On Well-k-Covered Graphs

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Abstract. A set S of vertices of a graph is k-independent if each vertex in S is adjacent to at most $k-1$ other vertices in S. A graph G is well-k-covered if every maximal k-independent set is maximum. We shall characterize the well-k-covered trees and for $k = 2$ all such graphs of girth 8 or more.

Introduction

The concept of a *well-covered graph*, one in which every maximal independent set of vertices is a maximum (and hence of the same size), was introduced by Plummer in 1970 (see [4]). Jacobson and Fink ([2],[3]) initiated work on *k-independent* sets of vertices, those in which each vertex in the set is adjacent to at most $k-1$ other vertices in the set. In this paper we extend the notion of well-covered to k-independence. That is, we call a graph *well-k-covered* if every maximal k-independent set of vertices is maximum. For instance, a path consisting of 2 edges is well-2-covered but a path with 3 edges is not.

A vertex of degree one will be called a leaf. We say a vertex v has a 2-path attached if there is a path of two edges joining a leaf to v where the intermediate vertex of the path is of degree two.

Let $N_i(w)$ represent the set of all vertices whose distance from w is exactly i. As usual $N(w)$ will represent $N_1(w)$. The star on $m+1$ vertices will be denoted by S_m and $\Delta(G)$ is used to denote the maximum degree of all vertices of G.

The following result will be useful in both main sections of the paper.

Lemma 1. *If G is well-k-covered, $k \geq 2$, and has a vertex v with $m \geq k$ leaves attached, then G is isomorphic to S_k.*

Proof: Assume G is well-k-covered and v has $m \geq k$ leaves, say $L_1, L_2, \cdots, L_m$, attached and $|V(G)| > m + 1$. Let another neighbour of v be w.

Extend $\{v, w, L_1, L_2, \cdots, L_{k-2}\}$ to a maximal k-independent set $S_1 = \{v, w, L_1, \cdots, L_{k-2}\} \cup S$. But $S_2 = \{w, L_1, L_2, ..., L_m\} \cup S$ is also k-independent (may not be maximal). This contradicts G being well-k-covered.

Well-2-covered graphs containing no small cycles

In this section we characterize the well-2-covered graphs of girth 8 or more. This is accomplished by a sequence of lemmas which show certain local structures to be forbidden in a well-2-covered graph.

Lemma 2. *If G is well-2-covered, then G cannot have a vertex v with 2 or more 2-paths attached.*

Proof: Say G is well-2-covered and v has two 2-paths, say (a_1, b_1) and (a_2, b_2), where the a_i are adjacent to v. Extend $\{v, a_1, b_2\}$ to a maximal 2-independent set $S_1 = \{v, a_1, b_2\} \cup S$. But $S_2 = \{a_1, a_2, b_1, b_2\} \cup S$ is also 2-independent (not necessarily maximal). This contradicts G being well-2-covered.

Lemma 3. *If G is well-2-covered, then G cannot have a vertex with a leaf and a 2-path attached.*

Proof: Say G is well-2-covered and v has a leaf L_1 and a 2-path (a_1, b_1), where a_1 is adjacent to v, attached. Extend $\{v, a_1\}$ to a maximal 2-independent set $S_1 = \{v, a_1\} \cup S$. But $S_2 = \{a_1, b_1, L_1\} \cup S$ is 2-independent. Contradiction.

Lemma 4. *If G is well-2-covered and the girth is at least 8, then a vertex of degree 3 or more cannot have a leaf attached.*

Proof: Say G is well-2-covered, the girth of G is 8 or more and v has a leaf L_1 attached where $d(v) \geq 3$. Let another neighbour of v be w.

For each $x_i \neq v$, where $x_i \in N(w)$, if x_i has at least one neighbour in $N_2(w)$ which in turn has at least one neighbour in $N_3(w)$, then pick such a pair, say y_i, z_i. That is, $y_i \in N_2(w)$, $z_i \in N_3(w)$ and y_i and z_i are adjacent and y_i is adjacent to x_i. Let the set of y_i be Y and the set of z_i be Z. By girth, $Z \cup Y$ is a 2-independent set consisting of a matching in G. Consider the component, say C, of G-$N(\{Z \cup Y\})$ containing v (and hence L_1 and w). This component C has to be well-2-covered and has 4 or more vertices.

By the definition of Y and Z and by Lemma 1, each neighbour of w in C-v is either a leaf or the center of a 2-path, and by virtue of the previous lemmas, w has at most one leaf and at most one 2-path attached (not both).

If w had neither a leaf nor a 2-path attached, then C is a well-2-covered graph with 2 leaves at v, namely L_1 & w. This is impossible by Lemma 1.

If w had a leaf attached, then C violates Lemma 3.

If w had a 2-path, say (a_1, b_1) attached, where a_1 was adjacent to w, then $\{v, w, b_1\}$ can be extended to a maximal 2-independent set $S_1 = \{v, w, b_1\} \cup S$ of C. But $S_2 = \{L_1, v, a_1, b_1\} \cup S$ of C is also a maximal 2-independent set. Hence C is not well-2-covered.

Lemma 5. *If G is of girth 8 or more and well-2-covered, then G must have a leaf.*

Proof: Assume G has no leaves. If all vertices are of degree 2 then G is not well-2-covered (for girth greater than 7).

Select a vertex v of degree 3 or more. Let one of the neighbours of v be w. As in Lemma 4, for each $x_i \neq v$ where $x_i \in N(w)$ select a pair $y_i \in N_2(w)$,

$z_i \in N_3(w)$ such that y_i is adjacent to x_i and z_i. Let the set of y_i be Y and the set of z_i be Z. By girth $Y \cup Z$ is 2-independent and the component C of G-$N(\{Y \cup Z\})$ containing v must be well-2-covered. But C violates Lemma 4 as w is now a leaf attached to v which is of degree 3 or more.

Lemma 6. *If G is well-2-covered and of girth 8 or more and v is a vertex with a 2-path attached, then all other neighbours of v must have a 2-path attached.*

Proof: Let G be well-2-covered and of girth 8 or more. Assume there is some vertex v with a 2-path, say (a_1, b_1), attached (where a_1 is adjacent to v) and v has another neighbour, say w, with no 2-path attached. By Lemma 3 w is not a leaf and by Lemmas 2 and 4 cannot have a leaf attached.

As in Lemma 4, for each $x_i \neq v$, where $x_i \in N(w)$, select a pair y_i, z_i where $y_i \in N_2(w)$, $z_i \in N_3(w)$, and y_i is adjacent to x_i and z_i. By Lemma 4 and the hypothesis that w does not have a 2-path attached y_i and z_i must exist. Let the set of y_i be Y and the set of z_i be Z. By girth, $Y \cup Z$ is 2-independent. The component, say C, of G-$N(\{Y \cup Z\})$ containing v has to be well-2-covered but it violates Lemma 3 since w is now a leaf of v which also has a 2-path, namely (a_1, b_1).

By applying Lemmas 5, 4 and then Lemma 6 repeatedly, we obtain

Theorem 1. *If G is of girth 8 or more, then G is well-2-covered if and only if G is either P_1 or P_2 or is obtained from an arbitrary graph of girth 8 or more by attaching a 2-path at each vertex.*

Note: C_7 shows the result to be sharp.

A characterization of well-k-covered trees

In this section we characterize the well-k-covered trees, for $k \geq 3$. For $k = 1$ see [1], [5] or [6]. The case $k = 2$ is included in the previous section.

The next lemma is useful in showing that certain subgraphs of a well-k-covered graph are also well-k-covered.

Lemma 7. *Let zw be a bridge of a well-k-covered graph G and let Z and W be the components of G-zw containing z and w respectively. If $\Delta(Z \cup \{w\}) \leq k-1$ and $d_G(w) \leq k-1$, then W is well-k-covered.*

Proof: Let I be a maximal k-independent set of W. Then $I \cup V(Z)$ is a maximal k-independent set in G and hence its order must be constant. Thus the cardinality of I must be also constant regardless of the choice of I.

We now define the family F_k of trees which we then show to be precisely the only well-k-covered ones. A tree T belongs to the family F_k if and only if $\Delta(T) \leq$

$k-1$ or there exists a partition P of $V(T)$ into stars S_k and trees of maximum degree at most $k-1$ satisfying:

(1) for each star of P, the degree in T of its center c is k and at most one endvertex, say e, either (i) has degree $\geq k$ or (ii) has a neighbour, other than c, of degree $\geq k$ (e may satisfy (i) and (ii) simultaneously) and

(2) each tree X of P is connected to T-X by an arbitrary number m of edges between m (not necessarily distinct) vertices a_i of X and m endvertices b_i of different stars of P such that the degrees in T of the a_i's and the b_i's are less than k.

We observe that for $k=1$, T belongs to F_1 if T is a single vertex or is obtained from any tree by adding a pendant edge at each vertex. That is, T can be partitioned into S_1's (or pendant edges) satisfying (1).

For $k=2$, T belongs to F_2 if it is a single vertex or K_2 or is obtained from an arbitrary tree by adding a pendant P_2 at each vertex. Again T can be partitioned into S_2's satisfying (1).

An example of a member of F_3 (F_4) is given in Figure 1 (2 respectively).

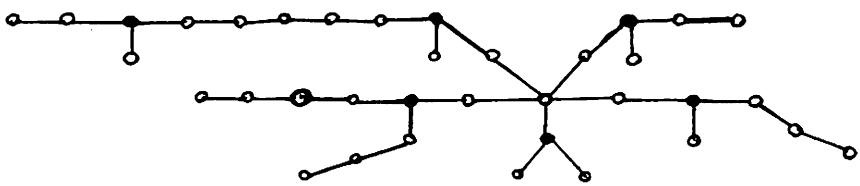

Figure 1

We first note that for T belonging to F_k, the partition P is unique.

Lemma 8. *Let T be a member of F_k, for some k. The partition P of T into stars and trees corresponding to the definition of F_k is unique.*

Proof: Let T be a member of F_k for some k. In any partition P, the centers of the stars have degree k and any vertices of degree greater than k are necessarily endvertices of stars. The only possibility of ambiguity is to recognize if a vertex of degree k is the center or an endvertex of a star. But this is not a problem since the center c of a star has at most one neighbour which either is of degree at least k or has neighbours (other than c) of degree at least k, whereas each of the neighbours of an endvertex of degree k of a star has degree k or has a neighbour of degree k. Thus the centers of the stars of any partition of T are well-defined and the partition is unique.

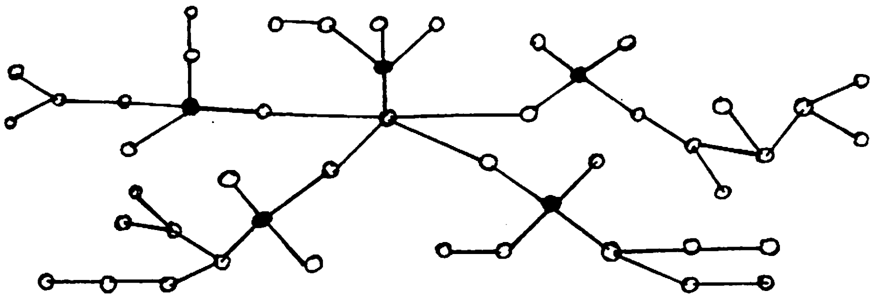

Figure 2

Theorem 2. *A tree T is well-k-covered if and only if T belongs to F_k.*

Proof: For $k = 1$ and 2 the result clearly holds. Henceforth $k \geq 3$.

In order to see that a tree T belonging to F_k is well-k-covered we observe that every maximal k-independent set of T consists of all the vertices of the trees in the partition P and of exactly k vertices of each star of P. In light of Lemma 8, the order of such a set is constant.

We use induction on the order of T to show that every well-k-covered tree belongs to F_k. Certainly if $|V(T)| \leq k$, then $\Delta(T) \leq k - 1$ and T belongs to F_k. Suppose the result holds for all T where $|V(T)| < n$ with $n \geq k + 1$ and let T be a well-k-covered tree of order n. If $\Delta(T) < k$ the result holds so we consider $\Delta(T) \geq k$.

Since T is a tree we can find a leaf L_1 whose neighbour v has at most one neighbour, say w, which is not a leaf (for instance, by considering a longest path). Let the neighbours of v be $\{L_1, L_2, L_3, \cdots, L_m, w\}$.

Say $m < k - 1$. Then $T - L_1$ is well-k-covered, by Lemma 7, and by the induction hypothesis belongs to F_k. It is straightforward to verify that T itself belongs to F_k with only a minor modification to the partition P of $T - L_1$. We first observe that since v is of degree less than $k - 1$ in $T - L_1$, v is not the center of a star in the partition P. If v is an endvertex of a star then in T, L_1 plays the role of an additional tree X satisfying (2) of the definition of F_k. If v is a part of a tree X in the partition of $T - L_1$, then in T, the partition is the same except that X has been replaced by $X \cup L_1$.

Now we consider $m \geq k - 1$. By Lemma 1 this means we need only consider $m = k - 1$.Extend $\{L_1, L_2, \cdots, L_{k-1}, v\}$ to a maximal k-independent set of T. Since w cannot be included in such a set, it follows that $T - N[v]$ is well-k-covered. By the induction hypothesis, $T - N[v]$ belongs to F_k. Note that there are $|V(T-N[v])|-s$ vertices in every maximal k-independent set of $T-N[v]$, where s is the number of stars in the partition P, since there is precisely one vertex from each star which must be excluded. Thus every maximal k-independent set of T must have $|V(T-N[v])|-s+k$ elements. Let this number be q. Consider the partition P of $T-N[v]$ and any component of $T-N[v]$. In T, w is joined precisely once to such a component since T is a tree.

The vertex w cannot be joined to the center, say c, of a star S, in the partition P else if we choose w and c to be in a maximal k-independent set I, only $k-2$ endvertices of S can be in I. That is, only $k-1$ vertices of S are in I which implies $|I| < q$.

If w were joined to an endvertex, say e_1, of a star S with center c then the partition P extended to include the star with center v shows T itself to be a member of F_k. The only possible problem would be if there were another endvertex, say e_2, of S such that $e_i (i = 1, 2)$ is either (i) of degree at least k or (ii) has a neighbour, say $y_i \neq c$, of degree at least k. But this cannot occur as shown by the following. If $e_i (i = 1, 2)$ satisfies (i) consider all of its neighbours. If it does not satisfy (i) but satisfies (ii) then consider y_i and $k-1$ of its neighbours, other than e_i. Take the union of these two sets and extend to a maximal k-independent set I of T. But I cannot include either e_1 or e_2 and thus $|I| < q$.

The remaining possibility is that w is joined to a vertex z belonging to a tree X in the partition P of $T-N[v]$. If the degree of z were k in T, then extend the set consisting of all neighbours of z to a maximal k-independent set I. Since z cannot be included, $|I| < q$. If the degree of w were k or more in T, then extend the set consisting of w and $k-1$ neighbours other than z to a maximal k-independent set I. But again, as z cannot be included, $|I| < q$. Hence both z and w have degree less than k in T. But then the partition P extended to include the additional star with center v shows T itself to be a member of F_k. This completes the proof.

We conclude by observing that this characterization allows us to determine in polynomial time ($0(n^3)$) if a given tree T is well-k-covered and if it is, to find its partition. We could proceed as follows:

For each vertex v of T, determine the degree of all the vertices of $v \cup N_1(v) \cup N_2(v)$. Select the stars whose centers are the vertices of degree k having at most one neighbour of degree at least k or having itself another neighbour of degree at least k. If these stars are not disjoint, then T is not well-k-covered. If they are disjoint, delete these stars and examine the remaining components X_i. If each tree X_i satisfies $\Delta(X_i) < k$ and condition 2 of the definition of F_k, then T is well-k-covered. If some tree does not, then T is not well-k-covered.

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