The Immersion Order, Forbidden Subgraphs and the Complexity of Network Integrity

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Abstract. The *vertex integrity* of a graph, $I(G)$, is given by $I(G) = \min_{V'}(|V'| + m(G-V'))$ where $V' \subseteq V(G)$ and $m(G-V')$ is the maximum order of a component of $G-V'$. The *edge integrity*, $I'(G)$, is similarly defined to be $I'(G) = \min_{E'}(|E'| + m(G-E'))$. Both of these are measures of the resistance of networks to disruption. It is shown that for each positive integer k, the family of finite graphs G with $I'(G) \leq k$ is a lower ideal in the partial ordering of graphs by immersions. The obstruction sets for $k \leq 4$ are determined and it is shown that the obstructions for arbitrary k are computable. For every fixed positive integer k it is decidable in time $O(n)$ for an arbitrary graph G of order n whether $I(G)$ is at most k, and also whether $I'(G)$ is at most k. For variable k, the problem of determining whether $I'(G)$ is at most k is shown to be NP-complete, complementing a similar previous result concerning $I(G)$.

1. Introduction

Several graph-theoretic parameters have been studied for the purpose of modeling the property of a network of being resistant to disruption. Connectivity is one such metric of network resilience. A systematic classification of such parameters has recently appeared [**BBLP1**].

The principal subject of this paper is the computational complexity of vertex and edge integrity, two "trade-off" metrics of network vulnerability [**BBLP2, BBLP3, BBLP4, BES1, BES2, CEF**].

Definition.: The *vertex integrity* of a graph G is $I(G) = \min_{V'}(|V'| + m(G - V'))$ where $V' \subseteq V(G)$ and $m(H)$ denotes the maximum order of a component of H.

Definition.: The *edge integrity* of a graph G is $I'(G) = \min_{E'}(|E'| + m(G-E'))$ where $E' \subseteq E(G)$.

It has previously been shown that for each positive integer k, the family of graphs

$$F_k = \{G | I(G) \leq k\}$$

is a lower ideal in the partial ordering of graphs by minors [**CEF**]. This has the consequence that for each fixed value of k, membership in F_k can be decided in time $O(n^2)$ for a graph of order n, by the deep results of Robertson and Seymour [**RS1,**

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RS2, RS3, RS4]. From a practical perspective, the small degree polynomial-time bound is somewhat deceptive. The algorithms are only proven to exist (the proof is nonconstructive, however, see [FL3] for a widely-applicable method of constructivization) and even if known, they would involve astronomical constants. It is an important open problem for many of the applications of the Robertson- Seymour theorems to determine whether alternative and "reasonable" small-degree polynomial-time algorithms can be found [FL1, FL2, Jo]. We show that this is indeed the case for vertex and edge integrity.

In the next section we show that for every k the family of graphs

$$F'_k = \{G | I'(G) \leq k\}$$

is a lower ideal in the partial ordering of graphs by immersions, recently proved to be a well-partial order by Robertson and Seymour [RS3]. The obstruction sets for $k \leq 4$ are determined, and are shown to be computable for arbitrary k.

In Section Three we show constructively that for each positive integer k, membership in F_k (and membership in F'_k) can be decided in time $O(n)$ for input a graph G of order n. The algorithms that we present identify a witnessing set of vertices $V' \subseteq V(G)$ (or edges $E' \subseteq E(G)$) when one exists.

In Section Four we show that the problem of determining, for a graph G and a positive integer k, whether $I'(G) \leq k$, is NP-complete. The reduction is from GRAPH PARTITIONING [GJ]. This complements a previous result that it is NP-complete to determine, for a graph G and a positive integer k, whether $I(G) \leq k$ [CEF].

All graphs in this paper are simple, without loops or multiple edges.

2. The Immersion Order and Obstruction Sets

Let G be a graph and let u, v, and w be vertices of G such that uv and uw are edges of G. A graph H is said to be obtained from G by a *lift* if $H = G - uv - uw + vw$.

Definition.: Given graphs H and G, $H \leq G$ in the immersion order if a graph isomorphic to H can be obtained from G by a sequence of operations of the following two kinds

(i) Replace G by a subgraph of G.
(ii) Lift two adjacent edges of G.

A *lower ideal* L in a partially ordered set $(S, \leq)$ is a subset $L \subseteq S$ such that if $x \in L$ and $y \leq x$ then $y \in L$. The *obstruction set* for L is the set of minimal elements of $\overline{L} = S - L$. The obstruction set O for L characterizes L in the sense that $x \in L$ if and only if there is no $y \in O$ such that $x \geq y$. The partially ordered set $(S, \leq)$ is a *well-partial order* if every obstruction set is finite.

Theorem 1. *The family of graphs F'_k is a lower ideal in the immersion order for all $k \geq 1$.*

Proof: Let $G \in F'_k$, G nontrivial. If v is an isolated vertex of G then $I'(G - v) = I'(G) \leq k$ and so $G - v \in F'_k$. Let $e \in E(G)$, $G' = G - e$ and let S be a set of edges of G such that $|S| + m(G - S) = I'(G)$. If $e \notin S$ then $m(G - S) \geq m(G' - S)$ so $I'(G') \leq I'(G) \leq k$, so $G' \in F'_k$. If $e \in S$ then let $S' = S - e$. Hence $|S'| = |S| - 1$ and $m(G - S) = m(G' - S')$ so $I'(G') \leq |S'| + m(G' - S') \leq |S| + m(G - S) = I'(G) \leq k$. Thus again $G' \in F'_k$. Therefore F'_k is closed under the operation of taking subgraphs.

Let uv and vw be adjacent edges of G and let $G' = G - uv - vw + uw$. If uv and vw are not in S then $m(G' - S) \leq m(G - S)$ so $I'(G') \leq I'(G) \leq k$ and $G' \in F'_k$. If $uv \in S$ or $vw \in S$ then let $S' = S - uv - vw + uw$. So $|S'| \leq |S|$ and $m(G' - S') \leq m(G - S)$ and hence $I'(G') \leq I'(G) \leq k$. Thus $G' \in F'_k$ and therefore F'_k is a lower ideal. ∎

Theorem 2. *For every positive integer k it is decidable in time $O(n^2)$ whether an arbitrary graph G of order n satisfies $I'(G) \leq k$.*

Proof: From [BES2], $I'(P_m) = \lceil 2\sqrt{m} \rceil - 1$ where P_m denotes the path with m vertices. So, for $m > (k+1)^2/4$, $P_m \notin F'_k$. Hence, by the theorems of Robertson and Seymour [RS4] we have our result. In particular, since the immersion order is a well-partial order, the obstruction set O'_k for F'_k is finite. The planar graph P_m is excluded, so for input G of order n, testing whether $G \geq H$ for each $H \in O'_k$ can be done in time $O(n^2)$. ∎

In the rest of this section we identify the obstruction sets O'_k for edge integrity for $k \leq 4$ and show that for every k, O'_k is computable.

Lemma 1. *If $G \in O'_k$ then $I'(G) = k + 1$.*

Proof: If $e \in E(G)$ and $G \in O'_k$ then, by the minimality of G, $I'(G - e) \leq k$. Let S be a set of edges in $G - e$ such that $I'(G - e) = |S| + m(G - e - S) \leq k$. Let $S' = S \cup \{e\}$. Thus $m(G - e - S) = m(G - S')$ and $|S'| = |S| + 1$ so $I'(G) \leq |S'| + m(G - S') = |S| + 1 + m(G - e - S) \leq k + 1$. ∎

Lemma 2. *If $G \in O'_k$ for $k \geq 1$ then G consists of at most $k + 2$ connected components and each of these components has order at most $k(k + 1)$.*

Proof: By Lemma 1 there is a set of edges S such that $|S| \leq k$ and $m(G - S) \leq k + 1 - |S|$. If C is a component of G of order greater than $k(k + 1)$ then $C - S$ must have a component of order $\geq k + 1$. But this implies that $I'(G) \geq k + 2$, contradicting Lemma 1.

If G has more than $k + 2$ components then for every set S of at most k edges there must be two components disjoint from S. Let C be a component of G of minimum order. We claim that $I'(G - C) \geq k + 1$ and therefore G is not minimal in $\overline{F}'_k$. Suppose not; then there is a set S of at most k edges in $G - C$ such that $|S| + m(G - C - S) \leq k$. There must be at least one component C' of $G - C$ disjoint from S, and therefore $|C'| \leq k - |S|$. Since $|C| \leq |C'|$, $m(G - S) \leq k - |S|$ which implies $I'(G) \leq k$, a contradiction. ∎

Theorem 3. *For each positive integer k the obstruction set O'_k is computable.*

$k \backslash i$	0	1	2	3
0	K_1	$\emptyset$	$\emptyset$	$\emptyset$
1	K_2	$\emptyset$	$\emptyset$	$\emptyset$
2	P_3	$\emptyset$	$\emptyset$	$\emptyset$
3	C_4 $K_{1,3}$ $P_4 \cup P_3$	P_5	$\emptyset$	$\emptyset$
4	C_5 $K_{1,4}$ $P_5 \cup P_4$ $P_5 \cup K_{1,3}$ $\cup P_4$ $\cup K_{1,3}$	P_7	$\cup P_3$	$\emptyset$

Table 1

Proof This follows immediately from Lemma 2, noting that it is straightforward to decide exhaustively for a graph H whether $H \in O'_k$. ∎

In what follows we describe a more efficient procedure for computing O'_k and we identify the obstruction sets for $k \leq 4$.

Define $O'_{k,i}$ to be the subset of O'_k consisting of those graphs H satisfying $\min\{|S| : I'(H) = k + 1 = |S| + m(H - S)\} = i$. Note that $O'_{k,i} = \emptyset$ for $i \geq k, k \geq 1$. Construct a graph H as follows

(i) Let $H - S$ consist of at most $k + i + 1$ components, each of which has order at most $k + 1 - i$ and at least one of which has order exactly $k + 1 - i$.

(ii) Add i edges to $H - S$, where each edge joins vertices in different components of $H - S$. These edges form the set S.

It is clear that any graph H in $O'_{k,i}$ can be constructed in this manner.

Theorem 4. *The sets* $O'_{k,i}, O \leq i \leq k \leq 4$, *are given in Table 1.*

Proof: Progressively determine the elements of $O'_{k,i}$ by identifying those graphs H constructed as above which satisfy the following conditions

(i) $I'(H) = k + 1$

(ii) The deletion of any edge or the lifting of any pair of adjacent edges of H decreases the edge-integrity.

(iii) H does not belong to $O'_{k,j}$ for $j < i$.

For example, let us determine the elements of $O'_{3,1}$. First note that any candidate H for membership in $O'_{3,1}$ must consist of at most 5 components of $H - S$ with order at most 3, with at least one component having order exactly 3, along with an edge joining two components of $H - S$. Thus $H - S$ consists of x_1 copies of K_1, x_2 copies of K_2, x_3 copies of P_3 and x_4 copies of K_3. Also $S = \{uv\}$, where u and v are in different components of $H - S$. If u or v is a vertex of degree 2 in $H - S$ then H contains a subgraph isomorphic to $K_{1,3}$. Note that $K_{1,3}$ is an element of $O'_{3,0}$ so that if H is isomorphic to $K_{1,3}$ then $H \notin O'_{3,1}$ and if $K_{1,3}$ is a proper subgraph of H then H is not minimal in $\overline{F}'_k$ and again $H \notin O'_{3,1}$. Thus if H is a graph in $O'_{3,1}$ then neither u nor v can belong to a copy of K_3 and neither u nor v can be the degree 2 vertex in a copy of P_3. If the largest component of H has less than five vertices then, for $S = \emptyset$, $|S| + m(H - S) \leq 4$ and so $H \in O'_{3,0}$ or $I'(H) \leq 3$. In either case $H \notin O'_{3,1}$. There must therefore be one component of H with at least 5 vertices. Thus, neither u nor v can be a copy of K_1. Hence $x_1 = 0$ since isolated vertices do not affect edge integrity. So uv must connect 2 copies of P_3 or a copy of P_3 to a copy of K_2. It follows that the largest component of H is either P_5 or P_6. Now $I'(P_5) = 4$ so $P_5 \in O'_{3,1}$. Thus, the largest component of any candidate for membership in $O'_{3,1}$ must be P_5. If H has any components besides the P_5 then H is not minimal in $\overline{F}'_k$ and therefore P_5 is the only graph in $O'_{3,1}$. ∎

3. Linear-time algorithms for fixed k

The fact that for each positive integer k the families of graphs F_k and F'_k are lower ideals in respectively, the minor and immersion orders, is enough to imply the existence of polynomial-time membership tests for these classes. The algorithms would be known if the obstruction sets for these classes were known. The Robertson-Seymour theorems establish only that the obstruction sets for F_k and F'_k are finite. We have seen in Section 2 that the obstruction sets for edge and vertex integrity are at least computable.

Because computing the vertex or edge integrity of a graph is *NP*-hard, it is to be expected that there would be a hidden constant exponential in k in our proof that, for all fixed k, membership in F_k and F'_k can be decided in time $O(n)$. The exponential hidden constant that we obtain is k^{3k}.

Algorithms based on obstruction testing exhibit the additional peculiarity that they do not generate any information like "natural evidence" for the decision. In other words, an algorithm for vertex integrity might respond "yes" to a graph G without giving any hint about what a vertex set $V' \subseteq V(G)$ might be that would satisfy the inequality $|V'| + m(G - V') \leq k$. The algorithm would only correctly establish whether or not such a subset $V' \subseteq V(G)$ exists.

The process of using a decision algorithm to construct evidence is termed *self-reducibility* and has been studied from a number of perspectives in structural complexity theory (see, for example [MP, S]). The starting point for our linear-time algorithms is the design of *efficient* self-reductions for k-vertex and k-edge integrity. In particular, we will show that, given to use as an oracle an algorithm that decides membership in F_k (F'_k), we can produce a witnessing set of vertices (edges), when one exists, in time $O(n)$ by making at most a constant k' number of queries to the oracle. The constant k' depends linearly on k.

In order to state the theorems that follow we must fix a model of computation, and in order to reflect linear time requirements on real machines we choose the random access framework [AHU]. A *random access oracle machine* is equipped with a special set of registers into which a word x can be written. The machine has a special query state $q_?$, and two associated states q_{yes} and q_{no}. Upon entering $q_?$ a transition is made to either q_{yes} or q_{no} depending upon whether $x \in A$, where A is the oracle language for the machine.

Lemma 3. *For every positive integer k there is a constant k' (that depends on k) and an oracle algorithm of time complexity $O(n)$ making at most k' calls to the oracle, that for input a graph G of order n satisfying $I(G) \leq k$ and $m(G) > k$, finds a vertex $v \in V(G)$ such that $I(G - v) \leq k - 1$.*

Proof: In time $O(n)$ it can be determined whether any vertex u of G has degree k or more. Since $I(G) \leq k$ there is some subset $V' \subseteq V(G)$ such that $|V'| + m(G - V') \leq k$. Suppose $u \notin V'$ and let t be the number of neighbors of u that belong to V'. The component of $G - V'$ containing u has order at least $k + 1 - t$, which implies that $|V'| + m(G - V') \geq k + 1$, a contradiction. Thus every such subset V' must contain u, and $I(G - u) \leq k - 1$. Otherwise, G has maximum degree at most $k - 1$. This implies that no component of G has order greater than or equal to k^3, since removing k vertices from a connected graph of maximum degree $k - 1$ yields a graph with no more than k^2 components.

In time $O(n)$ a component of G can be found with order at least $k + 1$. Such a component C exists, by the hypothesis that $m(G) > k$. Since C has order bounded by k^3, no more than $\log_2 k^3$ oracle calls are necessary to discover a vertex v satisfying the conclusion of the Lemma by binary search. We may take $k' = \lceil 3 \log_2 k \rceil$. ∎

Lemma 4. *For every positive integer k there is a constant k' (that depends on k) and an oracle algorithm of time complexity $O(n)$ that for input a graph G of order n satisfying $I'(G) \leq k$ and $m(G) > k$, finds an edge $e \in E(G)$ such that $I'(G - e) \leq k - 1$.*

Proof: The argument is similar. No component C of G can have order greater than k^2. Otherwise, if $t = |E'|$ for a witnessing set of edges E' then necessarily $t \leq k - 1$ and removing t edges from C yields at most $t + 1 \leq k$ components and one of these must have order at least k.

In time $O(n)$ a component C of order at least $k + 1$ can be found, and every witnessing set of edges E' must contain some edge of C. Thus in at most $\binom{k^2}{2}$ queries of the oracle an appropriate edge e can be identified. This can be improved by the observation that no vertex of G can have degree k or more. Thus $k' = k^3$ oracle calls suffice. ∎

Linear-time algorithms for integrity can be derived from the procedures embodied in the Lemmas above by replacing actual oracle consultations by exhaustive checking of all possible outcomes.

Theorem 5. *For all k in time $O(n)$ it can be determined for an arbitrary graph G of order n whether $I(G) \leq k$, and a set $V' \subseteq V(G)$ produced such that*

$$|V'| + m(G - V') \leq k$$

when one exists.

Proof: In time $O(n)$ it can be determined whether $m(G) \leq k$. If so, then $I(G) \leq k$ and V' can be taken to be the empty set. Otherwise, by Lemma 1, if such a set V' exists then by a procedure requiring time $O(n)$, a set of at most k^3 graphs can be obtained from G, each by removing a single vertex, and for at least one such graph $G' = G - v$ we must have $I(G') \leq k - 1$. Repeating this process, we construct a tree T, each vertex of which is labeled by a graph H, and by a subset S of V', with $H = G - S$. The root is labeled by G, the children of the root (there are at most k^3) are labeled with the graphs described above, assuming

$m(G) \geq k$, and in general a vertex of T at a distance r from the root is labeled with a graph obtained from G by removing r vertices, each in a stage of application of Lemma 1. A leaf of T at a distance r from the root is labeled by a graph H for which one of the following holds

(1) The largest component of H has order greater than $k - r$, but the $O(n)$ procedure of Lemma 1 fails to find either a vertex of degree at least $k - r$ or a component of order at most $(k - r)^3$.

(2) The largest component of H has order at most $k - r$. (In this case we may take V' to be the set S for which $G - S = H$, and conclude that $I(G) \leq k$.)

If at each leaf (1) holds, then $I(G) > k$. T has order bounded by a constant that depends on k, and thus the entire algorithm requires $O(n)$ time. The hidden constant is $(k^3)^k = k^{3k}$. ∎

By an entirely analogous argument using Lemma 2 we obtain

Theorem 6. *For all k, in time $O(n)$ it can be determined for an arbitrary graph G of order n whether $I'(G) \leq k$, and a witnessing set $E' \subseteq E(G)$ produced, when one exists.* ∎

4. The complexity of edge integrity for variable k.

The main result of this section is a proof that the following decision problem is NP-complete.

EDGE INTEGRITY

Input: A graph G and a positive integer k.

Question: Is $I'(G) \leq k$?

A similar result for the analogous problem VERTEX INTEGRITY has been previously established [CEF]. Our proof will make use of the following subproblem of GRAPH PARTITIONING [GJ] that is known to be NP-complete [HR].

GRAPH 3-PARTITION (G3P)

Instance: A graph G, and a positive integer k.

Question: Is there a subset $E' \subseteq E(G)$ with $|E'| \leq k$ and $m(G - E') \leq 3$?

Theorem 7. *EDGE INTEGRITY is NP-complete.*

Proof: The problem is plainly in NP. We assume that the order of G is at least 5. Given an instance (G, k) of G3P we show how to compute a graph G' and a positive integer k' such that $I'(G') \leq k'$ if and only if (G, k) is a "yes" instance of G3P. Let $k' = 4k$. Let G_1 be the graph obtained from G by identifying each vertex of G with a vertex of a copy of the complete graph on k vertices. Let G' be the disjoint union of G_1 and two copies of the complete graph on $3k$ vertices.

If there is a subset $E' \subseteq E(G)$ with $|E'| \leq k$ such that $G - E'$ has no component of order greater than 3, then removing the same set of edges from G' yields a graph in which no component has order more than $3k$, and so $I'(G') \leq 4k = k'$.

Conversely, suppose $I'(G') \leq 4k$, and let E' be a set of edges minimizing $|E'| + m(G' - E')$. Since at least $6k - 2$ edges must be removed from $2K_{3k}$ in order to produce a graph with largest component of order less than $3k$ and without loss of generality we may assume $E' \cap 2K_{3k} = \emptyset$, we must have $m(G' - E') \geq 3k$. $I'(G') \leq 4k$ immediately implies $|E'| \leq k$. If $G - E'$ has a component of order greater than 3 then $G' - E'$ has a component of order $\geq 4k$. If E' is nonempty then we are done. Otherwise, the order of G is 4, contrary to our assumption. ∎

5. Summary

The picture of the computational complexity of the vulnerability metrics of vertex and edge integrity appears now to be rather complete. For fixed k, the vertex integrity class is closed in the minor ordering, and the edge integrity class is closed in the immersion ordering. Thus encouraged to look for small degree polynomial-time algorithms we have seen that constructive algorithms with time bound $O(n)$ having reasonable hidden constants exponential in k are to be had, while for variable k both problems are NP-complete.

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