Finite Unitary Geometry and PBIB Designs (I)

Benfu Yang
Department of Mathematics
Teacher-training College of Chengdu, Chengdu
and
Wandi Wei
Department of Mathematics
Sichuan University
Chengdu, CHINA

Abstract. The conjugation relation among the subspaces of a finite unitary geometry and its properties are studied. Then they are used to find some enumeration formulas for the subspaces of the unitary geometry, to prove a type of transitivity of the unitary group, to construct PBIB designs, and to establish the isomorphism between some known PBIB designs.

1. Introduction.

Let q be a prime power, F_q the finite field with q elements, and F_{q^2} the finite field with q^2 elements which has F_q as a subfield. It is well known that the mapping

$$a \longrightarrow \overline{a} = a^q$$

is an automorphism of order 2 of F_{q^2}, and fixes the subfield F_q. The set of matrices T of order n over F_{q^2} such that $T\overline{T}' = I^{(n)}$, where $\overline{T}'$ is obtained from T by replacing its (i,j) entry t_{ij} by $\overline{t}_{ji}$ $(1 \leq i,j \leq n)$, is a group with matrix multiplication as the composition. This group is called the unitary group of degree n over F_{q^2} and denoted by $U_n(F_{q^2})$. Let $V_n(F_{q^2})$ denote the n-dimensional vector space over F_{q^2}. The unitary group $U_n(F_{q^2})$ can be viewed as a transformation group of $V_n(F_{q^2})\cdot V_n(F_{q^2})$ with the unitary group $U_n(F_{q^2})$ as its transformation group is called the n-dimensional unitary space or unitary geometry over F_{q^2}, and also denoted by $V_n(F_{q^2})$. Throughout the present paper, we will conduct our discussion in the n-dimensional unitary geometry $V_n(F_{q^2})$.

Two vectors α and β in $V_n(F_{q^2})$ are said to be orthogonal if $\alpha\overline{\beta}' = 0$. Let P be an m-dimensional subspace of $V_n(F_{q^2})$. We use P^* to denote the set of such vectors that are orthogonal to all the vectors in P. Obviously, P^* is an $(n-m)$-dimensional subspace, and is called the conjugate subspace of P.

Let Q be an $m \times n$ matrix of rank m over F_{q^2}. We will use the same symbol Q to denote the subspace that is spanned by the rows of Q. Q is said to be an m-dimensional subspace with index r if the rank of $Q\overline{Q}'$ is r, and is simply called an (m,r)-type subspace.

Wan *et al* [4] computed the number $N(m,r;n)$ of the (m,r)-type subspaces of $V_n(F_{q^2})$, and the number $N(m_1,r_1;m,r;n)$ of (m_1,r_1)-type subspaces that are included in a given (m,r)-type subspace. For convenience of reference, we restate these results as follows:

Theorem 1.

$$N(m,r;n)=\begin{cases}\frac{\prod_{i=n+r-2m+1}^{n}(q^i-(-1)^i)}{\prod_{i=1}^{r}(q^i-(-1)^i)\prod_{i=1}^{m-r}(q^{2i}-1)}\,q^{r(n+r-2m)}, & \text{if } r\le m \text{ and } n+r\ge 2m\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

Theorem 2. *Suppose that* $r \le m$ *and* $n + r \ge 2m$. *Then*

$$N(m_1,r_1;m,r;n)$$
$$=\sum_{k=\max\left(0,\left[\frac{-r-r_1+2m_1+1}{2}\right]\right)}^{\min(m-r,m_1-r_1)} \frac{\prod_{i=r+r_1-2m_1+2k+1}^{r}(q^i-(-1)^i)\prod_{i=m-r-k+1}^{m-r}(q^{2i}-1)}{\prod_{i=1}^{r_1}(q^i-(-1)^i)\prod_{i=1}^{m_1-k-r_1}(q^{2i}-1)\prod_{i=1}^{k}(q^{2i}-1)}$$
$$\cdot\, q^{r_1(r+r_1-2m_1+2k)+2(m_1-k)(m-r-k)}. \tag{2}$$

In the above formulas (and in the following) we adopt the usual convention

$$\prod_{i\in\phi} N_i = 1,\quad \sum_{i\in\phi} N_i = 0.$$

Wan [2], Yang [7], Shen [1] and Wei [5, 6] have studied the transitivity of the unitary group and some enumeration formulas, and then constructed a number of association schemes and a number of PBIB designs. In the present paper, we will study some properties of the conjugation relation in the unitary space $V_n(F_{q^2})$, and then obtain some relations among the enumeration formulas for some subspaces as well as a type of new transitivity of the unitary group $U_n(F_{q^2})$. Furthermore, we will point out that some known association schemes as well as some known PBIB designs are isomorphic to each other by giving the isomorphism mappings, and also give a new kind of PBIB designs.

The concepts and notations used but not defined in this paper are all adopted from [4].

2. Some properties of conjugation relation.

For two subspaces P and Q of the n-dimensional unitary space $V_n(F_{q^2})$, we use $P \cap Q$ to denote the intersection of P and Q, and $P \cup Q$ the subspace spanned by P and Q.

Theorem 3. *Let P and Q be two subspaces in the unitary space $V_n(F_{q^2})$, and $T \in U_n(F_{q^2})$. Then*

(i) $(P^*)^* = P$;
(ii) *If* $P \subseteq Q$, *then* $Q^* \subseteq P^*$;
(iii) $(PT)^* = P^*T$;
(iv) $(P \cup Q)^* = P^* \cap Q^*$;
(v) $(P \cap Q)^* = P^* \cup Q^*$.

Proof: From the definition of conjugation, it follows that $P \subseteq (P^\star)^\star$. Noting that $\dim P = \dim (P^\star)^\star$, we have (i).

(ii) and (iv) are two immediate consequences of the definition of conjugation.

Replacing P and Q by $P^\star$ and $Q^\star$, respectively, in (iv), we have (v) by (i).

Now we prove (iii). Since

$$(PT)(\overline{P^\star T})' = PT\overline{T}'(\overline{P^\star})' = P(\overline{P^\star})' = 0,$$

we have $P^\star T \subseteq (PT)^\star$. On the other hand, $\dim P^\star T = \dim(PT)^\star = n - \dim P$, so $P^\star T = (PT)^\star$. This completes the proof. ∎

Theorem 4. *Let $r \leq m$ and $n + r \geq 2m$. Then the conjugate subspace of an (m,r)-type subspace is of $(n-m,\ n+r-2m)$-type. And the conjugation mapping between the set $\mathcal{M}(m,r;n)$ consisting of all the (m,r)-type subspaces and the set $\mathcal{M}(n-m, n+r-2m; n)$ consisting of all the $(n-m, n+r-2m)$-type subspaces is a one-one mapping.*

Proof: Let

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & \lambda I & 0 \end{pmatrix} \begin{matrix} r \\ m-r \end{matrix}$$
$$\quad\ \ r \quad m-r \quad m-r \quad n+r-2m$$

$$P_1 = \begin{pmatrix} 0 & I & \lambda I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{matrix} m-r \\ n+r-2m \end{matrix}$$
$$\quad\ \ r \quad m-r \quad m-r \quad n+r-2m$$

where λ is an element of F_{q^2} such that $1 + \lambda\overline{\lambda} = 0$. For its existence, see Lemma 1 in Chapter 2 of [4]. Clearly, P is an (m,r)-type subspace, P_1 an $(n-m, n+r-2m)$-type subspace, and $P\overline{P_1}' = 0$, that is, $P_1 \subseteq P^\star$. On the other hand, $\dim P_1 = n - m = \dim P^\star$. Hence, $P_1 = P^\star$.

Let Q be an (m,r)-type subspace. Since the unitary group acts transitively on the set of the subspaces of the same type, there exists $T \in U_n(F_{q^2})$ such that $Q = PT$. By (iii) of Theorem 3, we have $Q^\star = P^\star T$. Hence $Q^\star$ and $P^\star$ are both of type $(n-m, n+r-2m)$, and so the former assertion of the theorem is true.

According to this assertion, the conjugate subspace of an $(n-m, n+r-2m)$-type subspace R is of (m,r)-type. And by (i) of Theorem 3, the conjugate subspace of this (m,r)-type subspace is the $(n-m, n+r-2m)$-type subspace R itself. So any $(n-m, n+r-2m)$-type subspace must be the conjugate subspace of some (m,r)-type subspace. Clearly, the conjugate subspaces of two distinct (m,r)-type subspaces are also distinct. Hence, the latter assertion of the theorem is also true. This proves the theorem. ∎

3. Some applications of conjugation relation.

In this section, we will apply the conjugation relation among the subspaces of the unitary space $V_n(F_{q^2})$ to the enumeration formulas for subspaces, to the study of a type of transitivity of the unitary group, and to the construction of a class of new PBIB designs. The application to establishing the isomorphism between some known PBIB designs will appear in the next section.

For the enumeration formulas for subspaces, we have

Theorem 5. *Let* $r \leq m, n + r - 2m \geq 0$, *and*

$$\max\left(0, \frac{-r - r_1 + 2m_1}{2}\right) \leq \min(m - r, m_1 - r_1),$$

then

(i) $N(m, r; n) = N(n - m, n + r - 2m; n)$;

(ii) $N^T(m_1, r_1; m, r; n) = N(n - m, n + r - 2m; n - m_1, n + r_1 - 2m_1; n)$; *where* $N^T(m_1, r_1; m, r; n)$ *denotes the number of the* $(m, r; n)$*-type subspaces which contain a given* $(m_1, r_1; n)$*-type subspace.*

(iii) $N(m, r; n) N(m_1, r_1; m, r; n) = N(m_1, r_1; n) N(n - m, n + r - 2m; n - m_1, n + r_1 - 2m_1; n)$.

Proof: The conclusion (i) is an immediate consequence of Theorem 4. By (i) and (ii) of Theorem 3, an (m_1, r_1)-type subspace P is included in an (m, r)-type subspace Q if and only if the $(n - m_1, n + r_1 - 2m_1)$-type subspace P^*, the conjugate subspace of P, contains the $(n - m, n + r - 2m)$-type subspace Q^*, the conjugate of Q. From this, we have (ii).

To prove (iii), we compute the number of the pairs (P, Q) of subspaces P and Q satisfying

$$P \in \mathcal{M}(m_1, r_1; n), \quad Q \in \mathcal{M}(m, r; n), \quad P \subseteq Q.$$

Considering first P and then Q, we know this number to be

$$N(m_1, r_1; n) \quad N^T(m_1, r_1; m, r; n).$$

On the other hand, considering first Q and then P, we know this number to be

$$N(m, r; n) N(m_1, r_1; m, r; n).$$

Therefore,

$$N(m_1, r_1; n) N^T(m_1, r_1; m, r; n) = N(m, r; n) N(m_1, r_1; m, r; n).$$

Substituting (ii) into it, we have (iii). This completes the proof. ∎

We can obtain some relations among the subspaces of higher dimensions from the ones among the subspaces of lower dimensions. For example, we have

Theorem 6. *Let $n \geq 4$. Then the intersection of two $(n-1, n-2)$-type subspaces in the unitary space $V_n(F_{q^2})$ must be either an $(n-2, n-4)$-type subspace or an $(n-2, n-2)$-type subspace, and must not be an $(n-2, n-3)$-type subspace.*

Proof: Let P and Q be two $(1,0)$-type subspaces. Then $P \cup Q$ is a $(2,0)$-type subspace when P and Q are orthogonal, and a $(2,2)$-type subspace when P and Q are not orthogonal. We know that the conjugate subspace of a $(1,0)$-type subspace is of $(n-1, n-2)$-type, the conjugate subspace of a $(2,0)$-type subspace is of $(n-2, n-4)$-type, and the conjugate subspace of a $(2,2)$-type subspace is of $(n-2, n-2)$-type. Therefore, by (iv) of Theorem 3 and Theorem 4, we have the conclusion of the theorem. ∎

Theorem 7. *Let $n \geq 4$. Then the unitary group $U_n(F_{q^2})$ acts transitively on the set of the $(n-1, n-2)$-type subspace pairs the two subspaces of each of which intersect in an $(n-2, n-4)$-type subspace, as well as transitively on the set of the $(n-1, n-2)$-type subspace pairs the two subspaces of each of which intersect in an $(n-2, n-2)$-type subspace.*

Proof: We know that the unitary group $U_n(F_{q^2})$ acts transitively on the set of the $(1,0)$-type subspace pairs the two subspaces of each of which span a $(2,0)$-type subspace, as well as transitively on the set of the $(1,0)$-type subspace pairs the two subspaces of each of which span a $(2,2)$-type subspace (see, for example, Chapters 7 and 8 of [4]). Therefore, by (iii) and (iv) of Theorem 3, and by Theorem 4, the conclusion of the theorem is obtained. ∎

Wan, Dai, Feng and Yang constructed in [4] the following association scheme:

Theorem 8. *Let $n \geq 4$. Taking the $(1,0)$-type subspaces of the unitary space $V_n(F_{q^2})$ as the treatments, and defining two treatments to be the first (resp. second) associates of each other if they as subspaces are orthogonal (resp. non-orthogonal), one obtains an association scheme with two associate classes. Its parameters are*

$$
\begin{aligned}
v &= \frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q^2-1} \\
n_1 &= \frac{(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})}{q^2-1}\, q^2 \\
p_{11}^1 &= (q^2-1) + \frac{(q^{n-4}-(-1)^{n-4})(q^{n-5}-(-1)^{n-5})}{q^2-1}\, q^4 \\
p_{11}^2 &= \frac{(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})}{q^2-1}.
\end{aligned}
\tag{3}
$$

By Theorems 6, 7 and 8 and by the conjugation relation among the subspaces, we have the following association scheme immediately:

Theorem 9. *Let $n \geq 4$. Taking the $(n-1, n-2)$-type subspaces of the unitary space $V_n(F_{q^2})$ as the treatments, and defining two treatments to be the first (resp. second) associates of each other if they as subspaces intersect in an $(n-2, n-4)$-type (resp. $(n-2, n-2)$-type) subspace, we obtain an association scheme with two associate classes and with the same parameters as in (3).*

We can use the association scheme in Theorem 9 to construct a new PBIB design when $n = 4$.

Theorem 10. *The $(3,2)$-type subspaces of the unitary space $V_4(F_{q^2})$ are taken as the treatments, and the $(2,1)$-type subspaces are taken as the blocks. Two treatments are defined to be the first (resp. second) associates of each other if they as subspaces intersect in a $(2,0)$-type (resp. $(2,2)$-type) subspace. A treatment is said to be arranged in a block if the $(3,2)$-type subspace which has been taken as the treatment intersects the $(2,1)$-type subspace which has been taken as the block in a $(1,0)$-type subspace. Then we obtain a PBIB design with two associate classes and with the parameters*

$$\begin{aligned} v &= (q^3+1)(q^2+1), & n_1 &= q^2(q+1), \\ p_{11}^1 &= q^2-1, & p_{11}^2 &= q+1, \\ b &= (q^3+1)(q^2+1)(q-1)q, & & \\ k &= (q+1)q^2, & r &= (q^2-1)q^3, \\ \lambda_1 &= (q^2-1)(q-1)q, & \lambda_2 &= (q^2-1)q. \end{aligned}$$

Proof: By the transitivity of the unitary group, we know that we certainly obtain a PBIB design. Its parameters can be computed as follows:

Setting $n = 4$ in (3), we have

$$\begin{aligned} v &= (q^3+1)(q^2+1), & n_1 &= q^2(q+1), \\ p_{11}^1 &= q^2-1, & p_{11}^2 &= q+1. \end{aligned}$$

By Theorem 1,

$$b = N(2,1;4) = (q^3+1)(q^2+1)(q-1)q.$$

Since k is the number of the $(3,2)$-type subspaces which intersect a given $(2,1)$-type subspace in the $(1,0)$-type subspaces, we have

$$\begin{aligned} k &= N(1,0;2,1;4)\left[\frac{N(3,2;4)N(1,0;3,2;4)}{N(1,0;4)} - \frac{N(3,2;4)N(2,1;3,2;4)}{N(2,1;4)}\right] \\ &= 1\cdot\left[(q+1)q^2 + 1 - \frac{(q^3+1)(q^2+1)(q-1)q}{(q^3+1)(q^2+1)(q-1)q}\right] = (q+1)q^2 \end{aligned}$$

and then

$$r = \frac{bk}{v} = (q^2 - 1)q^3.$$

We now compute λ_1. Let P and Q be two distinct $(3,2)$-type subspaces, and $D = P \cap Q$ a $(2,0)$-type subspace. then λ_1 is the number of the $(2,1)$-type subspaces (written as R) which intersect P as well as Q in the $(1,0)$-type subspaces. We assert that $R \cap P = R \cap Q$. Otherwise, $R \cap P$ and $R \cap Q$ are two distinct 1-dimensional subspaces, so they span a 2-dimensional subspace which is included in R. Then the subspace $(R \cap P) \cup (R \cap Q)$ must be R. But this contradicts the fact that two distinct $(1,0)$-type subspaces only span a $(2,0)$-type subspace or a $(2,2)$-type subspace. Hence, $R \cap P = R \cap Q$ and $R \cap P = R \cap Q \subseteq D$. On the other hand, the number of the $(2,1)$-type subspaces which include a fixed $(1,0)$-type subspace that is included in the subspace D is

$$\frac{N(2,1;4)N(1,0;2,1;4)}{N(1,0;4)} = (q-1)q.$$

Furthermore, two $(2,1)$-type subspaces which include, respectively, different $(1,0)$-type subspaces of D are distinct, so the number of the $(2,1)$-type subspaces, each intersecting D in a $(1,0)$-type subspace, is

$$N(1,0;2,0;4)(q-1)q = (q^2+1)(q-1)q.$$

Note that, a $(2,1)$-type subspace in P or in Q intersects D in a $(1,0)$-type subspace. So each of these $(2,1)$-type subspaces has been enumerated once. Therefore, the number of the $(2,1)$-type subspaces which intersect P as well as Q in the $(1,0)$-type subspaces, that is λ_1, is

$$\begin{aligned}\lambda_1 &= (q^2+1)(q-1)q - 2N(2,1;3,2;4)\\ &= (q^2+1)(q-1)q - 2(q-1)q = (q^2-1)(q-1)q.\end{aligned}$$

Finally, we compute λ_2. Let P and Q be two distinct $(3,2)$-type subspaces, $D = P \cap Q$ a $(2,2)$-type subspace, and R a $(2,1)$-type subspace which intersects P in a $(1,0)$-type subspace and Q in a subspace of the same type. In a similar way of computing λ_1, we have $R \cap P = R \cap Q \subseteq D$. Clearly, the number of the $(2,1)$-type subspaces which intersect D in the $(1,0)$-type subspaces is

$$N(1,0;2,2;4)(q-1)q = (q^2-1)q.$$

We now prove that it is λ_2. For this, we only need to show that any $(2,1)$-type subspace in P or in Q does not intersect D in a $(1,0)$-type subspace. Suppose the contrary is true. Let R be a $(2,1)$-type subspace in P, and $R_1 = D \cap R$ a $(1,0)$-type subspace. Write $D = \binom{R_1}{D_1}$ and $R = \binom{R_1}{R_2}$. Then

$$P = \begin{pmatrix} R_1 \\ D_1 \\ R_2 \end{pmatrix}.$$

Without loss of generality, we may suppose

$$\begin{pmatrix} R_1 \\ D_1 \end{pmatrix} \begin{pmatrix} \overline{R_1} \\ D_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & \star \end{pmatrix}, \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \begin{pmatrix} \overline{R_1} \\ R_2 \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$P\overline{P}' = \begin{pmatrix} R_1 \\ D_1 \\ R_2 \end{pmatrix} \begin{pmatrix} \overline{R_1} \\ D_1 \\ R_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \star & \star \\ 0 & \star & 1 \end{pmatrix}.$$

Hence, P is a $(3,3)$-type subspace, a contradiction with the fact that P is a $(3,2)$-type subspace. This proves the theorem. ∎

It is worth pointing out that although one can obtain a PBIB design isomorphic to the one in Theorem 10 by using Theorem 8 for $n = 4$, the computation of the parameters of the PBIB so obtained is not convenient.

4. Isomorphism between some known PBIB designs.

In the final section we will point out that some known PBIB designs are isomorphic to each other.

In [6], Wei gave the following association scheme, that will be denoted as Association Scheme B.

The 2-dimensional non-isotropic subspaces in the unitary space $V_3(F_{q^2})$ are taken as the treatments. Two treatments V_1 and V_2 are defined to be the ith ($1 \le i \le q-1$) associates of each other, denoted by $(V_1, V_2)_B = i$, if they as subspaces intersect in a non-isotropic vector D and if

$$\begin{pmatrix} D \\ P_1 \\ P_2 \end{pmatrix} \begin{pmatrix} \overline{D} \\ P_1 \\ P_2 \end{pmatrix}' = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & i = 1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & x_i \end{pmatrix}, & 2 \le i \le q-1, \end{cases}$$

where $x_2, x_3, \ldots$, and x_{q-1} are the $q-2$ elements of the set $F_q \backslash \{0,1\}$, and P_1 and P_2 are both orthogonal to D and satisfy

$$\begin{pmatrix} D \\ P_1 \end{pmatrix} = V_1, \quad \begin{pmatrix} D \\ P_2 \end{pmatrix} = V_2.$$

And two treatments V_1 and V_2 are defined to be the qth associates of each other, denoted by $(V_1, V_2)_B = q$, if they as subspaces intersect in an isotropic vector.

This association scheme is in fact isomorphic to one which can be obtained from a general association scheme given also by Wei in an earlier paper [5]. Setting $n = 3$ in [5], one obtains the following association scheme that will be denoted by Association Scheme A.

The $(1, 1)$-type subspaces in the unitary space $V_3(F_{q^2})$ are taken as the treatments. Two treatments V_1 and V_2 are defined to be the first associates of each other if V_1 and V_2 are orthogonal, that is,

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} \overline{V_1} \\ V_2 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

V_1 and V_2 are defined to be the ith $(2 \leq i \leq q-1)$ associates of each other if

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} \overline{V_1} \\ V_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & x_i, \end{pmatrix},$$

where $\{x_2, \ldots, x_{q-1}\} = F_q \backslash \{0, 1\}$; and V_1 and V_2 are defined to be the qth associates of each other if

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} \overline{V_1} \\ V_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The symbol $(V_1, V_2)_A = i$ denotes that V_1 and V_2 are the ith associates of each other in this association scheme.

Theorem 11. *The Association Schemes A and B described above are isomorphic to each other.*

Proof: Clearly, the subspaces taken as the treatments in Association Scheme A and the subspaces taken as the treatments in Association Scheme B are conjugate. Now we prove that the conjugation relation leads to an isomorphism between the two association schemes. It suffices to prove that for any two treatments in Association Scheme A, we have

$$(V_1, V_2)_A = i \text{ if and only if } (V_1^*, V_2^*)_B = i, \quad 1 \leq i \leq q.$$

It is easily seen that we only need to prove that for $1 \leq i \leq q$ we have

$$\text{if } (V_1, V_2)_A = i, \quad \text{then } (V_1^*, V_2^*)_B = i.$$

There are three cases to be examined.

(i) Case 1: $i = 1$. Take $V_1 = (1,0,0)$ and $V_2 = (0,1,0)$. Then $(V_1, V_2)_A = 1$ and

$$V_1^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_2^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write $D = V_1^* \cap V_2^* = (0,0,1)$. Then $V_1^* = \binom{D}{e_2}$, $V_2^* = \binom{D}{e_1}$, and

$$\begin{pmatrix} D \\ e_2 \\ e_1 \end{pmatrix} \begin{pmatrix} \overline{D} \\ e_2 \\ e_1 \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, D is non-isotropic. Hence, $(V_1^*, V_2^*)_B = 1$.

Let W_1 and W_2 be two treatments in Association Scheme A, and $(W_1, W_2)_A = 1$. Since the unitary group acts transitively on the set of the $(1,1)$-type subspace pairs the two subspaces of each of which are first associates, there exists $T \in U_3(F_{q^2})$ such that $W_1 = V_1T$ and $W_2 = V_2T$. Then

$$W_1^* = V_1^*T, \qquad W_2^* = V_2^*T,$$

that is,

$$W_1^* = \begin{pmatrix} DT \\ e_2T \end{pmatrix}, \quad W_2^* = \begin{pmatrix} DT \\ e_1T \end{pmatrix}.$$

Therefore, $W_1^* \cap W_2^* = DT$ is non-isotropic, and

$$\begin{pmatrix} DT \\ e_2T \\ e_1T \end{pmatrix} \begin{pmatrix} \overline{DT} \\ e_2T \\ e_1T \end{pmatrix}' = \begin{pmatrix} D \\ e_2 \\ e_1 \end{pmatrix} T\overline{T}' \begin{pmatrix} \overline{D} \\ e_2 \\ e_1 \end{pmatrix}' = \begin{pmatrix} D \\ e_2 \\ e_1 \end{pmatrix} \begin{pmatrix} \overline{D} \\ e_2 \\ e_1 \end{pmatrix}' = I. \tag{4}$$

Hence, $(W_1^*, W_2^*) = 1$.

(ii) Case 2: $2 \le i \le q-1$.

Take $V_1 = (1,0,0)$ and $V_2 = (1,x,0)$, where $x \in F_{q^2}$ such that $1 + x\overline{x} = x_i$. Clearly,

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \begin{pmatrix} \overline{V_1} \\ V_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix},$$

that is, $(V_1, V_2)_A = i$. On the other hand,

$$V_1^* = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_2^* = \begin{pmatrix} 0 & 0 & 1 \\ -\overline{x} & 1 & 0 \end{pmatrix}.$$

Therefore, $V_1^* \cap V_2^* = e_3$ is non-isotropic, and

$$\begin{pmatrix} e_3 \\ e_2 \\ \alpha \end{pmatrix} \begin{pmatrix} \overline{e_3} \\ e_2 \\ \alpha \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & i \end{pmatrix},$$

where $\alpha = (-\overline{x}, 1, 0)$. Hence, $(V_1^*, V_2^*)_B = i$.

Let W_1 and W_2 be two treatments in Association Scheme A, and $(W_1, W_2)_A = i$. Then there must be $T \in U_3(F_{q^2})$ such that $W_1 = V_1T$, $W_2 = V_2T$. By an argument similar to that used in Case 1, we have $(W_1^*, W_2^*)_B = i$.

(iii) Case 3: $i = q$.

Take $V_1 = e_1$ and $V_2 = (1, 1, \lambda)$, where $\lambda \in F_{q^2}$ such that $1 + \lambda\overline{\lambda} = 0$. Clearly, $(V_1, V_2)_A = q$. On the other hand,

$$V_1^* = \begin{pmatrix} 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}, \qquad V_2^* = \begin{pmatrix} 0 & 1 & \lambda \\ 1 & 0 & \lambda \end{pmatrix}.$$

Therefore, $V_1^* \cap V_2^* = (0, 1, \lambda)$ is isotropic. Hence, $(V_1^*, V_2^*)_B = q$.

Let W_1 and W_2 be two treatments in Association Scheme A, and $(W_1, W_2)_A = q$. By an argument similar to that used in Case 1, we have $(W_1^*, W_2^*)_B = q$. This completes the proof. ∎

The proof of the following two theorems is simpler than that in Theorem 11, so omitted.

Theorem 12. *The first PBIB design given in Theorem 5 of Chapter 8 of [4] is isomorphic to the first design given in Theorem 11 of the same chapter.*

Theorem 13. *The PBIB design with $n = 2\nu$ given in Theorem 14 of Chapter 8 of [4] is isomorphic to the design with $n = 2\nu$ and m and r replaced by $n - m$ and $n - 2m$, respectively, given in Theorem 13 of the same chapter.*

We incidentally point out that some similar work in finite symplectic and orthogonal geometries has been done. It is given in other two separate papers.

The authors would like to thank the referee for his helpful comments.

References

1. Hao Shen, *An enumeration theorem on unitary geometry over finite field and construction of PBIB designs*, Acta Math. Sinica **28** (1985), 747–755.
2. Zhexian Wan, *Notes on finite geometries and the construction of PBIB designs based on finite geometries*, Acta Math. Scientia **14**; **12** (1965), 1872–1876.
3. Zhexian Wan and Benfu Yang, *Notes on finite geometries and construction of PBIB designs (III) some enumeration theorems in finite unitary geometries and their applications*, Acta Math. Sinica **15** (1965), 533–544.
4. Zhexian Wan, Zongduo Dai, Xuning Feng and Benfu Yang, "Some studies on finite geometries and incomplete block desings", Science Press, Beijing, 1966.
5. Hongzeng Wei, *Using finite unitary geometry to construct a class of PBIB designs*, Chin. Ann. of Math. **4b**(3) (1983), 299–306.
6. Hongzeng Wei, *Using two-dimensional non-isotropic subspaces of finite unitary geometry to construct PBIB designs*, Chinese Ann. of Math. **8A**(3) (1987), 319–331.
7. Benfu Yang, *Note on finite geometries and the construction of PBIB designs (VIII) construction of some PBIB designs with a number of association schemes by using maximal totally isotropic subspaces in unitary geometries over finite fields*, Acta Math. Sinica **15** (1965), 826–841.