On Sum Graphs

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Abstract. We investigate the labellings of sum graphs, necessary conditions for a graph to be a sum graph, and the range of edge numbers of sum graphs.

1. Introduction

Definition 1. A simple graph G with vertex set $V(G) = \{V_1, V_2, \ldots, V_n\}$ is a sum graph if we can find a finite set $S = \{X_1, X_2, \ldots, X_n\} \subset N^+$ and assign X_i to V_i such that V_i and V_j are adjacent iff $(X_i + X_j) \in S$. We call S a labelling for G.

F. Harary introduced this concept in 1988 (see [4]) and left the characterization of sum graphs as a problem. The first version of this paper came out as preprint [3] in May, 1988. Recently I received the papers [1], [2], [4] and made some changes.

Besides the sum graphs themselves, I was interested in their manipulation on computers and their possible application to improved graph storage and edge searching. This does not turn out to be promising, however: as we will see later, the labels of some simple sum graphs grow exponentially.

For convenience, throughout this paper we reserve Nn for the empty graph on n vertices. Also, we assign to each edge $e = (V_i, V_j)$ a weight of $(X_i + X_j)$, and say that $(X_i + X_j)$ *contributes* the edge to V_i, V_j.

The following are some simple facts about sum graphs.

Facts:

1.1 ([3], Theorem 1) Let G be a graph with e edges. The graph $H = G + Ne$ is a sum graph.

(The same statement for connected graphs is given in [4].)

1.2 ([4]) A sum graph has at least one isolated vertex: the one assigned the largest integer.

1.3 ([4]) If $S = \{X_1, X_2, \ldots, X_n\}$ is a labelling for a sum graph G, then so is $k \cdot S = \{kX_1, kX_2, \ldots, kX_n\}$.

1.4 If G and H are sum graphs, then so is $(G + H)$. In particular, if G is a sum graph, then so is each $(G + Nk)$, $k = 1, 2 \ldots$.

2. Best labellings, and labellings for some simple sum graphs

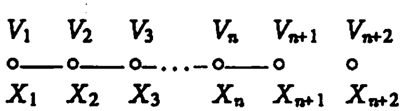
Although a sum graph can be given many different labellings, there is a unique and natural labelling for each sum graph.

Definition 2. Let G be a sum graph and let $S = \{X_1, X_2, \dots, X_n\}$ and $S' = \{Y_1, Y_2, \dots, Y_n\}$ be two labellings in ascending order. We say S is better than S' if the non-zero element $(X_i - Y_i)$ with the largest i is negative. We say that S is the best labelling if it is better than any other labelling for G.

The integers of a best labelling are obviously relatively prime. It seems to be much easier to label a sum graph than to best label it.

Facts:

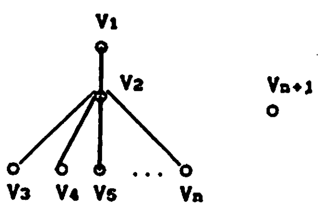
2.1 $Pn + N_1$ is a sum graph with the labelling:



$$\begin{cases} X_1 = a \\ X_2 = b \quad a < b \text{ and } a, b \in N^+ \\ X_i = X_{i-1} + X_{i-2} \ (3 \leq i \leq n+2) \end{cases}$$

A better labelling for $P_n + N_1$ is given in [3].

2.2 A star $+N_1$ is a sum graph with the labelling:



Label V_i with X_i

$$\begin{cases} X_1 = a \\ X_2 = b \quad a < b \text{ and } a, b \in N^+ \\ X_3 = X_1 + X_2 \\ X_i = X_{i-1} + X_2 \quad (4 \leq i \leq n+1) \end{cases}$$

Moreover, it can be shown that the choice $a = 1$ and $b = 2$ gives the best labelling.

2.3 ([1]) A tree $+N_1$ is a sum graph.

Generalized Fibonacci sequences such as those above are often used for labelling sum graphs (see [1],[3],[4]). Unfortunately, these sequences increase exponentially, and so are not suitable for computer storage and manipulation. We have not determined the best labelling for any class of sum graphs except a star $+N_1$; in particular, we do not know how to best label $P_n + N_1$ for all n. The following, found by computer search, are the best labellings of $(P_n + N_1)$ for

$3 \leq n \leq 12$.

1	2	3	4									
1	2	3	4	6								
1	2	4	5	6	8							
1	2	4	5	7	9	10						
1	2	4	6	7	9	12	13					
1	3	5	7	9	11	12	13	16				
1	3	5	7	9	11	12	13	17	20			
1	3	5	7	9	11	13	15	16	17	20		
1	3	5	7	9	11	13	15	16	17	19	24	
1	3	5	7	9	11	13	15	17	19	20	21	24

3. Isolation numbers and degree sequences

From Facts 1.1 and 1.4, we know that for any graph G there is a least integer $k = \text{in}(G)$ such that $(G + N_k)$ is a sum graph. In [4], for connected graphs, the number $\text{in}(G)$ is termed the isolation number. We extend the term to arbitrary graphs.

Definition 3. For any graph G (not necessarily connected), the isolation number is the smallest integer $k = \text{in}(G)$ (possibly negative) such that $(G + N_k)$ is a sum graph. Here if k is negative $(G + N_k)$ is interpreted as deleting (rather than adding) k isolated vertices.

In fact 1.1, we provided an upper bound on $\text{in}(G)$. Now we give a lower bound.

Theorem 1. *Let G be an arbitrary graph with degree sequence $D_1 \leq D_2 \leq \cdots \leq D_n$. Then $in(G) > \max_{\substack{1 \leq i \leq n \\ D_i \neq 0}} (D_i - i)$ if G is non-empty and $in(G) = -n + 1$ otherwise.*

To facilitate our proof, we use the following two lemmas.

Lemma 3.1. *([3], Theorem 2) Let G be a sum graph with $V(G) = \{V_1, V_2, \ldots, V_n\}$ and a labelling $X_1 < X_2 < \cdots < X_n$. Then $\deg_G(V_i) \leq n - i$ for all i. Furthermore each X_i contributes at most $\lfloor \frac{i-1}{2} \rfloor$ edges.*

Lemma 3.2. *([3], Lemma 3.2) Let $D_1 \leq D_2 \leq D_3 \leq \cdots \leq D_n$ be integers and $(L_1, L_2, L_3, \ldots, L_n)$ be any permutation of $(D_1, D_2, D_3, \ldots, D_n)$, then*

$$\max_{1 \leq i \leq n} (L_i - i) \geq \max_{1 \leq i \leq n} (D_i - i).$$

Proof of Theorem 1: If G is empty, the result is obvious. Let G be non-empty. First suppose G has no isolated vertex. Define $D = \max_{1 \leq i \leq n} (D_i - i)$ and $k = \text{in}(G)$. By definition, $H = G + N_k$ is a sum graph.

Let $X_1 < X_2 < \cdots < X_n$ be the labels on $V(G) = \{V_1, V_2, \ldots, V_n\}$ obtained from a labelling for H. By Lemma 3.1, $\deg_G(V_i) = \deg_H(V_i) \leq n + k - i$ as X_i is no less than the i-th element in the labelling for H. So $k \geq \max_{1 \leq i \leq n} \{\deg_G(V_i) - (n - i)\}$. Let $j = n - i + 1$. By lemma 3.2, $k \geq \max_{1 \leq j \leq n} \{\deg_G(V_{n-j+1}) - j\} + 1 \geq \max_{1 \leq j \leq n} \{D_j - j\} + 1 = D + 1$ because $(\deg_G(V_{n-j+1}) : 1 \leq j \leq n)$ is a permutation of $(D_1 D_2 \ldots D_n)$.

If G has isolated vertices, let G' be the graph obtained by deleting them. Apply the above result to G'. The inequality follows immediately. ∎

Corollary 3.1. *Let G be a graph with minimum vertex degree $\delta = \delta(G) > 0$. Then $G + N_{\delta-1}$ is not a sum graph.*

Example: $(C_n + N_1)$ is not a sum graph for $n \geq 3$ since $\delta = 2$.

We conclude this section with some easily verified facts about the degree sequences of sum graphs. Suppose G is a sum graph with $V(G) = \{V_1, V_2, \ldots, V_n\}$ and labelling $X_1 < X_2 < \cdots < X_n$.

Facts:

3.1 $\deg(V_1) \leq n - 2$. If $\deg(V_1) = n - 2$ and X_2/X_1 is specified then G is determined. In particular, if X_2/X_1 is not an integer, then G is a star $+N_1$.

3.2 If $\deg(V_2) = n - 2$ then G is a star $+N_1$.

3.3 There is no sum graph with at least two vertices of degree $(n-2)$ for $n \geq 4$.

4. Edge numbers of sum graphs

For convenience, we denote a sum graph G with labelling $X_1 < X_2 < \cdots < X_n$ by $G(X_1, X_2, \ldots, X_n)$. We call a vertex by the name of its label for the time being.

Lemma 4.1. *([3], Lemma 3.3) In the sum graph $G(1, 2, \ldots, N, X)$, X contributes $\lfloor \frac{2N-X+1}{2} \rfloor$ edges for $N + 1 \leq X \leq 2N - 1$ and 0 for all $X \geq 2N$. Hence $|E(G(1, 2, \ldots, N, X))| - |E(G(1, 2, \ldots, N, X + 1))| = 1$ or 0.*

Theorem 2. *If G is a sum graph, then $|E(G)| \leq \frac{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}{2}$. Furthermore, for any $0 \leq e \leq \frac{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}{2}$, there is at least one sum graph G with $|E(G)| = e$.*

Proof: Let $G = G(X_1, X_2, \ldots, X_n)$. By Lemma 3.1, X_i can contribute at most $\lfloor \frac{i-1}{2} \rfloor$ edges. Hence

if $n = 2k$, then $|E(G)| \leq 0 + 0 + 1 + 1 + \cdots + (k-1) + (k-1) = k(k-1)$;

if $n = 2k+1$, then $|E(G)| \leq 0 + 0 + 1 + 1 + \cdots + (k-1) + (k-1) + k = k^2$.

Thus $|E(G)| \leq \frac{\binom{n}{2} - \lfloor \frac{n}{2} \rfloor}{2}$ for any sum graph G.

Suppose now that $0 \leq e \leq \frac{\binom{n}{2}-\lfloor\frac{n}{2}\rfloor}{2}$. By Lemma 4.1, sum graphs $G(1,2,\ldots,n-1,X)$ for $n \leq X \leq 2^n$ have edge numbers ranging from

$$|E(G(1,2,\ldots,n-2,n-1,n))| \text{ to } |E(G(1,2,\ldots,n-2,n-1,2^n))|.$$

Similarly, sum graphs $G(1,2,\ldots,n-2,X,2^n)$ for $n-1 \leq X \leq 2^{n-1}$ have edge numbers from

$$|E(G(1,2,\ldots,n-2,n-1,2^n))| \text{ to } |E(G(1,2,\ldots,n-2,2^{n-1},2^n))|.$$

Therefore there are sum graphs on n vertices with edge numbers from

$$|E(G(1,2,\ldots,n-1,n))| \text{ to } |E(G(2^1,2^2,\ldots,2^{n-1},2^n))|.$$

So there is at least one sum graph G with $|E(G)| = e$, because $|E(G(1,2,\ldots,n-1,n))| = \frac{\binom{n}{2}-\lfloor\frac{n}{2}\rfloor}{2}$ and $|E(G(2^1,2^2,\ldots,2^{n-1},2^n))| = 0$. ∎

Furthermore we can prove the following.

Facts:

4.1 ([3]) Let H be a sum graph with n vertices and $e = \frac{\binom{n}{2}-\lfloor\frac{n}{2}\rfloor}{2}$ edges.
If $n = 1,2,3,4$, then $H \cong G(1,2,\ldots,n)$;
If $n \geq 5$ is odd, then H must have labelling $M \cdot \{1,2,\ldots,n\}$;
If $n \geq 6$ is even, then H must have labelling either $M \cdot \{1,2,\ldots,n\}$ or $M \cdot \{1,2,\ldots,n-1,n+1\}$.

4.2 If $X_1 < X_2 < \cdots < X_n$ is the best labelling for a sum graph then $X_i \leq 2X_{i-1}$ for $n-2 \leq i \leq n$.

4.3 If G and $(\overline{G} + N_1)$ are two sum graphs. Then G has $e = \frac{\binom{n}{2}-\lfloor\frac{n}{2}\rfloor}{2}$ edges where $|V(G)| = n$.

We note that in the constructive proof of Theorem 2, vertex X_i is labelled with $X_i \leq 2^i$. This leads to the following question.

Question 1. For a sum graph G, does G have a labelling $X_1 < X_2 < \cdots < X_n$ such that $X_n \leq 2^n$ (or, more strongly, $X_i \leq 2^i$ for $1 \leq i \leq n$)?

Suprisingly, no upper bound on X_n has been given so far.

5. Additional questions

Question 2. Suppose $(G_1 + N_1)$ and $(G_2 + N_1)$ are sum graphs. Is $(G_1 + G_2 + N_1)$ a sum graph ?

Question 3. Can we always find a labelling for a sum graph so that the smallest label is 1?

It is not difficult to see that a positive answer to Question 3 implies a positive answer to Question 2.

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