EXISTENCE OF INCOMPLETE GROUP DIVISIBLE DESIGNS

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Abstract. Incomplete group divisible designs (IGDDs) are the group divisible designs (GDDs) missing disjoint sub-GDDs, which need not exist. We denote by $\text{IGDD}_u^k(v,n)$ the design $\text{GDD}[k,1,v;uv]$ missing a sub-$\text{GDD}[k,1,n;un]$. In this paper we give the necessary condition for the existence of $\text{IGDD}_u^k(v,n)$ and prove that the necessary condition is also sufficient for $k=3$.

1. Introduction.

We suppose that the readers are familiar with the concept of pairwise balanced design (PBD), group divisible design (GDD), mutually orthogonal Latin squares, etc. (see [4], [12]). Here is the definition of IGDD missing one sub-GDD, adapted from [9].

Definition 1.1: Suppose X is a finite set, $\mathcal{G}=\{G_1,G_2,\dots,G_k\}$ is a partition of X, and $\mathcal{A}$ is a set of some subsets of X. Suppose also that $Y\subseteq X$, and $H_i=G_i\cap Y$ for $1\le i\le k$. Let $\mathcal{H}=\{H_1,\dots,H_k\}$. Then we say that we have an incomplete GDD $(X,\mathcal{G},\mathcal{A})$, briefly IGDD, missing a sub-GDD $(Y,\mathcal{H},-)$, provided that no block $A\in\mathcal{A}$ can contain two members of a group G_i, or two members of Y, and that every pair $\{x,y\}$ of elements with $\{x,y\}\not\subseteq Y$ and $\{x,y\}\not\subseteq G_i$ for $1\le i\le k$, is contained in a unique block.

Note that the "missing" sub-GDD needs not exist.

The singular indirect product construction plays an important role in various combinatorial design problems such as mutually orthogonal Latin squares, block designs, etc., and IGDD is instrumental in this construction.

The existence of $\text{IGDD}_4^4(v,n)$ is the same as the existence of $IA(v,n)$. Started from J. D. Horton [6] much work on this problem has been done by W. D. Wallis, K. Heinrich, L. Zhu ([5], [10], [13], [14]) and others. This problem is finally solved in [5]. In [9], D. R. Stinson has given a very general construction for GDDs using IGDDs. On the other hand, H. Hanani [4] and A. E. Brouwer, A. Schrijver, H. Hanani [3] have discussed the existence of GDDs for $k=3$ and 4. From all of these the natural and direct generalization is the existence problem of IGDDs.

By using recursive constructions of Wilson type, the direct construction of Bose's mixed differences, and other special constructions, we shall prove in this paper that the necessary condition for the existence of $\text{IGDD}_u^3(v,n)$ is also sufficient.

Now we give necessary condition for the existence of $\text{IGDD}_u^k(v,n)$ and $\text{IGDD}_u^3(v,n)$.

In Definition 1.1, suppose $x \in X - Y$ and $y \in Y$. Denote by r_x, r_y the numbers of blocks containing x, y, respectively. Denote by b the number of blocks of an $\mathrm{IGDD}_u^k(v,n)$. Let $r_{x,k-1}$ be the number of blocks each containing x and one element in Y. Let $r_{x,k}$ be the number of blocks each containing x and no element in Y. Then

$$\begin{cases} r_x = v(u-1)/(k-1) \\ r_y = (v-n)(u-1)/(k-1) \\ b = u(u-1)(v^2-u^2)/k(k-1) \\ r_{x,k-1} = (u-1)n \\ r_x = r_{x,k-1} + r_{x,k} \end{cases}.$$

Since $r_{x,k} \geq 0$, we have $v \geq (k-1)n$. Since r_x, r_y and b are all integers, we then have

Theorem 1.2. *The necessary condition for the existence of an* $IGDD_u^k(v,n)$ *is*

$$\begin{cases} v \geq (k-1)n \\ v(u-1) \equiv 0 \pmod{k-1} \\ (v-n)(u-1) \equiv 0 \pmod{k-1} \\ u(u-1)(v^2-n^2) \equiv 0 \pmod{k(k-1)} \end{cases}.$$

Corollary 1.3. *The necessary condition for the existence of an* $IGDD_u^3(v,n)$ *is* $v \geq 2n$, *and*

(i) $u \equiv 0,4 \pmod 6$ *and* $v, n \equiv 0 \pmod 2$, *or*

(ii) $u \equiv 1,3 \pmod 6$ *and* v, n *are any positive integers, or*

(iii) $u \equiv 2 \pmod 6$ *and* $\begin{cases} v \equiv 0 \pmod 6 \\ n \equiv 0 \pmod 6, \end{cases}$ *or* $\begin{cases} v \equiv 2,4 \pmod 6 \\ n \equiv 2,4 \pmod 6, \end{cases}$ *or*

(iv) $u \equiv 5 \pmod 6$ *and* $\begin{cases} v \equiv 0 \pmod 3 \\ n \equiv 0 \pmod 3, \end{cases}$ *or* $\begin{cases} v \equiv 1,2 \pmod 3 \\ n \equiv 1,2 \pmod 3. \end{cases}$

2. Constructions.

First we describe some recursive constructions. Let Z_n denote the set of integers $0, 1, \ldots, n-1$, which forms a ring of integers modulo n.

Theorem 2.1. *Suppose there exists a* $GDD[\{u_1, \ldots, u_s\}, 1, m; mu]$ *and an* $IGDD_h^k(v,n)$ *for every* $h = u_i$, $1 \leq i \leq s$. *Then there exists an* $IGDD_u^k(mv, mn)$.

Proof: Suppose the $\mathrm{GDD}[\{u_1, \ldots, u_s\}, 1, m; mu]$ is defined on $Z_m \times Z_u$ with groups $Z_m \times \{j\}$, $j = 0, 1, \ldots, u-1$. For each block B, where $|B| = u_i$, $1 \leq i \leq s$, construct an $\mathrm{IGDD}_{u_i}^k(v,n)$ on $B \times Z_v$. Then, all the blocks of these IGDDs form the block set of the required $\mathrm{IGDD}_u^k(mv, mn)$ based on the set

$Z_m \times Z_u \times Z_v$. If each input IGDD has its missing sub-GDD based on $B \times Z_n$, then the resulting IGDD has its missing sub-GDD based on $Z_m \times Z_u \times Z_n$. ∎

Since a $(u, \{u_1, \ldots, u_s\}, 1)$-PBD can be regarded as a GDD with all groups of size one, we then have from Theorem 2.1 the following.

Theorem 2.2. *Suppose there exists a $(u, \{u_1, \ldots, u_s\}, 1)$-PBD and an $IGDD^k_{u_i}(v, n)$ for every u_i, $1 \le i \le s$. Then there exists an $IGDD^k_u(v, n)$.*

Theorem 2.3. *Suppose there exists a TD$(u + 1, q)$. Suppose also there exist some non-negative integers t_i, $1 \le i \le q$, such that for a given positive integer m and any $1 \le i \le q$, an $IGDD^k_u(m+t_i, t_i)$ exists. Then there exists an $IGDD^k_u(mq + r, r)$, where $r = t_1 + \ldots + t_q$.*

Proof: Suppose $(Y, \mathcal{H}, \mathcal{B})$ is a TD$(u + 1, q)$, where u is a positive integer, $\mathcal{H} = \{H_1, H_2, \ldots, H_u, H_{u+1}\}$, $H_{u+1} = \{x_1, x_2, \ldots, x_q\}$. We define u weight functions $W_i: Y \to Z^+ \cup \{0\}$, for $1 \le i \le u$, such that

$$W_i(x) = \begin{cases} m & \text{if } x \in H_i, \\ 0 & \text{if } x \in H_j,\ j \le u \text{ and } j \ne i, \\ t_j & \text{if } x = x_j \in H_{u+1}. \end{cases}$$

Then, (1) for each block $B \in \mathcal{B}$, there exists an IGDD$^k_u(m + t_j, t_j)$, where $t_j = W_i(x_j)$, $x_j \in B \cap H_{u+1}$ and $1 \le i \le u$; (2) for each group H_i, $1 \le i \le u$, there exists a trivial GDD with group size vector $(0, \ldots, 0, \overset{m}{i}, 0, \ldots, 0)$ and empty block set. From the general construction for GDDs in Stinson [9], we obtain an IGDD with one missing sub-GDD, which has the group size vector $(r, r, \ldots, r)$. The IGDD has the group size vector $(mq + r, mq + r, \ldots, mq + r)$ and is the required IGDD$^k_u(mq + r, r)$. ∎

The following lemma is obvious and we shall use it quite often without mentioning.

Lemma 2.4. *Suppose there exist an $IGDD^k_u(v, v_1)$ and an $IGDD^k_u(v_1, n)$. Then there exists an $IGDD^k_u(v, n)$.*

Next, we give some further recursive constructions.

Theorem 2.5. *Suppose there exists a $GDD[k, 1, m; mt]$. If there exist an $IGDD^k_u(m + n, n)$ and an $RTD(k, u)$, then there exist an $IGDD^k_u(tm + n, m + n)$ and an $IGDD^k_u(tm + n, n)$.*

Proof: Suppose the GDD is based on $Z_m \times Z_t$. Give each element weight u and use RTD(k, u) as input designs, we obtain a GDD$[k, 1, mu; mtu]$. Since the TD(k, u) is resolvable, for any block B of the initial GDD the input design contains blocks $B \times \{i\}$ for $0 \le i \le u - 1$. We break the groups of size mu by constructing IGDD$^k_u(m + n, n)$ on each group such that for group $Z_m \times \{j\} \times Z_u$

the IGDD has groups $Z_m \times \{j\} \times \{i\} \cup \{\infty_i^1, \ldots, \infty_i^n\}, i = 0, 1, \ldots, u-1$. In the resulting design we delete, for each $i \in Z_u$, all the blocks in the set $Z_m \times Z_t \times \{i\}$ and obtain the required $\text{IGDD}_u^k(tm+n, n)$. For some j if we further delete all the blocks in the set $Z_m \times \{j\} \times Z_n$, we obtain an $\text{IGDD}_u^k(tm+n, m+n)$. ∎

Theorem 2.6. *Suppose there exists a* $GDD[3, 1, m; mt]$. *If there exists an* $IGDD_5^3(m+n, n)$, *then there exist an* $IGDD_5^3(tm+n, m+n)$ *and an* $IGDD_5^3(tm+n, n)$.

Proof: It is obvious that an RTD(3, 5) exists. Then apply Theorem 2.5. ∎

Corollary 2.7. *If an* $IGDD_5^3(m+n, n)$ *exists, then an* $IGDD_5^3(3m+n, m+n)$ *and an* $IGDD_5^3(3m+n, n)$ *exist.*

Proof: The existence of a $\text{GDD}[3, 1, m; 3m]$ is obvious, then apply Theorem 2.6. ∎

The following is a generalization of Theorem 2.5.

Theorem 2.8. *Suppose there exists an* $IGDD_t^k(m+r, r)$. *If there exist an* $IGDD_u^k(m+r+h, r+h)$ *and an* $RTD(k, u)$, *then there exists an* $IGDD_u^k(tm+tr+h, tr+h)$.

Proof: Suppose the IGDD is based on $Z_{m+r} \times Z_t$. Give each element weight u and use RTD(k, u) as input designs like we did in Theorem 2.5, we obtain an IGDD. Break its groups with $\text{IGDD}_u^k(m+r+h, r+h)$, based on $(Z_{m+r} \times \{j\} \cup \{\infty^1, \ldots, \infty^h\}) \times Z_u$, and delete all the blocks in each set $Z_{m+r} \times Z_t \times \{j\}$ for $0 \le j \le u-1$, we obtain the required $\text{IGDD}_u^k(tm+tr+h, tr+h)$, based on $(Z_{m+r} \times Z_t \cup \{\infty^1, \ldots, \infty^h\}) \times Z_u$. ∎

Theorem 2.9. *For any positive integer* m, *if there exists an* $IGDD_5^3(m+1+n, 1+n)$, *then there exists an* $IGDD_5^3(3m+3+n, 3+n)$.

Proof: Apply Theorem 2.7 with $t = 3, k = 3, u = 5, r = 1$ and $h = n$. Since $N(m+1) \ge 1$, an $\text{IGDD}_3^3(m+1, 1)$ exists. An RTD(3, 5) is obvious and the conclusion then follows. ∎

Theorem 2.10. *If there exists an* $IGDD_5^3(v, n)$, *then an* $IGDD_5^3(vm, nm)$ *exists for any positive integer* m.

Proof: Give each point of the $\text{IGDD}_5^3(v, n)$ weight m. Since a $\text{GDD}[3, 1, m; 3m]$ exists, we obtain the required IGDD. ∎

Finally, we use Bose's mixed difference method to give some direct construction for $\text{IGDD}_5^3(v, n)$, which will be used in Section 5 and Section 6.

Suppose $(X, \mathcal{G}, \mathcal{A})$ is an IGDD missing sub-GDD $(Y, \mathcal{H}, -)$ with $\mathcal{G} = \{G_1, \ldots, G_5\}$, $G_i = (Z_{v-n} \cup \{\infty^1, \ldots, \infty^n\}) \times \{i\}$, where $v - 2n \equiv 0 \pmod 3$, $\mathcal{H} = \{H_1, \ldots, H_5\}$, $H_i = \{\infty^1, \ldots, \infty^n\} \times \{i\}$. For brevity we simply write

$(x)_i$ or x_i for $\{x\} \times \{i\}$ and ∞_i^j for $\{\infty^j\} \times \{i\}$. Consider initial blocks defined on $(Z_{v-n} \cup \{\infty^1, \dots, \infty^n\}) \times Z_5$ by filling the gaps in the following patterns:

$$\frac{v-2n}{3}\begin{cases} \{(\ )_0,(\ )_1,(\ )_3\} \\ \{(\ )_0,(\ )_1,(\ )_3\} \\ \vdots \\ \{(\ )_0,(\ )_1,(\ )_3\} \end{cases} (I); \quad \frac{v-2n}{3}\begin{cases} \{(\ )_0,(\ )_1,(\ )_4\} \\ \{(\ )_0,(\ )_1,(\ )_4\} \\ \vdots \\ \{(\ )_0,(\ )_1,(\ )_4\} \end{cases} (II);$$

$$2n\begin{cases} \{\infty_0^1,(\ )_1,(\ )_4\} \\ \{\infty_0^1,(\ )_2,(\ )_3\} \\ \vdots \\ \{\infty_0^n,(\ )_1,(\ )_4\} \\ \{\infty_0^n,(\ )_2,(\ )_3\} \end{cases} (III).$$

We develop these initial blocks modulo $v - n$ for the elements and modulo 5 for the indices. Then we only need to verify the $(1,0)$-mixed differences and $(4,1)$-mixed differences. Once part (I) and part (II) are constructed properly, that is, they produce different $(1,0)$-mixed differences and different $(4,1)$-mixed differences, part (III) can be constructed freely. For example, if d does not appear as a $(4,1)$-mixed difference from part (I) and part (II), we can take some block $\{\infty_0^j,(0)_1,(d)_4\}$ in part (III) in order that d does appear as a $(4,1)$-mixed difference from this block. In this way we can obtain an $IGDD_5^3(v,n)$ iff the $(1,0)$-mixed differences from part (I) and part (II) are all different and so do $(4,1)$-mixed differences.

3. Preliminaries.

In order to prove the main necessary and sufficient condition in Section 7 we give some preliminary results here, which will be used in the subsequent sections.

By A. E. Brouwer [2], T. Beth, D. Jungnickel and H. Lenz [1], R. Roth and M. Peters [7], we have

Lemma 3.1. *Denote* $B = \{10,14,18,22,26,30,34,38,42\}$, $C = \{20,28,44,52\}$, *and* $A = \{n \mid n \text{ is an integer} \geq 5\} - \{6\} - B - C$. *If* $q \in A$, *then there exists a* $TD(6,q)$. *If* $q \in B$, *then there exists a* $TD(6,\frac{1}{2}q)$. *If* $q \in C$, *then there exists a* $TD(6,\frac{1}{4}q)$.

Lemma 3.2. *There exists an* $IGDD_5^3(3k,0)$ *for any positive integer* k.

Proof: This IGDD is equivalent to a $GDD[3,1,3k;15k]$ and the latter exists from Theorem 6.2 of H. Hanani [4]. ∎

The following theorem is essentially Theorem 2.2 in Chapter 6 of [8].

Theorem 3.3. *The necessary and sufficient condition for the existence of IGDD$_3^3$ (v, n) is $v \geq 2n$.*

In the remaining part of this section we shall show the existence of IGDD$_5^3$ (v, n) for some special parameters v and n. In the following lemma we list the initial blocks for an IGDD$_5^3$ (v, n) based on the set $X = (Z_{v-n} \cup \{\infty^1, \ldots, \infty^n\}) \times Z_5$, where the element is briefly written as x_i or ∞_i^j. By (mod $v - n$, mod 5) we mean that the initial blocks are developed mod $v - n$ for the elements and mod 5 for the indices. By (———, mod 5) we mean that development is done only to indices. The notation (mod $v - n$,———) has the similar meaning. In each case the groups G_i are $(Z_{v-n} \cup \{\infty^1, \ldots, \infty^n\}) \times \{i\}$ and the missing sub-GDD has groups $H_i = \{\infty^1, \ldots, \infty^n\} \times \{i\}$.

Lemma 3.4. *There exists an IGDD$_5^3$$(v, n)$, where $(v, n) \in \{(5,1), (5,2), (7,2), (9,3), (21,9), (11,4), (16,7), (17,8)\}$.*

Proof: For each case the initial blocks are listed in Appendix. ∎

If $k \notin K$, we denote by $(v, K \cup \{k^*\}, 1)$-PBD the PBD containing a unique block of size k. Let $B(K \cup \{k^*\})$ denote the set $\{v \mid \exists (v, K \cup \{k^*\}, 1)\text{-PBD}\}$. By R. M. Wilson [11] we have

Lemma 3.5. *If $u \equiv 5 \pmod 6$, then $u \in B(5^*, 3)$.*

Corollary 3.6. *There exists an IGDD$_5^3$$(2, 1)$.*

Proof: Since $11 \in B(5^*, 3)$, there exists a $(11, \{5^*, 3\}, 1)$-PBD. Delete one point not belonging to the block of size 5, and delete also that block. This gives an IGDD$_5^3$$(2, 1)$. ∎

Lemma 3.7. *There exists an IGDD$_5^3$$(v, n)$ for $(v, n) = (13,4), (14,5)$.*

Proof: Apply Theorem 2.10 with the IGDD$_5^3$$(2, 1)$ in Corollary 3.6, we obtain an IGDD$_5^3$$(6,3)$. Apply Theorem 2.9 with this IGDD and the IGDD$_5^3$$(5,2)$ in Lemma 3.4, we get the required IGDDs. ∎

Lemma 3.8. *There exists an IGDD$_5^3$$(v, n)$ for $(v, n) \in \{(31, 13), (13, 1), (13,5), (11,5), (7,1), (19,7), (17,5), (25,7), (23,11)\}$.*

Proof: Apply Corollary 2.7 with IGDD$_5^3$$(v, n)$ for $(v, n) = (13,4), (5,1)$ and $(5,2)$, we obtain the first four IGDDs. Since IGDD$_5^3$$(7,2)$ and IGDD$_5^3$$(2,1)$ exist, we have from Lemma 2.4 and IGDD$_5^3$$(7,1)$. Then apply Corollary 2.7, we get the sixth IGDD. Give respectively weight 2 and 3 to each point of a GDD[3, 1, 2; 8], we have a GDD[3, 1, 4; 16] and a GDD[3, 1, 6; 24]. Applying Theorem 2.6 with IGDD$_5^3$$(5, 1)$ and IGDD$_5^3$$(7, 1)$ gives the next two IGDDs. Apply Corollary 2.7 with IGDD$_5^3$$(11,5)$, we obtain the last IGDD. ∎

Lemma 3.9. *There exists an* $IGDD_5^3(v,n)$ *for* $(v,n) \in \{(4,2),(6,3),(8,4),(12,6),(16,8),(24,12),(10,5),(10,4),(20,8),(18,6),(14,2),(22,10),(15,3),(28,10),(34,16)\}$.

Proof: Apply Theorem 2.10 with the known IGDDs shown above. ∎

Lemma 3.10. *There exists an* $IGDD_5^3(v,n)$ *for* $(v,n) \in \{(4,1),(8,2),(16,4),(10,1)\}$.

Proof: Apply Lemma 2.4 with $\mathrm{IGDD}_5^3(v,n)$, where $(v,n) = (4,2),(2,1),(8,4),(16,8),(10,5)$ and $(5,1)$. ∎

4. Existence of $\mathrm{IGDD}_5^3(v,n)$ for $n \equiv 0 \pmod 3$.

We are now in a position to prove the existence of $\mathrm{IGDD}_5^3(v,n)$. In this section we deal with the case $n \equiv 0 \pmod 3$. By Corollary 1.3, we must now have $v \geq 2n$ and $v \equiv 0 \pmod 3$. Let $n = 3k$, $v - n = 3q$. Obviously, we have $q \geq k$. Let A, B and C be the sets in Lemma 3.1.

Lemma 4.1. *If* $q \in A$, *then there exists an* $IGDD_5^3(v,n)$, *where* $v \equiv n \equiv 0 \pmod 3$ *and* $v \geq 2n$.

Proof: Since $q \in A$, we have from Lemma 3.1 a TD$(6,q)$. Let $t_1 = t_2 = \ldots = t_k = 3$, $t_{k+1} = \ldots = t_q = 0$ in Theorem 2.3. By Lemma 3.2 and Lemma 3.9 there exist $\mathrm{IGDD}_5^3(3,0)$ and $\mathrm{IGDD}_5^3(6,3)$, then we have from Theorem 2.3 an $\mathrm{IGDD}_5^3(3q+3k,3k)$, that is, an $\mathrm{IGDD}_5^3(v,n)$, where $v \equiv n \equiv 0 \pmod 3$ and $v \geq 2n$. ∎

Lemma 4.2. *If* $q \in B \cup C$, *then there exists an* $IGDD_5^3(3q+3k,3k)$.

Proof: For $q \in B$, we have a TD$(6,\frac{1}{2}q)$. If k is even, let $t_1 = \ldots = t_{\frac{1}{2}k} = 6$, $t_{\frac{1}{2}k+1} = \ldots = t_{\frac{1}{2}q} = 0$ in Theorem 2.3. By the existence of $\mathrm{IGDD}_5^3(12,6)$ and $\mathrm{IGDD}_5^3(6,0)$, we have an $\mathrm{IGDD}_5^3(3q+3k,3k)$. If k is odd, since $\frac{1}{2}q \geq \frac{1}{2}(k-1)+1$, we can take $t_1 = \ldots = t_{\frac{1}{2}(k-1)} = 6$, $t_{\frac{1}{2}(k+1)} = 3$, $t_{\frac{1}{2}(k-1)+2} = \ldots = t_{\frac{1}{2}q} = 0$ in Theorem 2.3. By the existence of an $\mathrm{IGDD}_5^3(9,3)$, there exists an $\mathrm{IGDD}_5^3(3q+3k,3k)$.

For $q \in C$, we have a TD$(6,\frac{1}{4}q)$. Let $t_i \in \{0,1,\ldots,12\}$, where $1 \leq i \leq q$. By the lemmas in Section 3 we have an $\mathrm{IGDD}_5^3(12+t_i,t_i)$ for any t_i. As $\frac{1}{4}q \geq 3k/12$, we could choose suitable t_i such that $t_1 + \ldots + t_{\frac{1}{4}q} = 3k$. Then we have an $\mathrm{IGDD}_5^3(3q+3k,3k)$. ∎

Theorem 4.3. *When* $n \equiv 0 \pmod 3$, *the necessary and sufficient condition for the existence of* $IGDD_5^3(v,n)$ *is* $v \equiv 0 \pmod 3$ *and* $v \geq 2n$.

Proof: By Lemma 4.1 and Lemma 4.2 we need only check the cases when $q \in \{1,2,3,4,6\}$ and $q \geq k$. We list these parameters $(v,n) = (3q+3k,3k)$ as follows.

$q=1$,	$(3,0)$	$q=4$,	$(12,0)$	$q=6$,	$(18,0)$
	$(6,3)$		$(15,3)$		$(21,3)$
$q=2$,	$(6,0)$		$(18,6)$		$(24,6)$
	$(9,3)$		$(21,9)$		$(27,9)$
	$(12,6)$		$(24,12)$		$(30,12)$.
$q=3$,	$(9,0)$				$(33,15)$
	$(12,3)$				$(36,18)$
	$(15,6)$				
	$(18,9)$				

The existence of $IGDD_5^3(v,n)$ for these v and n can be obtained directly from the results in Section 3 or by applying Theorem 2.10 with those known results.

5. Existence of $IGDD_5^3(v,n)$ for $n \equiv 1 \pmod 3$.

According to Corollary 1.3, we have $v \geq 2n$ and $v \equiv 1,2 \pmod 3$. First, we consider the case when $v \equiv 1 \pmod 3$, that is, $v - n \equiv 0 \pmod 3$. Let $n = 3k+1$ and $v - n = 3q$. Since $v \geq 2n$, we have $q \geq k+1$.

Theorem 5.1. *Suppose $n \equiv v \equiv 1 \pmod 3$ and $v \geq 2n$. Then there exists an $IGDD_5^3(v,n)$.*

Proof: Apply Theorem 2.3 with appropriate t_i and IGDDs. For $q \in A$ and a $TD(6,q)$, take $t_1 = \ldots = t_k = 3$, $t_{k+1} = 1$, $t_{k+2} = \ldots = t_q$. Since there are $IGDD_5^3(v,n)$ for $(v,n) = (6,3),(4,1),(3,0)$, we obtain an $IGDD_5^3(3q+3k+1, 3k+1)$, that is, $IGDD_5^3(v,n)$.

For $q \in B$, there exists a $TD(6, \frac{1}{2}q)$. Since $\frac{1}{2}q \geq \frac{1}{2}(k+1)$, we have $\frac{1}{2}q \geq k_1 + 1$ if $k = 2k_1$ is even, and $\frac{1}{2}q \geq k_2 + 1$ if $k = 2k_2 + 1$ is odd. Write $3k+1 = k_1 \cdot 6 + 1 \cdot 1$ or $3k+1 = k_2 \cdot 6 + 1 \cdot 4$. Then the conclusion follows from the existence of the input $IGDD_5^3(v,n)$ for $(v,n) = (12,6),(7,1),(6,0),(10,4)$.

For $q \in C$, there exists a $TD(6, \frac{1}{4}q)$. Since $12 \cdot \frac{1}{4}q > 3k+1$ and $IGDD_5^3(12+t_i, t_i)$ exists for any $i \in \{0,1,\ldots,12\}$ and $1 \leq i \leq \frac{1}{4}q$, we could choose suitable t_i such that $t_1 + \ldots + t_{\frac{1}{4}q} = 3k+1$ and get the required IGDD.

To complete the proof we need only check the cases when $q \in \{1,2,3,4,6\}$ and $q \geq k+1$. This leaves the parameters $(v,n) = (3q+3k+1, 3k+1)$ as follows:

$q=1$,	$(4,1)$	$q=4$,	$(13,1)$	$q=6$,	$(19,1)$
$q=2$,	$(7,1)$		$(16,4)$		$(22,4)$
	$(10,4)$		$(19,7)$		$(25,7)$
$q=3$,	$(10,1)$		$(22,10)$		$(28,10)$.
	$(13,4)$				$(31,13)$
	$(16,7)$				$(34,16)$

The existence of those IGDDs can be obtained from the results in Section 3 and Lemma 2.4. ∎

Next, we consider the case when $v \equiv 2 \pmod 3$, that is, $v-n \equiv 1 \pmod 3$. Let $n = 3k+1$ and $v-n = 3q+1$. Since $v \geq 2n$, we have $q \geq k$.

Lemma 5.2. *Suppose $k \leq q < \frac{1}{2}(5k+1)$. Then there exists an $IGDD_5^3(3q+3k+2, 3k+1)$.*

Proof: We use Bose's mixed difference method described in Section 2. We need only list the part (I) and part (II) and show that the $(1,0)$-mixed differences are different, and so do the $(4,1)$-mixed differences. Based on $(Z_{3q+1} \cup \{\infty^i \mid 1 \leq i \leq 3k+1\}) \times Z_5$, the initial blocks for the first two parts are:

$$q{-}k\begin{cases} 0_0\ (3q)_1\, 0_3 \\ 0_0\ (3q-1)_1\, 1_3 \\ \vdots \\ 0_0\ (2q+k+1)_1 (q-k-1)_3 \end{cases} (I);\quad q{-}k\begin{cases} (q+2k+1)_0\ 0_1\ (q-k)_4 \\ (q+2k+3)_0\ 0_1\ (q-k+1)_4 \\ \vdots \\ (3q-1)_0\ 0_1\ (2q-2k-1)_4 \end{cases} (II).$$

First, we consider the $(1,0)$-mixed differences. From the first two columns of part (I) we have differences $2q+k+1,\ 2q+k+2, \ldots, 3q$. Part (II) yields differences $2, 4, \ldots, 2q-2k$ and $3k+1, 3k+2, \ldots, q+2k$. Since $k \leq q < \frac{1}{2}(5k+1)$, we have $2q-2k < 3k+2$ and $q+2k < 2q+k+1$. This guarantees that all these differences are different.

Next, we consider the $(4,1)$-mixed differences. From part (I) we have $0, 1, \ldots, q-k-1$ and $q+2k+2, q+2k+4, \ldots, 3q$. Part (II) yields differences $q-k, q-k+1, \ldots, 2q-2k-1$. Since $q < \frac{1}{2}(5k+1)$, we have $2q-2k-1 < q+2k+2$. Therefore, these differences are also different. This completes the proof. ∎

Lemma 5.3. *Suppose $\frac{1}{2}(5k+1) < q \leq 4k$. Then there exists an $IGDD_5^3(3q+3k+2, 3k+1)$.*

Proof: We construct on $(Z_{3q+1} \cup \{\infty^i \mid 1 \leq i \leq 3k+1\}) \times Z_5$ initial blocks similar to Lemma 5.2. The first two parts are:

$$q{-}k\begin{cases} 0_0\ (2q-2k)_1 (3k+2)_3 \\ 0_0\ (2q-2k+1)_1 (3k+4)_3 \\ \vdots \\ 0_0\ (3q-3k-1)_1 (2q+k)_3 \end{cases} (I);\quad q{-}k\begin{cases} 0_0\ 0_1\ (2q+k+1)_4 \\ 0_0\ 1_1\ (2q+k)_4 \\ \vdots \\ 0_0\ (q-k-1)_1 (q+2k+2)_4 \end{cases} (II).$$

For the $(1,0)$-mixed differences, we have from part (I) the differences $2q-2k$, $2q-2k+1, \ldots, 3q-3k+1$, and from part (II) the differences $0, 1, \ldots, q-k-1$ and $q-k, q-k+1, \ldots, 2q-2k-1$. These differences are obviously different.

For the $(4,1)$-mixed differences, we have from part (I) the differences $3k+2$, $3k+4, \ldots, 2q+k$ and $q-4k-1, q-4k, \ldots, 2q-5k-2$. Part (II) yields the differences $3k+3, 3k+5, \ldots, 2q+k+1$. Here, $q-4k-1 \equiv 4q-4k \pmod{3q+1}$. In other words, we have differences $3k+2, 3k+3, \ldots, 2q+k+1$ and $4q-4k, 4q-4k+1, \ldots, 2q-5k-2$. Since $\frac{1}{2}(5k+1) < q \leq 4k$, we have $2q+k+1 < 4q-4k$ and $2q-5k-2 < 3k+2$. Therefore, these differences are different. And the proof is complete. ∎

Lemma 5.4. *There exists an* $IGDD_5^3(3q+3k+2, 3k+1)$ *for* $q = \frac{1}{2}(5k+1)$.

Proof: The initial blocks are:

$$q-k\begin{cases} 0_0\ 0_1\ (5k+2-2q)_3 \\ 0_0\ 1_1\ (5k+4-2q)_3 \\ \vdots \\ 0_0\ (q-k-1)_1(3k)_3 \end{cases} (I);\quad q-k\begin{cases} 0_0\ (q-k)_1(q+2k+1)_4 \\ 0_0\ (q-k+1)_1(q+2k)_4 \\ \vdots \\ 0_0\ (2q-2k-1)_1(3k+2)_4 \end{cases} (II).$$

The $(1,0)$-mixed differences are:

$$\begin{aligned} &0, 1, \ldots, q-k-1; \\ &q-k, q-k+1, \ldots, 2q-2k-1; \\ &2q-2k, 2q-2k+1, \ldots, 3q-3k-1. \end{aligned}$$

They are obviously different. The $(4,1)$-mixed differences are:

$$\begin{aligned} &5k-2q+2, 5k-2q+4, \ldots, 3k; \\ &q-4k-1, q-4k, \ldots, 2q-5k-2; \\ &5k-2q+3, 5k-2q+5, \ldots, 3k+1. \end{aligned}$$

The first and third lines can be combined as

$$5k-2q+2,\ 5k-2q+3, \ldots, 3k+1.$$

Since $q = \frac{1}{2}(5k+1)$ and $q-4k-1 \equiv 4q-4k \pmod{3q+1}$, we have $3k+1 < 4q-4k$. So, these differences are different and the proof is complete. ∎

Combining Lemmas 5.2, 5.3 and 5.4 gives the following.

Lemma 5.5. *Suppose* $k \leq q \leq 4k$. *Then there exists an* $IGDD_5^3(3q+3k+2, 3k+1)$.

Lemma 5.6. *If* $q \geq 4k+1$, *then there exists an* $IGDD_5^3(3q+3k+2, 3k+1)$.

Proof: In the case when $q-k \in A$, there is a $TD(6, q-k)$. Since $q-k \geq 3k+1$, we can take $t_1 = \ldots = t_{3k+1} = 2$ and $t_{3k+2} = \ldots = t_{q-k} = 0$ in

Theorem 2.3. The input $IGDD_5^3(v,n)$ for $(v,n) = (3,0),(5,2)$ come from Section 3. Then an $IGDD_5^3(3q+3k+2, 6k+2)$ exists. Applying Lemma 2.4 with $IGDD_5^3(6k+2, 3k+1)$ produces the required IGDD.

For $q-k \in B$, there exists a $TD(6, \frac{1}{2}(q-k))$. Since $q \geq 4k+1$, we have $\frac{1}{2}(q-k) \geq k+1$. Taking $t_1 = \ldots = t_k = 6$, $t_{k+1} = 2$, $t_{k+2} = \ldots = t_{q-k} = 0$, we obtain an $IGDD_5^3(3q-3k+6k+2, 6k+2)$ and then the required $IGDD_5^3(3q+3k+2, 3k+1)$.

For $q-k \in C$, there exists a $TD(6, \frac{1}{4}(q-k))$. Since $q \geq 4k+1$, we have $\frac{1}{4}(q-k) \geq \frac{1}{2}(k-1)+1$ if k is odd and $\frac{1}{4}(q-k) \geq \frac{1}{2}k+1$ if k is even. Write $6k+2 = \frac{1}{2}(k-1)\cdot 12 + 1\cdot 8$ and take $t_1 = \ldots = t_{\frac{1}{2}(k-1)} = 12$, $t_{\frac{1}{2}(k+1)} = 8$, $t_{\frac{1}{2}(k-1)+2} = \ldots = t_{\frac{1}{4}(q-k)} = 0$ for the former. Write $6k+2 = \frac{1}{2}k\cdot 12 + 2$ and take the corresponding t_i for the latter. We obtain the required IGDD.

Now, there remains the case $q-k \in \{0,1,2,3,4,6\}$ to be considered. Since $3k+1 \leq q-k \leq 6$, we have $0 \leq k \leq 1$. More specifically, we have $q-k = 4$, 6 if $k = 1$, and $q-k = 1,2,3,4,6$ if $k = 0$. That is, we need consider those $IGDD_5^3(v,n)$ where $(v,n) = (20,4),(26,4),(5,1),(8,1),(11,1),(14,1),(20,1)$. All these can be handled by Theorem 2.10, Lemma 2.4 and the results in Section 3. The proof is now complete. ∎

Combining Theorem 5.1, Lemma 5.5 and Lemma 5.6 we obtain the main result of this section.

Theorem 5.7. *When $n \equiv 1 \pmod 3$, the necessary and sufficient condition for the existence of an $IGDD_5^3(v,n)$ is $v \equiv 1,2 \pmod 3$ and $v \geq 2n$.*

6. Existence of $IGDD_5^3(v,n)$ for $n \equiv 2 \pmod 3$.

By Corollary 1.3 we have in this case that $v \geq 2n$ and $v \equiv 1,2 \pmod 3$. We first consider the case when $v \equiv 2 \pmod 3$, that is, $v - n \equiv 0 \pmod 3$. Let $n = 3k+2$ and $v - n = 3q$. Since $v \geq 2n$, we know that $q \geq k+1$.

Theorem 6.1. *Suppose $v \equiv n \equiv 2 \pmod 3$ and $v \geq 2n$. Then there exists an $IGDD_5^3(v,n)$.*

Proof: Apply Theorem 2.3 again like we did in the proof of Theorem 5.1. For $q \in A$ and a $TD(6,q)$, take $t_1 = \ldots = t_k = 3$, $t_{k+1} = 2$, $t_{k+2} = \ldots = t_q = 0$. For $q \in B$, there exists a $TD(6, \frac{1}{2}q)$. Write $3k+2 = \frac{1}{2}(k-1)\cdot 6 + 1\cdot 5$ if k is odd, and $3k+2 = \frac{1}{2}k\cdot 6 + 1\cdot 2$ if k is even. Using $IGDD_5^3(v,n)$ for $(v,n) = (12,6),(11,5),(6,0)$ and $(8,2)$ we obtain the required IGDD. For $q \in C$, we have a $TD(6, \frac{1}{4}q)$. Since there exists an $IGDD_5^3(12+t_i, t_i)$ where $t_i \in \{0,1,\ldots,12\}$, $i = 1,2,\ldots,\frac{1}{4}q$, and $\frac{1}{4}q \geq (3k+2)/12$, we could choose suitable t_i such that $t_1+\ldots+t_{\frac{1}{4}q} = 3k+2$. Hence, an $IGDD_5^3(3q+3k+2, 3k+2)$ exists.

Now, we consider the remaining cases $q = 1, 2, 3, 4, 6$. We list the parameters $(v, n) = (3q + 3k + 2, 3k + 2)$ as follows:

$q = 1$, $(5,2)$	$q = 4$, $(14,2)$	$q = 6$, $(20,2)$
$q = 2$, $(8,2)$	$(17,5)$	$(23,5)$
$(11,5)$	$(20,8)$	$(26,8)$
$q = 3$, $(11,2)$	$(23,11)$	$(29,11)$
$(14,5)$		$(32,14)$
$(17,8)$		$(35,17)$

All these IGDDs, except two with $(v, n) = (29, 11)$ and $(35, 17)$, exist from Theorem 2.10, Lemma 2.4 and the results in Section 3. Applying Corollary 2.7 with $m = 9$ and $n = 2, 8$, we get an $\mathrm{IGDD}_5^3(29, 11)$ and an $\mathrm{IGDD}_5^3(35, 17)$. Therefore, the proof is complete. ∎

We now turn to consider the case when $v \equiv 1 \pmod 3$, that is, $v - n \equiv 2 \pmod 3$. Let $n = 3k + 2$ and $v - n = 3q + 2$. Since $v \geq 2n$, we have $q \geq k$.

Lemma 6.2. *Suppose $k \leq q \leq 4k$. Then there exists an $IGDD_5^3(3q + 3k + 4, 3k + 2)$.*

Proof: When $q = k$, the required IGDD comes from the $\mathrm{IGDD}_5^3(2, 1)$. When $k < q < \frac{1}{2}(5q) + 2$, we use Bose's mixed difference method and list the first two parts of initial blocks as follows:

$$q{-}k\begin{cases} 0_0 \ (3q+1)_1 0_3 \\ 0_0 \ (3q)_1 1_3 \\ \vdots \\ 0_0 \ (2q+k+2)_1 (q-k-1)_3 \end{cases} (I); \quad q{-}k\begin{cases} (q+2k+3)_0 \ 0_1 \ (q-k)_4 \\ (q+2k+5)_0 \ 0_1 \ (q-k+1)_4 \\ \vdots \\ (3q+1)_0 \ 0_1 \ (2q-2k-1)_4 \end{cases} (II).$$

When $\frac{1}{2}(5k) + 2 \leq q \leq 4k$, we take

$$q{-}k\begin{cases} 0_0 \ (2q-2k)_1 (3k+3)_3 \\ 0_0 \ (2q-2k+1)_1 (3k+5)_3 \\ \vdots \\ 0_0 \ (3q-3k-1)_1 (2q+k+1)_3 \end{cases} (I), \quad q{-}k\begin{cases} 0_0 \ 0_1 \ (2q+k+2)_4 \\ 0_0 \ 1_1 \ (2q+k+1)_4 \\ \vdots \\ 0_0 \ (q-k-1)_1 (q+2k+3)_4 \end{cases} (II).$$

It is a routine matter to verify that the $(1, 0)$-mixed differences in each case are different and so do the $(4, 1)$-mixed differences. Thus the proof is complete. ∎

Lemma 6.3. *Suppose $q \geq 4k + 1$. Then there exists an $IGDD_5^3(3q + 3k + 4, 3k + 2)$.*

Proof: We mainly use Theorem 2.3. In the case when $q - k \in A$, a $\mathrm{TD}(6, q - k)$ exists. Since $3q - 3k \geq 3(3k + 1) \geq 6k + 4$, there exists an $\mathrm{IGDD}_5^3(3q - 3k + 6k + 4, 6k + 4)$, and then an $\mathrm{IGDD}_5^3(3q + 3k + 4, 3k + 2)$.

For $q-k \in B$, there exists a TD$(6, \frac{1}{2}(q-k))$. Since $\frac{1}{2}(q-k)\cdot 6 \geq 6k+4$, there exists an IGDD$_5^3(3q+3k+4, 3k+2)$. If $q-k \in C$, there is a TD$(6, \frac{1}{4}(q-k))$. Write $6k+4 = \frac{1}{2}(k-1)\cdot 12 + 1\cdot 10$ if k is odd, and $6k+4 = \frac{1}{2}k\cdot 12 + 1\cdot 4$ if k is even. Since $\frac{1}{4}(q-k)\cdot 12 \geq 6k+4$, we also have the required IGDD.

If $q-k \in \{1,2,3,4,6\}$, we have $3k+1 \leq q-k \leq 6$, which leads to the parameters $(v,n) = (3q+3k+4, 3k+2)$ as follows,

$$(7,2), (10,2), (13,2), (16,2), (22,2), (22,5), (28,5).$$

All these IGDDs exist from the result in Section 3 and Lemma 2.4 and Theorem 2.10. The proof is complete. ∎

Combining Theorem 6.1, Lemma 6.2 and Lemma 6.3 we have

Theorem 6.4. *When $n \equiv 2 \pmod 3$, the necessary and sufficient condition for the existence of an IGDD$_5^3(v,n)$ is $v \equiv 1,2 \pmod 3$ and $v \geq 2n$.*

7. Existence of IGDD$_u^3(v,n)$.

From Corollary 1.3 we know the necessary condition for the existence of IGDD$_u^3$ (v,n). In this section we shall prove that this necessary condition is also sufficient.

Theorem 7.1. *For $u \equiv 1,3 \pmod 6$, the necessary and sufficient condition for the existence of IGDD$_u^3(v,n)$ is $v \geq 2n$.*

Proof: Here $u \in B(3)$. The conclusion follows from Theorem 3.3, Theorem 2.2 and Corollary 1.3. ∎

Theorem 7.2. *For $u \equiv 5 \pmod 6$, the necessary and sufficient condition for the existence of IGDD$_u^3(v,n)$ is $v \geq 2n$ and*

$$\begin{cases} v \equiv 0 \pmod 3 \\ n \equiv 0 \pmod 3, \end{cases} \quad or \quad \begin{cases} v \equiv 1,2 \pmod 3 \\ n \equiv 1,2 \pmod 3. \end{cases}$$

Proof: By Lemma 3.5, we have $u \in B(5^*, 3)$. From Theorem 4.3, Theorem 5.7 and Theorem 6.4 there exists an IGDD$_5^3(v,n)$ for the given parameters v and n. Therefore, an IGDD$_u^3(v,n)$ exists from Theorem 2.2 and Theorem 3.3. This proves the sufficiency. The conclusion then follows from Corollary 1.3. ∎

Theorem 7.3. *For $u \equiv 0,4 \pmod 6$, the necessary and sufficient condition for the existence of IGDD$_u^3(v,n)$ is $v \geq 2n$ and $v \equiv n \equiv 0 \pmod 2$.*

Proof: Since $v \equiv n \equiv 0 \pmod 2$ and $v \geq 2n$, there exists an IGDD$_3^3(\frac{1}{2}v, \frac{1}{2}n)$ from Theorem 3.3. Since $u \equiv 0,4 \pmod 6$, $2u+1 \equiv 1,3 \pmod 6$, and then $2u+1 \in B(3)$. Deleting one point from a $(2u+1,3,1)$-BIBD, we obtain a GDD$[3,1,2;2u]$. By Theorem 2.1 anc Corollary 1.3 we have the result. ∎

Theorem 7.4. *For $u \equiv 2 \pmod 6$, the necessary and sufficient condition for the existence of $IGDD^3_u(v,n)$ is $v \geq 2n$, and*

$$\begin{cases} v \equiv 0 \pmod 6 \\ n \equiv 0 \pmod 6), \end{cases} \quad or \quad \begin{cases} v \equiv 2,4 \pmod 6 \\ n \equiv 2,4 \pmod 6). \end{cases}$$

Proof: Here, $2u+1 \equiv 5 \pmod 6$. So, we have from Lemma 3.5 that $2u+1 \in B(5^*,3)$. Delete one point from a $(2u+1,\{5^*,3\},1)$-PBD, not belonging to the block of size 5, we obtain a GDD$[\{5^*,3\},1,2;2u]$. Since an IGDD$^3_5(\frac{1}{2}v,\frac{1}{2}n)$ exists from Theorem 7.2, and an IGDD$^3_3(\frac{1}{2}v,\frac{1}{2}n)$ exists from Theorem 3.3, we then obtain the sufficiency from Theorem 2.2. The conclusion follows from Corollary 1.3. ∎

Combining the above theorems of this section we have the main theorem of this paper.

Theorem 7.5. *The necessary condition for the existence of an $IGDD^3_u(v,n)$, shown in Corollary 1.3, is also sufficient.*

References

1. T. Beth, D. Jungnickel, H. Lenz, "Design Theory", Bibliographisches Institut, Zürich, 1985.
2. A.E. Brouwer, *The number of mutually orthogonal Latin squares — a table up to order 10,000*, Math. Centrum Report **ZW123** (June, 1979), Amsterdam.
3. A.E. Brouwer, A. Schrijver, H. Hanani, *Group divisible designs with block-size four*, Discrete Math. **20** (1977), 1-10.
4. H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. **11** (1975), 255-369.
5. K. Heinrich, L. Zhu, *Existence of orthogonal Latin squares with aligned subsquares*, Discrete Math. **59** (1986), 69-78.
6. J.D. Horton, *Sub-latin squares and incomplete orthogonal arrays* J.Combinatorial Theory (A) **16** (1974), 23-33.
7. R. Roth, M. Peters, *Four pairwise orthogonal Latin squares of order 24*, J. Combinatorial Theory (A) **44** (1987), 152-155.
8. H.J. Ryser, *Combinatorial Mathematics*, Carus Math. Monographs **14** (1963).
9. D.R. Stinson, *A general construction for group divisible designs*, Discrete Math. **33** (1981), 89-94.
10. W.D. Wallis, L. Zhu, *Orthogonal Latin squares with small subsquares*, Lecture Notes in Math., 1036, Combinatorial Mathematics X (Adelaide, 1982), 398-409, Springer, Berlin-New York (1983).
11. R.M. Wilson, *Some partition of all triples into Steiner triple systems*, Hypergraph seminar, Lecture Notes in Math. **411** (1974), 267-277, Springer-Verlag.

12. R.M. Wilson, *Constructions and uses of pairwise balanced designs*, Math. Centre Tracts **55** (1974), 18-41.
13. L. Zhu, *Orthogonal Latin squares with subsquares*, Discrete Math. **48** (1984), 315-321.
14. L. Zhu, *Some results on orthogonal Latin squares with orthogonal subsquares*, Utilitas Math. **25** (1984), 241-248.

Appendix

$\mathrm{IGDD}_5^3(5,1)$, $X = (Z_4 \cup \{\infty\}) \times Z_5$,

$\infty_0\ 0_1\ 2_4 \qquad 0_0\ 1_1\ 2_4$

$\infty_0\ 0_2\ 0_3 \qquad 0_0\ 3_1\ 0_3 \qquad (\mathrm{mod}\ 4, \mathrm{mod}\ 5)$

$\mathrm{IGDD}_5^3(5,2)$, $X = (Z_3 \cup \{\infty^1, \infty^2\}) \times Z_5$,

$\infty_0^1\ 0_1\ 2_4 \qquad \infty_0^2\ 0_1\ 0_4 \qquad 0_0\ 0_1\ 1_3$

$\infty_0^1\ 1_1\ 1_4 \qquad \infty_0^2\ 1_1\ 2_4 \qquad 1_0\ 2_1\ 2_4$

$\infty_0^1\ 2_1\ 0_4 \qquad \infty_0^2\ 2_1\ 1_4$

$\infty_0^1\ 0_2\ 1_3 \qquad \infty_0^2\ 0_2\ 2_3$

$\infty_0^1\ 1_2\ 0_3 \qquad \infty_0^2\ 1_2\ 1_3$

$\infty_0^1\ 2_2\ 2_3 \qquad \infty_0^2\ 2_2\ 0_3 \qquad (\text{———}, \mathrm{mod}\ 5)$

$\mathrm{IGDD}_5^3(7,2)$, $X = (Z_5 \cup \{\infty^1, \infty^2\}) \times Z_5$,

$\infty_0^1\ 0_1\ 0_4 \qquad \infty_0^2\ 0_1\ 2_4 \qquad 0_0\ 0_1\ 1_3$

$\infty_0^1\ 0_2\ 2_3 \qquad \infty_0^2\ 0_2\ 1_3 \qquad 0_0\ 3_1\ 1_4$

$(\mathrm{mod}\ 5, \mathrm{mod}\ 5)$

$\mathrm{IGDD}_5^3(9,3)$, $X = (Z_6 \cup \{\infty^1, \infty^2, \infty^3\}) \times Z_5$,

$\infty_0^1\ 0_1\ 0_4 \qquad \infty_0^3\ 0_1\ 2_4$

$\infty_0^1\ 0_2\ 0_3 \qquad \infty_0^3\ 0_2\ 5_3$

$\infty_0^2\ 0_1\ 1_4 \qquad 0_0\ 1_1\ 3_3$

$\infty_0^2\ 0_2\ 2_3 \qquad 2_0\ 0_1\ 5_4 \qquad (\mathrm{mod}\ 6, \mathrm{mod}\ 5)$

$\mathrm{IGDD}_5^3(21,9)$, $X = (Z_{12} \cup \{\infty^1, \ldots, \infty^9\}) \times Z_5$,

$\infty_0^1\ 0_1\ 0_4$	$\infty_0^4\ 0_1\ 3_4$	$\infty_0^7\ 0_1\ 6_4$
$\infty_0^1\ 0_2\ 2_3$	$\infty_0^4\ 0_2\ 5_3$	$\infty_0^7\ 0_2\ 9_3$
$\infty_0^2\ 0_1\ 1_4$	$\infty_0^5\ 0_1\ 4_4$	$\infty_0^8\ 0_1\ 7_4$
$\infty_0^2\ 0_2\ 3_3$	$\infty_0^5\ 0_2\ 6_3$	$\infty_0^8\ 2_2\ 10_3$
$\infty_0^3\ 0_1\ 2_4$	$\infty_0^6\ 0_1\ 5_4$	$\infty_0^9\ 0_1\ 8_4$
$\infty_0^3\ 0_2\ 4_3$	$\infty_0^6\ 0_2\ 8_3$	$\infty_0^9\ 0_2\ 11_3$
$0_0\ 7_1\ 9_3$	$0_0\ 0_1\ 11_4$	(mod 12, mod 5)

$\mathrm{IGDD}_5^3(11,4)$, $X = (Z_7 \cup \{\infty^1, \ldots, \infty^4\}) \times Z_5$,

$\infty_0^1\ 0_1\ 0_4$	$\infty_0^3\ 0_1\ 2_4$	$0_0\ 4_1\ 5_3$
$\infty_0^1\ 0_2\ 0_3$	$\infty_0^3\ 0_2\ 2_3$	$0_0\ 5_1\ 1_4$
$\infty_0^2\ 0_1\ 1_4$	$\infty_0^4\ 0_1\ 4_4$	
$\infty_0^2\ 0_2\ 1_3$	$\infty_0^4\ 0_2\ 3_3$	(mod 7, mod 5)

$\mathrm{IGDD}_5^3(16,7)$, $X = (Z_9 \cup \{\infty^1, \ldots, \infty^7\}) \times Z_5$,

$\infty_0^1\ 0_1\ 2_4$	$\infty_0^2\ 0_1\ 3_4$	$\infty_0^3\ 0_1\ 4_4$	$\infty_0^4\ 0_1\ 5_4$	$\infty_0^5\ 0_1\ 6_4$
$\infty_0^6\ 0_1\ 7_4$	$\infty_0^2\ 0_2\ 2_3$	$\infty_0^3\ 0_2\ 3_3$	$\infty_0^4\ 0_2\ 4_3$	$\infty_0^5\ 0_2\ 5_3$
$\infty_0^6\ 0_2\ 6_3$	$\infty_0^7\ 0_2\ 7_3$		(mod 9, mod 5)	
$\infty_0^7\ 0_1\ 8_4$	$\infty_0^7\ 1_1\ 1_4$	$\infty_0^7\ 2_1\ 3_4$	$\infty_0^7\ 3_1\ 2_4$	$\infty_0^7\ 4_1\ 4_4$
$\infty_0^7\ 5_1\ 6_4$	$\infty_0^7\ 6_1\ 5_4$	$\infty_0^7\ 7_1\ 7_4$	$\infty_0^7\ 8_1\ 0_4$	
$\infty_0^1\ 0_2\ 1_3$	$\infty_0^1\ 1_2\ 0_3$	$\infty_0^1\ 2_2\ 2_3$	$\infty_0^1\ 3_2\ 4_3$	$\infty_0^1\ 4_2\ 3_3$
$\infty_0^1\ 5_2\ 5_3$	$\infty_0^1\ 6_2\ 7_3$	$\infty_0^1\ 7_2\ 6_3$	$\infty_0^1\ 8_2\ 8_3$	
$0_0\ 0_1\ 1_3$	$1_0\ 2_1\ 2_4$	$1_0\ 1_1\ 2_3$	$2_0\ 3_1\ 3_4$	$3_0\ 3_1\ 4_3$
$4_0\ 5_1\ 5_4$	$4_0\ 4_1\ 5_3$	$5_0\ 6_1\ 6_4$	$6_0\ 6_1\ 7_3$	$7_0\ 8_1\ 8_4$
$7_0\ 7_1\ 8_3$	$8_0\ 0_1\ 0_4$		(———, mod 5)	

IGDD$^3_5$(17,8), $X = (Z_9 \cup \{\infty^1, \dots, \infty^8\}) \times Z_5$,

$\infty_0^1\ 0_1\ 0_4$	$\infty_0^1\ 1_1\ 1_4$	$\infty_0^1\ 2_1\ 8_4$	$\infty_0^1\ 3_1\ 3_4$	$\infty_0^1\ 4_1\ 4_4$
$\infty_0^1\ 5_1\ 2_4$	$\infty_0^1\ 6_1\ 6_4$	$\infty_0^1\ 7_1\ 7_4$	$\infty_0^1\ 8_1\ 5_4$	
$\infty_0^2\ 0_1\ 2_4$	$\infty_0^2\ 1_1\ 3_4$	$\infty_0^2\ 2_1\ 1_4$	$\infty_0^2\ 3_1\ 5_4$	$\infty_0^2\ 4_1\ 6_4$
$\infty_0^2\ 5_1\ 4_4$	$\infty_0^2\ 6_1\ 8_4$	$\infty_0^2\ 7_1\ 0_4$	$\infty_0^2\ 8_1\ 7_4$	
$\infty_0^3\ 0_1\ 3_4$	$\infty_0^3\ 1_1\ 2_4$	$\infty_0^3\ 2_1\ 4_4$	$\infty_0^3\ 3_1\ 6_4$	$\infty_0^3\ 4_1\ 5_4$
$\infty_0^3\ 5_1\ 7_4$	$\infty_0^3\ 6_1\ 0_4$	$\infty_0^3\ 7_1\ 8_4$	$\infty_0^3\ 8_1\ 1_4$	
$\infty_0^4\ 0_1\ 4_4$	$\infty_0^4\ 1_1\ 5_4$	$\infty_0^4\ 2_1\ 3_4$	$\infty_0^4\ 3_1\ 7_4$	$\infty_0^4\ 4_1\ 8_4$
$\infty_0^4\ 5_1\ 6_4$	$\infty_0^4\ 6_1\ 1_4$	$\infty_0^4\ 7_1\ 2_4$	$\infty_0^4\ 8_1\ 0_4$	
$\infty_0^5\ 0_1\ 5_4$	$\infty_0^5\ 1_1\ 4_4$	$\infty_0^5\ 2_1\ 6_4$	$\infty_0^5\ 3_1\ 8_4$	$\infty_0^5\ 4_1\ 7_4$
$\infty_0^5\ 5_1\ 0_4$	$\infty_0^5\ 6_1\ 2_4$	$\infty_0^5\ 7_1\ 1_4$	$\infty_0^5\ 8_1\ 3_4$	
$\infty_0^6\ 0_1\ 6_4$	$\infty_0^6\ 1_1\ 7_4$	$\infty_0^6\ 2_1\ 5_4$	$\infty_0^6\ 3_1\ 0_4$	$\infty_0^6\ 4_1\ 1_4$
$\infty_0^6\ 5_1\ 8_4$	$\infty_0^6\ 6_1\ 3_4$	$\infty_0^6\ 7_1\ 4_4$	$\infty_0^6\ 8_1\ 2_4$	
$\infty_0^7\ 0_1\ 7_4$	$\infty_0^7\ 1_1\ 8_4$	$\infty_0^7\ 2_1\ 0_4$	$\infty_0^7\ 3_1\ 1_4$	$\infty_0^7\ 4_1\ 2_4$
$\infty_0^7\ 5_1\ 3_4$	$\infty_0^7\ 6_1\ 4_4$	$\infty_0^7\ 7_1\ 5_4$	$\infty_0^7\ 8_1\ 6_4$	
$\infty_0^8\ 0_1\ 8_4$	$\infty_0^8\ 1_1\ 6_4$	$\infty_0^8\ 2_1\ 7_4$	$\infty_0^8\ 3_1\ 2_4$	$\infty_0^8\ 4_1\ 0_4$
$\infty_0^8\ 5_1\ 1_4$	$\infty_0^8\ 6_1\ 5_4$	$\infty_0^8\ 7_1\ 3_4$	$\infty_0^8\ 8_1\ 4_4$	
$\infty_0^1\ 0_2\ 1_3$	$\infty_0^1\ 1_2\ 0_3$	$\infty_0^1\ 2_2\ 8_3$	$\infty_0^1\ 3_2\ 4_3$	$\infty_0^1\ 4_2\ 3_3$
$\infty_0^1\ 5_2\ 2_3$	$\infty_0^1\ 6_2\ 7_3$	$\infty_0^1\ 7_2\ 6_3$	$\infty_0^1\ 8_2\ 5_3$	
$\infty_0^2\ 0_2\ 2_3$	$\infty_0^2\ 1_2\ 1_3$	$\infty_0^2\ 2_2\ 3_3$	$\infty_0^2\ 3_2\ 5_3$	$\infty_0^2\ 4_2\ 4_3$
$\infty_0^2\ 5_2\ 6_3$	$\infty_0^2\ 6_2\ 8_3$	$\infty_0^2\ 7_2\ 7_3$	$\infty_0^2\ 8_2\ 0_3$	
$\infty_0^3\ 0_2\ 3_3$	$\infty_0^3\ 1_2\ 4_3$	$\infty_0^3\ 2_2\ 2_3$	$\infty_0^3\ 3_2\ 6_3$	$\infty_0^3\ 4_2\ 7_3$
$\infty_0^3\ 5_2\ 5_3$	$\infty_0^3\ 6_2\ 0_3$	$\infty_0^3\ 7_2\ 1_3$	$\infty_0^3\ 8_2\ 8_3$	
$\infty_0^4\ 0_2\ 4_3$	$\infty_0^4\ 1_2\ 3_3$	$\infty_0^4\ 2_2\ 5_3$	$\infty_0^4\ 3_2\ 7_3$	$\infty_0^4\ 4_2\ 6_3$
$\infty_0^4\ 5_2\ 8_3$	$\infty_0^4\ 6_2\ 1_3$	$\infty_0^4\ 7_2\ 0_3$	$\infty_0^4\ 8_2\ 2_3$	
$\infty_0^5\ 0_2\ 5_3$	$\infty_0^5\ 1_2\ 6_3$	$\infty_0^5\ 2_2\ 4_3$	$\infty_0^5\ 3_2\ 8_3$	$\infty_0^5\ 4_2\ 0_3$
$\infty_0^5\ 5_2\ 7_3$	$\infty_0^5\ 6_2\ 2_3$	$\infty_0^5\ 7_2\ 3_3$	$\infty_0^5\ 8_2\ 1_3$	
$\infty_0^6\ 0_2\ 6_3$	$\infty_0^6\ 1_2\ 5_3$	$\infty_0^6\ 2_2\ 7_3$	$\infty_0^6\ 3_2\ 0_3$	$\infty_0^6\ 4_2\ 8_3$
$\infty_0^6\ 5_2\ 1_3$	$\infty_0^6\ 6_2\ 3_3$	$\infty_0^6\ 7_2\ 2_3$	$\infty_0^6\ 8_2\ 4_3$	
$\infty_0^7\ 0_2\ 7_3$	$\infty_0^7\ 1_2\ 8_3$	$\infty_0^7\ 2_2\ 0_3$	$\infty_0^7\ 3_2\ 1_3$	$\infty_0^7\ 4_2\ 2_3$
$\infty_0^7\ 5_2\ 3_3$	$\infty_0^7\ 6_2\ 4_3$	$\infty_0^7\ 7_2\ 5_3$	$\infty_0^7\ 8_2\ 6_3$	
$\infty_0^8\ 0_2\ 8_3$	$\infty_0^8\ 1_2\ 7_3$	$\infty_0^8\ 2_2\ 6_3$	$\infty_0^8\ 3_2\ 2_3$	$\infty_0^8\ 4_2\ 1_3$
$\infty_0^8\ 5_2\ 0_3$	$\infty_0^8\ 6_2\ 5_3$	$\infty_0^8\ 7_2\ 4_3$	$\infty_0^8\ 8_2\ 3_3$	
$0_0\ 0_1\ 1_3$	$1_0\ 2_1\ 2_4$	$3_0\ 3_1\ 4_3$	$4_0\ 5_1\ 5_4$	$6_0\ 6_1\ 7_3$
$7_0\ 8_1\ 8_4$		(———, mod 5)		