On Incomplete Group Divisible Designs

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Abstract. In this paper we deal with recursive constructions for incomplete group divisible designs (IGDDs). Denoting GD[k,1,v; uv]-GD[k,1,n; un] by (u,k)-IGD[v,n], we will prove, as an application, that a (7,4)-IGD[v,n] exists if and only if $v \geq 3n$ and $v - n \equiv 0 \pmod 2$.

1. Preliminaries.

We assume that the concepts of PBDs, BIBDs, GDDs, Latin squares and their orthogonality, are known. A GD[k,1,v; kv] is called a transversal design and denoted by TD[k,v]. By N(v) we mean the number of mutually orthogonal Latin squares of order v. It is well known that $N(v) \geq k - 2$ is equivalent to a TD[k,v]. A TD[k,v] whose block-family can be partitioned into parallels is called a *resolvable* TD[k,v] and denoted by RT[k,v]. It is also well known that the existence of a RT[k,v] is equivalent to $N(v) \geq k - 1$.

Definition 1.1 A GD[k,1,v;uv]-GD[k,1,n;un] is a quadruple (X,G,H,A) satisfying the conditions:

(1) X is a set of uv elements;

(2) $G = \{G_i : |G_i| = v, 1 \leq i \leq u\}$ is a partition of X;

(3) $H = \{H_i : G_i \supset H_i, |H_i| = n, 1 \leq i \leq u\}$;

(4) A is a collection of k-subsets (called blocks) of X and satisfies the condition: if $x \in G_i$ and $y \in G_j$, then $\{x, y\}$ occurs in one and only one block if $i \neq j$ and at least one of $\{x, y\}$ occurs in $\cup_{1 \leq i \leq u}(G_i \setminus H_i)$.

G is called the group family; A is called the block family. (X,G,H,A) is denoted by (u,k)-IGD[v,n].

By simple calculation, we obtain the following result.

Theorem 1.2 The necessary conditions for the existence of a (u,k)-IGD[v,n] are:

$(v - n)(u - 1) \equiv 0 \pmod{k - 1}$;
$v(u - 1) \equiv 0 \pmod{k - 1}$;
$u(u - 1)(v^2 - n^2) \equiv 0 \pmod{k(k - 1)}$;
$v \geq n(k-1)$.

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When $v = (k - 1)n$, a (u,k)-IGD[v,n] is equivalent to a *k-frame* (cf. [6]). And a (k,k)-IGD[v,n] is nothing but k-2 mutually orthogonal Latin squares of order v with missing subsquares of order n (cf. [4]).

2. Recursive Constructions

Construction I to Construction VI are commonly used techniques; so we omit proofs.

Construction I. If both a (u,k)-IGD[v,m] and a (u,k)-IGD[m,n] exist, then a (u,k)-IGD[v,n] exists.

Construction II. If a (u_i,k)-IGD[v,n] exists for $i = 1, 2, \dots, t$, and if a $(u,\{u_1,\dots u_t\},1)$-PBD exists, then a (u,k)-IGD[v,n] exists.

Construction III. If a (u,k)-IGD[v,n] exists and if $N(t) \geq k - 2$, then a (u,k)-IGD[tv,tn] exists.

Construction IV. If a GD[k,1,n;un] exists and if $N(t) \geq k - 2$, then a (u,k)-IGD[nt,n] exists.

Construction V. If (1) $N(t) \geq u - 1$, and (2) a $(u,k)\text{-IGD}[m+l_i, l_i]$ exists for all i in $1 \leq i \leq t$, then(1) a $(u,k)\text{-IGD}[mt+\Sigma_{1\leq i\leq t}l_i, \Sigma_{1\leq i\leq t}l_i]$ exists; (2) a $(u,k)\text{-IGD}[mt+\Sigma_{1\leq i\leq t}l_i, m+l_r]$ exists, provided that a $(u,k)\text{-IGD}[\Sigma_{1\leq i\leq t}l_i, l_r]$ exists.

Construction VI. If we have the conditions (1) $N(t) \geq u + s - 2$ for $s \geq 0$; (2) a $(u,k)\text{-IGD}[m+l_i, l_i]$ exists for $1 \leq i \leq s$; (3) a GD[k,1,m; um] exists; then a $(u,k)\text{-IGD}[mt+\Sigma_{1\leq i\leq s}l_i, m+\Sigma_{1\leq i\leq s}l_i]$ exists.

We now develop four further constructions.

Construction VII. If (1) a (u,k)-IGD[v',n'] exists, (2) a (k,k)-IGD[v,n] exists, and (3) a (u,k)-IGD[n'v,n'n] exists, then a (u,k)-IGD[v'v,v'n] exists.

Proof. Let $G_i = \{a_1^{(i)},a_2^{(i)},\dots,a_{v'-n'}^{(i)},\infty_1^{(i)},\dots,\infty_{n'}^{(i)}\}$, $G = \{G_i: 1 \leq i \leq u\}$, and $X = \cup_{1\leq i\leq u}G_i$. According to (1), there is a (u,k)-IGD[v',n'] on X such that G is the group family and $\mathbf{B} = \{B_1,B_2,\dots,B_s\}$ is the block family, and the condition that $\{\infty_j^{(p)},\infty_k^{(q)}\}$ is not in any block is satisfied.

Let $Y = \{y_1,...,y_{v-n},\infty_1,...,\infty_n\}$. By (3), on $(\cup_{1\le i\le u}\{\infty_1{}^{(i)},...,\infty_{n'}{}^{(i)}\}) \times Y$ we may form a (u,k)-IGD[n'v,n'n] with group family $\{\{\infty_1{}^{(i)},...,\infty_{n'}{}^{(i)}\} \times Y: 1 \le i \le u\}$ and such that its block family **B'** satisfies the condition that every pair in $(\cup_{1\le i\le u}\{\infty_1{}^{(i)},\infty_2{}^{(i)},...,\infty_{n'}{}^{(i)}\}) \times \{\infty_1,\infty_2,...,\infty_n\}$ is not contained in any block.

For $B_i \in$ **B**, on $B_i \times Y$ we construct a (k,k)-IGD[v,n] such that its group family is $\{\{b\}\times Y: b \in B_i\}$ and such that its block family $\mathbf{B}_i$ satisfies the condition that no block contains any pair from $B_i \times \{\infty_1,\infty_2,...,\infty_n\}$.

Now let $\{G_i \times Y: 1 \le i \le u\}$ be the group family and $\mathbf{B}'\cup(\cup_{1\le i\le s}\mathbf{B}_i)$ be the block family; we obtain a (u,k)-IGD[v'v, v'n] on $X\times Y$.

Construction VIII. If (1) a GD[k,1,m; um] exists, (2) a (k,k)-IGD[v,n] exists, then a (u,k)-IGD[mv,mn] exists.

Proof. We merely take (v', n') to be (m, 0) in Construction VII.

Construction IX. If (1) $N(u) \ge k - 1$, (2) a (k,k)-IGD[v,n] exists, and (3) a (u,k)-IGD[v+n',n+n'] exists, then a (u,k)-IGD[kv+n',kn+n'] exists.

Proof. Suppose $A = \{a_1,a_2,...,a_u\}$, $B = \{b_1,b_2,...,b_k\}$. Since $N(u) \ge k - 1$, we form an RT[k,u] on $X = A\times B$ with group family $\{A_i: A_i = A\times\{b_i\}, 1 \le i \le k\}$ and with block family $\mathbf{B}_1 = \{B_1, B_2,..., B_{u^2}\}$ where $B_1, B_2,..., B_u$, form a parallel class. Now we have already obtained a GD[{k,u},1,k;uk] with group family $\{B_1, B_2,..., B_u\}$ and block family $\mathbf{B}_2 = \{A_1,...,A_k,B_{u+1},...B_{u^2}\}$. Set $Y = \{y_1,...,y_{v-n}, \infty'_1,...,\infty'_n\}$, $Z = \{\infty_1,...,\infty_{n'}\}$.

For $B_i \in \mathbf{B}_2$, we construct on $B_i\times Y$ a (k,k)-IGD[v,n] such that its group family is $\{\{b\}\times Y: b \in B_i\}$ and such that its block family $\mathbf{B}'_i$ satisfies the condition that no block contains any pair from $B_i\times\{\infty'_1,...,\infty'_n\}$.

On $\{A_i\times Y\}\cup\{A\times Z\}$, we construct a (u,k)-IGD[v+n', n+n'] such that its group family is $\{((\{(a_j,b_i)\}\times Y) \cup (\{a_j\}\times Z): 1 \le j \le u\}$ and such that its block family $\mathbf{B}''_i$ satisfies the condition that no block contains any pair chosen from $(A_i\times \{\infty'_1,...,\infty'_n\})\cup (A\times Z)$.

Now on $(X\times Y)\cup(A\times Z)$ we take $\{(\{a_j\}\times B\times Y)\cup(\{a_j\}\times Z): 1 \le j \le u\}$ to be the group family and $B_3 = \{\cup_{u+1\le i\le u^2} B'_i\}\cup\{\cup_{1\le i\le k} B''_i\}$ to be the block family; then we obtain a (u,k)-IGD[kv+n',kn+n'].

Construction X. If we have conditions (1) $N(u) \ge k - 1$, (2) a GD[k,1,v; kv] exists, (3) a (u,k)-IGD[v+n',n'] exists, then a (u,k)-IGD[kv+n',v+n'] exists.

Proof. In the proof given for Construction IX, we set n = 0 and we take $B_3 = \{\cup_{u+1\le i\le u^2} B'_i\}\cup\{\cup_{2\le i\le k} B''_i\}$. The conclusion follows.

3. An application.

For k = 4, there are many results about Incomplete Group Divisible Designs.

Theorem 3.1. ([8]) If v = 3n, then a (u,4)-IGD[v,n] exists if and only if we have $n(u - 1) \equiv 0 \pmod 3$.

The following important result is obtained in [5].

Theorem (Heinrich-Zhu). A (4,4)-IGD[v,n] exists if and only if $v \ge 3n$ and $(v,n) \ne (6,1)$.

In this paper, we will use our results to prove the following theorem.

Theorem 3.3. A (7,4)-IGD[v,n] exists if and only if $v \ge 3n$ and v-n is even.

We will use several Lemmas. First, we record the necessary condition.

Lemma 3.4. The necessary condition for the existence of a (7,4)-IGD[v,n] is that $v \ge 3n$ and v-n is even.

Proof. This is just a corollary of Theorem 1.1.

Lemma 3.5. ([3]) A GD[4,1,v;uv] exists if and only if we have the conditions (1) $v(u - 1) \equiv 0 \pmod 3$, and (2) $v^2u(u - 1) \equiv 0 \pmod{12}$, except for (v,u) = (2,4) or (6,4).

Corollary 3.6. A GD[4,1,v; 7v] exists if and only if $v \equiv 0 \pmod 2$.

Lemma 3.7. If $v - n \le 12$, then Lemma 3.4 is sufficient for the existence of a (7,4)-IGD[v,n].

Proof. From the appendix, we only need to construct a (7,4)-IGD[v,n] for (v,n) ∈ {(3,1), (9,1), (6,2), (8,2), (10,2), (14,2), (9,3), (15,3), (12,4), (16,4), (15,5), (18,6)}. By Theorem 3.1, we know that, if (v,n) ∈ {(3,1), (6,2), (9,3), (12,4), (15,5), (18,6)}, then a (7,4)-IGD[v,n] exists. Since a (7,4)-IGD[9,3] and a (7,4)-IGD[3,1] exist, a (7,4)-IGD[9,1] exists by Construction I. By Corollary 3.6, a GD[4,1,2;14] exists. Hence we may set n = 2, t = 4, 5, 7, in Construction IV; this produces a (7,4)-IGD[8,2], a (7,4)-IGD[10,2], and a (7,4)-IGD[14,2], respectively. Since a (7,4)-IGD[5,1] exists (cf. the appendix), we can take t = 3 to get a (7,4)-IGD[15,3] by Construction III. By Corollary 3.6, a GD[4,1,4; 28] exists; so we may set t = 4 to get a (7,4)-IGD[16,4] (IV). This completes the proof.

Lemma 3.8. If $v - n > 12$ and $v - n \notin F$, then Lemma 3.4 is sufficient for the existence of a (7,4)-IGD[v,n], where F = {20, 24, 30, 40, 60, 120}.

Proof. From [1], we have $N(t) \geq 6$ if t = 70, 72, or t > 77. Hence, if $v - n > 12$ and $v - n \notin F$, then v - n can be represented as $t \times m$ where m is even, $2 \leq m \leq 12$, and $N(t) \geq 6$. Since $v \geq 3n$, we have $n \leq (v - n)/2$ and $(v - n)/2 = t \times (m/2)$; thus n can be written as $n = \sum_{1 \leq i \leq t} l_i$ with $0 \leq l_i \leq m/2$. By Lemma 3.7, a (7,4)-IGD[$m+l_i, l_i$] exists, since $m \leq 12$. Using Construction V, and taking t and m to be the same as here, we get a (7,4)-IGD[v,n].

Lemma 3.9. If v - n = 20, then Lemma 3.4 is sufficient for the existence of a (7,4)-IGD[v,n].

Proof. Since a (7,4)-IGD[21,7] and a (7,4)-IGD[7,1] exist, a (7,4)-IGD[21,1] exists by Construction I. Let m = 2, t = 11, $l_1 = l_2 = \dots = l_s = 1$. By using construction VI and taking s to be 0, 1, 2, 3, 4, 5, respectively, we can obtain a(7,4)-IGD[v,n] for (v,n) equal to (22,2), (23,3), (24,4), (25,5), (26,6), (27,7), respectively. Since a GD[4,1,2;14] exists by Corollary 3.6, and a (4,4)-IGD[14,4] exists by Theorem 3.2, a (7,4)-IGD[28,8] exists by Construction VIII. A (7,4)-IGD[30,10] exists by Theorem 3.1. In the Appendix, a (7,4)-IGD[29,9] is given. This completes the proof.

Lemma 3.10. If v - n = 40, then Lemma 3.4 is sufficient for the existence of a (7,4)-IGD[v,n].

Proof. If n ∈ {1, 2, 3, 4, 5}, then a (7,4)-IGD[40+n,3n] exists by previous results. Since a (7,4)-IGD[3n,n] exists by Theorem 3.1, a (7,4)-IGD[40+n,n] exists for n ∈ {1, 2, 3, 4, 5} by Construction I. By taking (u,k,v) = (7,4,10+n) and using Construction IX, we get a (7,4)-IGD[46,6] when (n,n') = (1,2); a (7,4)-IGD[47,7] when (n,n') = (1,3); a (7,4)-IGD[48,8] when (n,n') = (1,4); a (7,4)-IGD[49,9] when (n,n') = (2,1); a (7,4)-IGD[50,10] when (n,n') = (2,2); a (7,4)-IGD[51,11] when (n,n') = (2,3); a (7,4)-IGD[52,12] when (n,n') = (3,0);

a (7,4)-IGD[53,13] when (n,n') = (3,1); a (7,4)-IGD[54,14] when (n,n') = (3,2); a (7,4)-IGD[56, 16] when (n,n') = (4,0); a (7,4)-IGD[57,17] when (n,n') = (4,1).

Since a (7,4)-IGD[11,3] exists, a (7,4)-IGD[55,15] exists by Construction III. Since a GD[4,1,2;14] exists by Corollary 3.6, and a (4,4)-IGD[29,9] exists by Lemma 3.9, a (7,4)-IGD[58,18] exists by Construction VIII. Also, a (7,4)-IGD[60,20] exists by Theorem 3.1, and a (7,4)-IGD[59,19] exists (cf. the Appendix). This completes the proof.

Lemma 3.11. If $v - n \in \{24, 30, 60, 120\}$, then Lemma 3.4 is sufficient for the existence of a (7,4)-IGD[v,n].

Proof. Since a (7,4)-IGD[25,3] and a (7,4)-IGD[3,1] exist, a (7,4)-IGD[25,1] exists by Construction I. Set $m = 2, t = 13, l_1 = l_2 = ... = l_s = 1, 0 \le s \le 7$, and use Construction VI to give a (7,4)-IGD[v,n] for $26 \le v \le 33$ and $v - n = 24$. By taking $u = 7, k = 4, v = 8, n' \in \{2, 3, 4\}$, and using Construction X, we get a (7,4)-IGD[v,n] for $(v,n) \in \{(34,10), (35,11), (36,12)\}$.

Since a (7,4)-IGD[31,3] and a (7,4)-IGD[3,1] exist, a (7,4)-IGD[31,1] exists. By taking $m = 2, t = 16, l_1 = l_2 = ... = l_s = 1, 0 \le s \le 10$, and using Construction VI, we obtain a (7,4)-IGD[v,n] for $32 \le v \le 42$ and $v - n = 30$. By taking $u = 7$, $k = 4, v = 10, n' \in \{3, 4, 5\}$ and using Construction X, we obtain a (7,4)-IGD[v,n] for $(v,n) \in \{(43,13), (44,14), (45,15)\}$.

Since a (7,4)-IGD[61,3] and a (7,4)-IGD[3,1] exist, a (7,4)-IGD[61,1] exists. By taking $m = 2, t = 31, l_1 = l_2 = ... = l_s = 1, 0 \le s \le 25$, and using Construction VI, we obtain a (7,4)-IGD[v,n] for $62 \le v \le 87$ and $v - n = 60$. By taking $u = 7$, $k = 4, v = 20, n' \in \{8, 9, 10\}$ and using Construction X, we obtain a (7,4)-IGD[v,n] for $(v,n) \in \{(88,28), (89,29), (90,30)\}$.

Since a (7,4)-IGD[121,3] and a (7,4)-IGD[3,1] exist, a (7,4)-IGD[121,1] exists. By taking $m = 2, t = 61, l_1 = l_2 = ... = l_s = 1, 0 \le s \le 55$, and using Construction VI, we get a (7,4)-IGD[v,n] for $122 \le v \le 177$ and $v - n = 120$. By taking $u = 7$, $k = 4, v = 40, n' \in \{18, 19, 20\}$, and using Construction X, we get a (7,4)-IGD[v,n] for $(v,n) \in \{(178,58), (179,59), (180,60)\}$. This completes the proof.

Lemma 3.4 and Lemmas 3.7 to3.11 establish Theorem 3.3.

We have also obtained an interesting example of a (13,4)-IGD[6,1] (cf. Appendix (12)). Since a (4,4)-IGD[v,n] exists if and only if $v \ge 3n$ and $(v,n) \ne (6,1)$, and since $13 \in B[4]$, we have the following result by Construction II.

Theorem 3.12. A (13,4)-IGD[v,n] exists if and only if $v \ge 3n$.

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Appendix

In what follows we assume, unless indicated otherwise, that the group family is $\{A\times\{j\}: j\in Z_7\}$.

(1) A (7,4)-IGD[5,1] exists. $A = Z_4\cup\{\infty_1\}$. The base blocks are:

$((\infty_1,0),(0,1),(3,2),(1,4))$, $((\infty_1,0),(0,6),(2,5),(2,3))$,
$((0,0),(0,1),(3,3),(3,6))$, (mod 4, mod 7).

(2) A (7,4)-IGD[7,1] exists. $A = Z_6\cup\{\infty_1\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(4,4))$, $((\infty_1,0),(0,6),(1,5),(1,3))$,
$((0,0),(3,1),(2,3),(2,6))$, $((0,0),(1,1),(3,3),(4,6))$,
(mod 6, mod 7).

(3) A (7,4)-IGD[11,1] exists. $A = Z_{10} \cup \{\infty_1\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(4,4))$, $((\infty_1,0),(0,6),(4,5),(9,3))$,
$((0,0),(9,1),(6,3),(6,6))$, $((0,0),(8,1),(8,3),(7,6))$
$((0,0),(7,1),(5,3),(8,6))$, $((0,0),(5,1),(7,3),(9,6))$,
(mod 10, mod 7).

(4) A (7,4)-IGD[13,1] exists. $A = Z_{12} \cup \{\infty_1\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(4,4))$, $((\infty_1,0),(0,6),(1,5),(7,3))$,
$((0,0),(10,1),(6,3),(9,6))$, $((0,0),(9,1),(2,3),(10,6))$,
$((0,0),(8,1),(11,3),(11,6))$, $((0,0),(7,1),(9,3),(7,6))$,
$((0,0),(6,1),(1,3),(8,6))$, (mod 12, mod 7).

(5) A (7,4)-IGD[12,2] exists. $A = Z_{10} \cup \{\infty_1,\infty_2\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(4,4))$, $((\infty_1,0),(0,6),(1,5),(0,3))$,
$((\infty_2,0),(0,1),(8,2),(1,4))$, $((\infty_2,0),(0,6),(4,5),(4,3))$,
$((0,0),(7,1),(5,3),(8,6))$, $((0,0),(5,1),(7,3),(9,6))$,
$((0,0),(4,1),(9,3),(7,6))$, (mod 10, mod 7).

(6) A (7,4)-IGD[11,3] exists. $A = Z_8 \cup \{\infty_1,\infty_2,\infty_3\}$. The base blocks are:

$((\infty_1,0),(0,1),(1,2),(3,4))$, $((\infty_1,0),(0,6),(6,5),(1,3))$,
$((\infty_2,0),(0,1),(3,2),(1,4))$, $((\infty_2,0),(0,6),(3,5),(2,3))$,
$((\infty_3,0),(0,1),(6,2),(5,4))$, $((\infty_3,0),(0,6),(1,5),(6,3))$,
$((0,0),(0,1),(0,3),(4,6))$, (mod 8, mod 7).

(7) A (7,4)-IGD[13,3] exists. $A = Z_{10} \cup \{\infty_1,\infty_2,\infty_3\}$.The base blocks are:
$((\infty_1,0),(0,1),(0,2),(0,4))$, $((\infty_1,0),(0,6),(8,5),(1,3))$,
$((\infty_2,0),(0,1),(3,2),(6,4))$, $((\infty_2,0),(0,6),(6,5),(,8,3))$,
$((\infty_3,0),(0,1),(5,2),(7,4))$, $((\infty_3,0),(0,6),(3,5),(2,3))$,
$((0,0),(8,1),(4,3),(9,6))$, $((0,0),(9,1),(3,3),(4,6))$,
(mod 10, mod 7).

(8) A(7,4)-IGD[14,4] exists. $A = Z_{10} \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$. The base blocks are :

$((\infty_1,0),(0,1),(0,2),(1,4))$, $((\infty_1,0),(0,6),(2,5),(8,3))$,
$((\infty_2,0),(0,1),(7,2),(4,4))$, $((\infty_2,0),(0,6),(5,5),(5,3))$,
$((\infty_3,0),(0.1),(4,2),(7,4))$, $((\infty_3,0),(0,6),(7,5),(1,3))$,
$((\infty_4,0),(0,1),(2,2),(0,4))$, $((\infty_4,0),(0,6),(9,5),(7,3))$,
$((0,0),(9,1),(8,3),(4,6))$, (mod 10, mod 7).

(9) A (7,4)-IGD[17,5] exists. $A = Z_{12} \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(8,4))$, $((\infty_1,0),(0,6),(8,5),(9,3))$,
$((\infty_2,0),(0,1),(5,2),(5,4))$, $((\infty_2,0),(0,6),(6,5),(0,3))$,
$((\infty_3,0),(0,1),(8,2),(9,4))$, $((\infty_3,0),(0,6),(3,5),(8,3))$,
$((\infty_4,0),(0,1),(10,2),(2,4))$, $((\infty_4,0),(0,6),(1,5),(11,3))$,
$((\infty_5,0),(0,1),(1,2),(10,4))$, $((\infty_5,0),(0,6),(10,5),(5,3))$,
$((0,0),(3,1),(6,3),(5,6))$, (mod 12, mod 7).

(10) A (7,4)-IGD[29,9] exists. $A = Z_{20} \cup \{\infty_1,\infty_2,\ldots,\infty_9\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(10,4))$, $((\infty_1,0),(0,6),(19,5),(1,3))$,
$((\infty_2,0),(0,1),(2,2),(1,4))$, $((\infty_2,0),(0,6),(17,5),(17,3))$,
$((\infty_3,0),(0,1),(5,2),(17,4))$, $((\infty_3,0),(0,6),(14,5),(9,3))$,
$((\infty_4,0),(0,1),(7,2),(18,4))$, $((\infty_4,0),(0,6),(12,5),(6,3))$,
$((\infty_5,0),(0,1),(9,2),(2,4))$, $((\infty_5,0),(0,6),(10,5),(8,3))$,
$((\infty_6,0),(0,1),(12,2),(13,4))$, $((\infty_6,0),(0,6),(7,5),(4,3))$,
$((\infty_7,0),(0,1),(14,2),(8,4))$, $((\infty_7,0),(0,6),(5,5),(16,3))$,
$((\infty_8,0),(0,1),(16,2),(0,4))$, $((\infty_8,0),(0,6),(3,5),(15,3))$,
$((\infty_9,0),(0,1),(18,2),(15,4))$, $((\infty_9,0),(0,6),(1,5),(14,3))$,
$((0,0),(11,1),(7,3),(16,6))$, (mod 20, mod 7).

(11) A(7,4)-IGD[59,19] exists. A = $Z_{40} \cup \{\infty_1, \infty_2, \ldots, \infty_{19}\}$. The base blocks are:

$((\infty_1,0),(0,1),(0,2),(3,4))$,
$((\infty_2,0),(0,1),(2,2),(1,4))$,
$((\infty_3,0),(0,1),(4,2),(13,4))$,
$((\infty_4,0),(0,1),(6,2),(38,4))$,
$((\infty_5,0),(0,1),(8,2),(36,4))$,
$((\infty_6,0),(0,1),(10,2),(34,4))$,
$((\infty_7,0),(0,1),(12,2),(17,4))$,
$((\infty_8,0),(0,1),(14,2),(30,4))$,
$((\infty_9,0),(0,1),(16,2),(28,4))$,
$((\infty_{10},0),(0,1),(18,2),(26,,4))$,
$((\infty_{11},0),(0,1),(22,2),(20,4))$,
$((\infty_{12},0),(0,1),(24,2),(18,4))$,
$((\infty_{13},0),(0,1),(26,2),(21,4))$,
$((\infty_{14},0),(0,1),(28,2),(25,4))$,
$((\infty_{15},0),(0,1),(30,2),(5,4))$,
$((\infty_{16},0),(0,1),(32,2),(23,4))$,
$((\infty_{17},0),(0,1),(34,2),(15,4))$,
$((\infty_{18},0),(0,1),(36,2),(6,4))$,
$((\infty_{19},0),(0,1),(38,2),(27,4)$,
$((0,0),(21,1),(22,3),(21,6))$,
$((\infty_1,0),(0,6),(39,5),(32,3))$,
$((\infty_2,0),(0,6),(37,5),(31,3))$,
$((\infty_3,0),(0,6),(35,5),(33,3))$,
$((\infty_4,0),(0,6),(33,5),(3,3))$,
$((\infty_5,0),(0,6),(31,5),(5,3))$,
$((\infty_6,0),(0,6),(29,5),(7,3))$,
$((\infty_7,0),(0,6),(27,5),(9,3))$,
$((\infty_8,0),(0,6),(25,5),(11,3))$
$((\infty_9,0),(0,6),(23,5),(36,3))$,
$((\infty_{10},0),(0,6),(20,5),(16,3))$,
$((\infty_{11},0),(0,6),(17,5),(21,3))$,
$((\infty_{12},0),(0,6),(15,5),(38,3))$,
$((\infty_{13},0),(0,6),(13,5),(28,3))$,
$((\infty_{14},0),(0,6),(11,5),(0,3))$,
$((\infty_{15},0),(0,6),(9,5),(29,3))$,
$((\infty_{16},0),(0,6),(7,5),(24,3))$,
$((\infty_{17},0),(0,6),(5,5),(26,3))$,
$((\infty_{18},0),(0,6),(3,5),(30,3))$,
$((\infty_{19},0),(0,6),(1,5),(8,3))$,
(mod 40, mod 7).

(12) A(13,4)-IGD[6,1] exists. We take $\{A\times\{j\}: j\in Z_{13}\}$ as group family, where A = $Z_5 \cup \{\infty\}$. The base blocks are:

$((\infty,0),(0,1),(2,2),(4,11))$,
$((\infty,0),(0,3),(0,7),(4,9))$,
$((0,0),(2,5),(0,10),(3,6))$,
$((0,0), (0,1),(4,4),(1,6))$,
$((\infty,0),(0,4),(0,10),(0,12))$,
$((\infty,0),(0,5),(4,6),(2,8))$,
$((0,0),(3,1),(1,4),(2,6))$,
(mod 5, mod 13).