude as exhibited by the example that preceded Theorem D. We now describe an example to illustrate that the inequality $\delta(G) > (n - (k - m + 1)(d-2) + m - 2)/2$ also has the correct order of magnitude. For $(n + (d+3)(m-1))/(d+4) < k \leq (n-m)/d + m - 1$, and the appropriate divisibility conditions for n, consider the graph

$$W\left(m-1, d-2; \ell, \frac{n - (d-2)\ell - m - 1}{2}, 1, 1, \frac{n - (d-2)\ell - m - 1}{2}\right),$$

where $\ell = k - m + 1$. The graph L obtained from this graph by deleting the edge between the two complete graphs on the rim with a single vertex has connectivity

k, $\delta(L) = (n-(d-2)(k-m+1)+m-3)/2$, and does not satisfy $P_{d,m}(L)$. Thus, the results of Theorem 1 cannot be substantially improved.

Both Theorem 4 and Theorem 5 will follow from some of the preliminary results already stated in section 2 and the following theorems. These next results give some relationships between degree and neighborhood conditions on all nonadjacent pairs of vertices and property $P_{d,m}$.

THEOREM 2. *Let $d \geq 3$ and m be positive integers, and let G be an m-connected graph of order n. Then, G satisfies $P_{d,m}(G)$ if*

$$NC(G) > (5n-m-d-4)/(d+2)+2m-2.$$

PROOF: Let $t = (5n-m-d-4)/(d+2)+2m-2$. Suppose that G is an m-connected graph with $NC(G) > t$ that does not satisfy $P_{d,m}(G)$. We will show that this leads to a contradiction.

Let x and y be a pair of vertices for which there does not exist m openly disjoint paths between them, each of length at most d. Select m openly disjoint paths between x and y such that the sum of their lengths is a minimum. Denote these paths be $P_1, P_2, \cdots, P_m$, and assume that the lengths of these paths are $r_1+1, r_2+1, \cdots, r_m+1$ respectively with $r_1 \leq r_2 \leq \cdots \leq r_m$. Let R by the vertices in this system of paths, and let S be the remaining vertices. Thus, the number of vertices in R is

$$r = 2+\sum_{i=1}^{m} r_i,$$

and S has $n-r$ vertices. Note that by assumption $r_i \geq 1$ for $1 < i < m$, and $r_m \geq d$. Hence, $r \geq m+d$.

Consider the subgraph L of G induced by the vertices in the paths P_i and P_m, and note that the sum of the lengths of these paths cannot be shortened in L. Lemma F applied twice (to the graph L) implies that the number of edges between the r_m interior vertices of P_m and the r_i interior vertices of P_i is at most r_m+2r_i-2 for each $i (1 \leq i < m)$. Since P_m has r_m edges and x and y each have degree m relative to R, the sum of the degrees of the vertices of P_m relative to R is at most

$$\sum_{i=1}^{m-1}(r_m+2r_i-2)+2m+2r_m = (m-1)r_m+2r-2.$$

Let $P_m = (x = x_0, x_1, \cdots, x_{r_m}, x_{r_m+1} = y)$. For $0 \leq i < j \leq r_m+1$, let

$$N_{ij} = N(x_i) \cup N(x_j),$$
$$R_{ij} = R \cap N_{ij}, \text{ and}$$
$$S_{ij} = S \cap N_{ij}.$$

Hence, N_{ij} is the disjoint union of R_{ij} and S_{ij}, and if x_i and x_j are nonadjacent, then

$$|R_{ij}| + |S_{ij}| = |N_{ij}| > t.$$

Therefore,

$$(r_m+2)t < \sum_{i=0}^{r_m-1}(|R_{i,i+2}|+|S_{i,i+2}|)+|R_{0,r_m}|+|S_{0,r_m}|+|R_{1,r_{m+1}}|+|S_{1,r_{m+1}}|. \quad (1)$$

Note that no $z \in S$ can be adjacent to both x_i and x_j for any $|j - i| \geq 3$, so each vertex of S will be in at most five of the S_{ij}'s in (1). Also, the neighborhood of a vertex in P_m will contribute to precisely two of the R_{ij}'s in (1). Thus, a consequence of (1) is the following:

$$\begin{aligned}(r_m + 2)t &< 5(n - r) + 2((m - 1)r_m + 2r - 2). \\ &< 5n - r - 4 + 2(m - 1)r_m.\end{aligned}$$

Hence,

$$\begin{aligned}t &< (5n - r - 4 + 2(m - 1)r_m)/(r_m + 2) \\ &< (5n - m - d - 4)/(d + 2) + 2(m - 1)\end{aligned}$$

This contradicts the restriction on t, and completes the proof of Theorem 2. ∎

Let t be even and $n = (d + 4)t + m - 1$. Consider the graph H obtained from the generalized wheel

$$W(m - 1, 2; 2t, d; t)$$

by deleting all of the edges between the two complete graphs on the rim with $2t$ vertices. This graph has order n, connectivity $t + m - 1$, satisfies $NC(H) \geq 5t + m - 3$, but does not have property $P_{d,m}(H)$. Thus the condition in Theorem 3 is the correct order of magnitude for d and m fixed and n sufficiently large.

The next theorem involves both degree and neighborhood conditions for pairs of nonadjacent vertices. The graph L derived from the generalized wheel

$$W\left(m - 1, d - 2; \ell, \frac{n - (d - 2)\ell - m - 1}{2}, 1, 1, \frac{n - (d - 2)\ell - m - 1}{2}\right),$$

(where $\ell = k - m - 1$) which was described after the proof of Theorem 1 indicates that the conditions of this next result cannot be lowered significantly.

THEOREM 3. *Let $d \geq 3$, m, and $k > (d - 1)(m - 1)$ be positive integers and let G be a k-connected graph of order n. Then, G satisfies property $P_{d,m}$ if either*

$$NC(G) > n - 2 - (d - 3)(k - (d - 1)(m - 1)), \quad or$$
$$DC(G) > n - 2 - (d - 3)(k - (d - 1)(m - 1)) + m.$$

PROOF: We will deal with neighborhood condition and the degree condition at the same time. Let $t = n - 2 - (d - 3)(k - (d - 1)(m - 1))$, and suppose that G is an k-connected graph ($k > (d - 1)(m - 1)$) with $NC(G) > t$ (or respectively $DC(G) > t + m$) that does not satisfy $P_{d,m}(G)$. We will show that this leads to a contradiction. We can assume that G is edge maximal with respect to not satisfying $P_{d,m}(G)$, so the addition of any edge will generate a graph that satisfies $P_{d,m}(G)$.

Let x and y be a pair of vertices for which there does not exist m openly disjoint paths between them, each of length at most d. By the edge maximality of G we can assume that $m - 1$ paths of length at most d do exist, and we will denote the vertices in the interior of these paths by C. There are at most $(m - 1)(d - 1)$ vertices in C.

Select $k-(d-1)(m-1)$ openly disjoint paths between x and y that are disjoint from C, and such that the sum of their lengths is a minimum. Denote these paths by $P_1, P_2, \cdots P_{k-(m-1)(d-1)}$. Each of these paths has at least $d + 2$ vertices. Let x' be a vertex on one of these paths that is adjacent to x. Then, clearly x' and y are nonadjacent vertices. The minimality of the sum of the lengths of these paths and the fact that $P_{d,m}(G)$ is not satisfied implies that neither x' nor y is adjacent to the $d - 3$ vertices on each of these paths that precede the predecessor of y for each of these paths. Thus, there are at least $n' = (d - 3)(k - (d - 1)(m - 1))$ vertices that are not adjacent to either x' or y. For the neighborhood condition this gives the following inequality:

$$t < |N(x') \cup N(y)| \leq n - n' - 2 = t.$$

For the degree condition this gives the inequality:

$$t + m < d(x') + d(y) \leq n - 2 - (d - 3)(k - (d - 1)(m - 1)) + m = t + m.$$

In either case this gives a contradiction that completes the proof of Theorem 3. ▮

Theorem 4 is a survey result that is not a direct consequence of Theorem B, Theorem E, Theorem 3 and Theorem C. For a graph G of order n, the neighborhood condition $NC(G) \geq n - 1$ implies that G is a complete graph, while $NC(G) \geq n - 2$ implies no connectivity in G (i.e. there are disconnected graphs that satisfy $NC(G) \geq n - 2$). This observation along with Theorem 2, Theorem 3, and Theorem C verifies Theorem 5. The fact that the conditions in Theorems 2, 4 and 5 cannot be lowered significantly follows from the fact that this is true for each of the results used to verify these theorems.

We now deal with the special case $d = 2$.

THEOREM 6. *For any positive integer m, a graph G of order n and connec-*

tivity k satisfies $P_{2,m}(G)$ *if at least one of the following inequalities are satisfied:*

$$k > \frac{n+m}{2} - 1$$
$$\delta(G) > \frac{n+m}{2} - 1$$
$$DC(G) > \frac{3n+m}{2} - 3$$
$$NC(G) > n-2.$$

Also, each of the inequalities is sharp.

PROOF: The sharpness of the inequalities can be observed by considering the graph H obtained from a K_{n-2} by adding two adjacent vertices that have precisely $m-2$ common adjacencies in the K_{n-2} and whose neighborhoods equally share (as much as is possible, the remaining vertices of the K_{n-2}. Thus H has order n, connectivity $\lfloor (n+m)/2 \rfloor - 1, \delta(H) = \lfloor (n+m)/2 \rfloor - 1, DC(H) = \lfloor (3n+m)/2 \rfloor - 3$, and $NC(H) = n-2$, but H does not satisfy $P_{2,m}(H)$.

Assume the result is not true, and that x and y is a pair of vertices in a graph G satisfying the hypothesis of Theorem 6 that does not have m paths of length at most 2 between them. By Theorem C we know that the connectivity of G does not exceed $(n+m)/2 - 1$. We can assume with no loss of generality that $d(x) \leq d(y)$.

First, consider the case when x and y are nonadjacent. Thus, by assumption, $|N(x) \cap N(y)| \leq m-1$. Thus, $|N(x) \cup N(y)| + |\{x, y\}| \geq 2|N(x)| - |N(x) \cap N(y)| + 2 > n$, if $d(x) > (n+m-3)$, and so clearly

$$d(x) \leq (n+m-3)/2$$
$$d(x) + d(y) \leq n+m-3, \text{ and}$$
$$|N(x) \cup N(y)| \leq n-2.$$

All of these inequalities contradict the assumptions, so we can assume that x and y are adjacent. Then, also by assumption, $|N(x) \cap N(y)| \leq m-2$. Any vertex z of G that is not adjacent to x has degree at most $n-2$, so we have

$$d(x) \leq (n+m)/2 - 1$$
$$d(x) + d(z) \leq (3n+m)/2 - 3, \text{ and}$$
$$|N(x) \cup N(z)| \leq n-2.$$

This also contradicts the inequalities, and completes the proof of Theorem 6. ∎

4. PROBLEMS

There are several interesting questions and natural extensions related to the results presented. In Theorems 2 and 3 the relationship between neighborhood conditions on nonadjacent pairs of vertices and property $P_{d,m}$ was investigated. One need not consider only nonadjacent pairs of vertices, but adjacent pairs or all pairs of vertices could be considered in neighborhood conditions implying $P_{d,m}$. In addition, one need not restrict consideration to just the union of the neighborhoods of pairs of vertices. For any fixed integer $t \geq 2$, the number of vertices in the union of the neighborhoods of any set of t (nonadjacent) vertices can be considered in the neighborhood condition. Examples of results of this nature can be found in [AFF], [F] and [FGJL2]. Also, the relationship between property $P_{d,m}$ and degree closure conditions of the type considered in [BC] or neighborhood closure conditions considered in [FGJL1] should be investigated. For all of these possibilities of a neighborhood condition or a closure condition, the generalized wheel and the cube have significantly different properties, so many interesting problems of this type remain.

A natural question which is a basic extremal problem at the opposite end of the spectrum is to determine the minimum number of edges in a graph that satisfies property $P_{d,m}$.

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