Partial-Fraction Decompositions and Harmonic Number Identities

Wenchang Chu

Dipartimento di Matematica, Università degli Studi di Lecce
Lecce-Arnesano: P. O. Box 193, 73100 Lecce, ITALY
Email address: *chu.wenchang@unile.it*

Abstract: *By means of partial fraction method, we investigate the decomposition of rational functions. Several striking identities on harmonic numbers and generalized Apéry numbers will be established, including the binomial-harmonic number identity associated with Beukers' conjecture on Apéry numbers.*

1. Introduction

The generalized harmonic numbers are defined to be partial sums of the Riemann-Zeta series:

$$H_0^{\langle m\rangle} = 0 \quad \text{and} \quad H_n^{\langle m\rangle} = \sum_{k=1}^{n} \frac{1}{k^m} \quad \text{for} \quad m,\, n = 1, 2, \cdots. \tag{1}$$

When $m = 1$, they reduce to the classical ones, shortened as $H_n = H_n^{\langle 1\rangle}$.

If the shifted factorial is defined by

$$(c)_0 \equiv 1 \quad \text{and} \quad (c)_n = c(c+1)\cdots(c+n-1) \quad \text{for} \quad n = 1, 2, \cdots \tag{2}$$

then we can establish, by means of the standard partial-fraction decompositions, the following algebraic identities:

$$\frac{n!}{(x)_{n+1}} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{x+k} \tag{3}$$

$$\frac{(n!)^2}{(x)_{n+1}^2} = \sum_{k=0}^{n} \binom{n}{k}^2 \left\{ \frac{1}{(x+k)^2} + \frac{2}{x+k}(H_k - H_{n-k}) \right\} \tag{4}$$

$$\frac{(n!)^3}{(x)_{n+1}^3} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \left\{ \frac{1}{(x+k)^3} + \frac{3}{(x+k)^2}(H_k - H_{n-k}) \right. \tag{5a}$$

$$\left. + \frac{3}{2(x+k)} \Big[3(H_k - H_{n-k})^2 + \big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) \Big] \right\}. \tag{5b}$$

This work has been carried out during my visit to Center for Combinatorics (LPMC), Nankai University (2005).

Multiplying (5a-5b) across by x and then letting $x \to \infty$, we recover one identity among the hardest challenges claimed in [4, Eq 16], [5, Eq 12] and [10, Eq 20]:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big\{3(H_k - H_{n-k})^2 + \big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big)\Big\} = 0. \tag{6}$$

This has best exemplified the power of partial fraction method. For more general rational functions, we will investigate their partial fraction decompositions in the second section, which involve the complete Bell polynomials (or cyclic indicators of symmetric groups) on the generalized harmonic numbers. Several further examples and miscellaneous formulae will be collected in the third and last section. In order to facilitate consultation for readers, three short tables of the complete Bell polynomials on the generalized harmonic numbers will be presented in the appendices.

2. Partial Fraction Decompositions

For two natural numbers n and k with $0 \le k \le n$, define two functions related to harmonic numbers by

$$H_\ell(x) := \sum_{\iota=1}^{n}\frac{1}{(\iota - x)^\ell} \quad \Longrightarrow \quad H_\ell(-k) = H_{n+k}^{\langle\ell\rangle} - H_k^{\langle\ell\rangle} \tag{7a}$$

$$\mathcal{H}_\ell(x) := \sum_{\substack{\iota=0\\ \iota\neq k}}^{n}\frac{1}{(\iota + x)^\ell} \quad \Longrightarrow \quad \mathcal{H}_\ell(-k) = H_{n-k}^{\langle\ell\rangle} + (-1)^\ell H_k^{\langle\ell\rangle}. \tag{7b}$$

They come respectively from the logarithmic derivatives of the binomial coefficients

$$h(x) = \frac{(1-x)_n}{n!} = \binom{n-x}{n} \tag{8a}$$

$$\hbar(x) = \frac{n!\times(x+k)}{(x)_{n+1}} = \frac{\binom{n}{k}}{\binom{x+k-1}{k}\binom{x+n}{n-k}}. \tag{8b}$$

Let $\sigma(\ell)$ be the set of partitions $\{1^{m_1}, 2^{m_2}, \cdots, \ell^{m_\ell}\}$ of ℓ represented by ℓ-tuples of nonnegative integers $(m_1, m_2, \cdots, m_\ell)$ such that $\sum_{k=1}^{\ell} km_k = \ell$. Its subset of ℓ-partitions into m parts with $\sum_{k=1}^{\ell} m_k = m$ is denoted by $\sigma_m(\ell)$.

Theorem (Partial fraction decomposition). *Let λ, μ and n be three natural numbers with $\lambda + (\lambda - \mu)n > 0$. Then there holds the algebraic identity:*

$$\frac{(n!)^{\lambda-\mu}(1-x)_n^{\mu}}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n}(-1)^{k\lambda}\binom{n}{k}^{\lambda}\binom{n+k}{k}^{\mu}\sum_{\ell=0}^{\lambda-1}\frac{\Omega_\ell(\lambda,\mu,-k)}{\ell!\,(x+k)^{\lambda-\ell}} \tag{9}$$

with the Ω-coefficients being determined by the Bell polynomials (or the cyclic indicators of symmetric groups)*:*

$$\Omega_\ell(\lambda,\mu,x) = (-1)^\ell \ell! \sum_{\sigma(\ell)}\prod_{i=1}^{\ell}\frac{\left\{\lambda\mathcal{H}_i(x) - (-1)^i\mu H_i(x)\right\}^{m_i}}{m_i!\, i^{m_i}} \tag{10}$$

where the multiple sum runs over $\sigma(\ell)$, the set of ℓ-partitions represented by ℓ-tuples of nonnegative integers $(m_1, m_2, \cdots, m_\ell)$ such that $\sum_{k=1}^{\ell} km_k = \ell$.

In particular, the Ω-coefficients read explicitly as

$$\Omega_\ell(\lambda,\mu,-k) = \ell! \sum_{\sigma(\ell)}\prod_{i=1}^{\ell}\frac{\left\{\lambda\left[H_k^{(i)}+(-1)^i H_{n-k}^{(i)}\right]+\mu\left[H_k^{(i)}-H_{n+k}^{(i)}\right]\right\}^{m_i}}{m_i!\, i^{m_i}}. \tag{11}$$

Proof. By means of partial fraction decomposition, we can formally write

$$\frac{\hbar^\lambda(x)h^\mu(x)}{(x+k)^\lambda} = \frac{(n!)^{\lambda-\mu}(1-x)_n^{\mu}}{(x)_{n+1}^{\lambda}} = \sum_{k=0}^{n}\sum_{\ell=0}^{\lambda-1}\frac{C(k,\ell)}{(x+k)^{\lambda-\ell}}$$

where the coefficients $C(k,\ell)$ are to be determined. Letting $\mathcal{D}_x = \frac{d}{dx}$ stand for the derivative operator with respect to x and then noting that

$$\hbar(-k) = (-1)^k\binom{n}{k}$$

$$h(-k) = \binom{n+k}{k}$$

we first demonstrate that for $0 \le \ell < \lambda$ there holds:

$$C(k,\ell) = \hbar^\lambda(-k)h^\mu(-k) \times \frac{\Omega_\ell(\lambda,\mu,-k)}{\ell!} \tag{12}$$

where the Ω-coefficients are given by the following logarithmic derivatives:

$$\Omega_\ell(\lambda,\mu,x) = \frac{\mathcal{D}_x^\ell\left\{\hbar^\lambda(x)h^\mu(x)\right\}}{\hbar^\lambda(x)h^\mu(x)}. \tag{13}$$

For $\ell = 0$, we have obviously $\Omega_0(\lambda,\mu,x) \equiv 1$ and that

$$\begin{aligned} C(k,0) &= \lim_{x\to -k} \hbar^{\lambda}(x)h^{\mu}(x) \times \Omega_0(\lambda,\mu,x) \\ &= \hbar^{\lambda}(-k)h^{\mu}(-k) \times \Omega_0(\lambda,\mu,-k). \end{aligned}$$

Next for $\ell = 1$, we can check (12-13) through L'Hôpital's rule that

$$\begin{aligned} C(k,1) &= \lim_{x\to -k} (x+k)^{\lambda-1}\Big\{\frac{\hbar^{\lambda}(x)h^{\mu}(x)}{(x+k)^{\lambda}} - \frac{C(k,0)}{(x+k)^{\lambda}}\Big\} \\ &= \lim_{x\to -k} \frac{\hbar^{\lambda}(x)h^{\mu}(x) - C(k,0)}{x+k} \\ &= \lim_{x\to -k} \mathcal{D}_x\Big\{\hbar^{\lambda}(x)h^{\mu}(x)\Big\} \\ &= \hbar^{\lambda}(-k)h^{\mu}(-k) \times \Omega_1(\lambda,\mu,-k). \end{aligned}$$

Supposing now the truth of (12-13) for $\ell = 0, 1, \cdots, m-1$ with $m < \lambda$, then we have to verify it also for $\ell = m$. Applying again the L'Hôpital rule m-times, we can determine the coefficient

$$\begin{aligned} C(k,m) &= \lim_{x\to -k} (x+k)^{\lambda-m}\Big\{\frac{\hbar^{\lambda}(x)h^{\mu}(x)}{(x+k)^{\lambda}} - \sum_{\ell=0}^{m-1} \frac{C(k,\ell)}{(x+k)^{\lambda-\ell}}\Big\} \\ &= \lim_{x\to -k} \frac{1}{(x+k)^m}\Big\{\hbar^{\lambda}(x)h^{\mu}(x) - \sum_{\ell=0}^{m-1} C(k,\ell) \times (x+k)^{\ell}\Big\} \\ &= \lim_{x\to -k} \hbar^{\lambda}(x)h^{\mu}(x)\frac{\mathcal{D}_x^m\Big\{\hbar^{\lambda}(x)h^{\mu}(x)\Big\}}{m!\, f^{\lambda}(x)} \\ &= \hbar^{\lambda}(-k)h^{\mu}(-k) \times \frac{\Omega_m(\lambda,\mu,-k)}{m!}. \end{aligned}$$

Based on the induction principle, we have confirmed that the coefficients in partial fraction decomposition are determined by (12-13).

To complete the proof of the theorem, it remains to show that these coefficients can be calculated explicitly through equation (10) and therefore (11) (Bell polynomials and/or the cyclic indicators of symmetric groups).

Manipulating the differential operation

$$\frac{\mathcal{D}_x^{1+\ell}\Big\{\hbar^{\lambda}(x)h^{\mu}(x)\Big\}}{\hbar^{\lambda}(x)h^{\mu}(x)} = \frac{\mathcal{D}_x\Big\{\hbar^{\lambda}(x)h^{\mu}(x)\Big\}}{\hbar^{\lambda}(x)h^{\mu}(x)}\frac{\mathcal{D}_x^{\ell}\Big\{\hbar^{\lambda}(x)h^{\mu}(x)\Big\}}{\hbar^{\lambda}(x)h^{\mu}(x)}$$

we can derive for (13) the recurrence relation

$$\Omega_{1+\ell}(\lambda,\mu,x) \;=\; \Big\{\mathcal{D}_x - \lambda\mathcal{H}_1(x) - \mu H_1(x)\Big\}\,\Omega_\ell(\lambda,\mu,x). \qquad (14)$$

It is trivial to see that $\Omega_\ell(\lambda,\mu,x)$ defined by (10) admits the initial condition $\Omega_0(\lambda,\mu,x)\equiv 1$. If we can check that $\Omega_\ell(\lambda,\mu,x)$ defined by (10) satisfies the same recurrence relation (14), then the validity of (10) would be confirmed for all the natural numbers ℓ.

Now substituting the RHS of (10) into the RHS of (14) and then noticing the differential relations

$$\begin{aligned}\mathcal{D}_x H_j(x) &= +jH_{j+1}(x)\\ \mathcal{D}_x \mathcal{H}_j(x) &= -j\mathcal{H}_{j+1}(x)\end{aligned}$$

we get the following expression

$$(-1)^{1+\ell}\ell!\Bigg\{\Big[\lambda\mathcal{H}_1(x)+\mu H_1(x)\Big]\sum_{\sigma(\ell)}\prod_{i=1}^{\ell}\frac{\{\lambda\mathcal{H}_i(x)-(-1)^i\mu H_i(x)\}^{m_i}}{m_i!\,i^{m_i}} \qquad (15a)$$

$$+\sum_{\sigma(\ell)}\prod_{i=1}^{\ell}\frac{\{\lambda\mathcal{H}_i(x)-(-1)^i\mu H_i(x)\}^{m_i}}{m_i!\,i^{m_i}}\sum_{j=1}^{\ell} jm_j\,\frac{\lambda\mathcal{H}_{1+j}(x)+(-1)^j\mu H_{j+1}(x)}{\lambda\mathcal{H}_j(x)-(-1)^j\mu H_j(x)}\Bigg\}. \qquad (15b)$$

In accordance with the combinatorial structure, each ℓ-partition enumerated by $\sigma(\ell)$ becomes a $(1+\ell)$-partition with a "j"-part being shifted to a "$1+j$"-part for $0\le j\le\ell$. Vice versa, every $(1+\ell)$-partition enumerated by $\sigma(1+\ell)$ reduces to a ℓ-partition with a "$1+j$"-part being replaced by a "j"-part for $0\le j\le\ell$. Then the sum over partitions should be reformulated accordingly.

First, the line (15a) with an extra part "1" yields a new factor $M_1 := m_1+1$. Then if $m_\ell = 0$, for each j corresponding to the shift from part "j" to part "$1+j$" displayed in line (15b), the coefficient jm_j is replaced by $(1+j)M_{j+1}$ under two index substitution $M_j := m_j - 1$ and $M_{1+j} := m_{1+j}+1$ for $1\le j<\ell$. Lastly if $m_\ell = 1$, the coefficient ℓm_ℓ will be replaced by $(1+\ell)M_{\ell+1}$ with two summation index being substituted by $M_\ell := m_\ell - 1$ and $M_{1+\ell} := m_\ell$. Summing up, we may combine (15a) with (15b) and obtain the following expression

$$(-1)^{1+\ell}\ell!\sum_{\sigma(1+\ell)}\prod_{i=1}^{1+\ell}\frac{\Big\{\lambda\mathcal{H}_i(x)-(-1)^i\mu H_i(x)\Big\}^{M_i}}{M_i!\,i^{M_i}}\sum_{j=1}^{1+\ell} jM_j.$$

According to (10), the last expression becomes $\Omega_{1+\ell}(\lambda, \mu, x)$, i.e., the left member of (14) thanks for the $(1+\ell)$-partition $1+\ell = \sum_{j=1}^{1+\ell} jM_j$. This confirms that the RHS of (10) satisfies indeed the recurrence relation (14). We therefore have established equality (10) and (11). This completes the proof of the theorem. □

In the theorem, multiplying the partial fraction decomposition by x and then letting $x \to \infty$, we derive the following harmonic number identity.

Corollary (Harmonic number identity). *Let λ, μ and n be three natural numbers with $\lambda + (\lambda - \mu)n > 1$. Then there holds the algebraic identity:*

$$\sum_{k=0}^{n} (-1)^{k\lambda} \binom{n}{k}^{\lambda} \binom{n+k}{k}^{\mu} \Omega_{\lambda-1}(\lambda, \mu, -k) = 0 \tag{16}$$

where the Ω-coefficients are given by the Bell polynomials (10).

Defining further two sequences by

$$\varpi_\ell(\lambda, x) = \Omega_\ell(\lambda, 0, x) = \frac{\mathcal{D}_x^\ell \hbar^\lambda(x)}{\hbar^\lambda(x)} = \ell! \sum_{\sigma(\ell)} (-1)^\ell \lambda^m \prod_{i=1}^{\ell} \frac{\mathcal{H}_i^{m_i}(x)}{m_i!\, i^{m_i}} \tag{17a}$$

$$\omega_\ell(\mu, x) = \Omega_\ell(0, \mu, x) = \frac{\mathcal{D}_x^\ell h^\mu(x)}{h^\mu(x)} = \ell! \sum_{\sigma(\ell)} (-1)^m \mu^m \prod_{i=1}^{\ell} \frac{H_i^{m_i}(x)}{m_i!\, i^{m_i}} \tag{17b}$$

and then applying the Leibniz rule to (13), we find the following convolution formula:

$$\Omega_\ell(\lambda, \mu, x) = \sum_{\iota=0}^{\ell} \binom{\ell}{\iota} \varpi_\iota(\lambda, x) \omega_{\ell-\iota}(\mu, x). \tag{18}$$

Putting $x = -k$, we write down the corresponding relation as follows:

$$\Omega_\ell(\lambda, \mu, -k) = \sum_{\iota=0}^{\ell} \binom{\ell}{\iota} \varpi_\iota(\lambda, -k) \omega_{\ell-\iota}(\mu, -k) \tag{19}$$

where ϖ and ω are explicitly provided by the following formulae:

$$\varpi_\ell(\lambda, -k) = \Omega_\ell(\lambda, 0, -k) = \ell! \sum_{\sigma(\ell)} \lambda^m \prod_{i=1}^{\ell} \frac{\left\{H_k^{\langle i \rangle} + (-1)^i H_{n-k}^{\langle i \rangle}\right\}^{m_i}}{m_i!\, i^{m_i}} \tag{20a}$$

$$\omega_\ell(\mu, -k) = \Omega_\ell(0, \mu, -k) = \ell! \sum_{\sigma(\ell)} \mu^m \prod_{i=1}^{\ell} \frac{\left\{H_k^{\langle i \rangle} - H_{n+k}^{\langle i \rangle}\right\}^{m_i}}{m_i!\, i^{m_i}}. \tag{20b}$$

3. Examples: Harmonic Number Identities

By means of the theorem and the corollary, we will display several examples of partial fraction decompositions and the corresponding harmonic number identities.

Example 1 ($\lambda = 1$). *For $\mu = 0,\ 1$, there hold partial fraction expansions*

$$(n!)^{1-\mu}\frac{(1-x)_n^{\mu}}{(x)_{n+1}} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}^{\mu}\frac{(-1)^k}{x+k}$$

and the two corresponding harmonic number identities:

$$(-1)^n\mu = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}^{\mu}.$$

When $\mu = 0$, the corresponding partial fraction expansion reads as the formula displayed in (3). For an alternative derivation, refer to the recent paper [7]. When $\mu = 1$, the last binomial identity is a special case of the Chu-Vandermonde convolution formula on binomial coefficients:

$$\binom{u+v}{n} = \sum_{k=0}^{n}\binom{u}{k}\binom{v}{n-k}.$$

Example 2 ($\lambda = 2$). *For $\mu = 0, 1, 2$, there hold partial fraction expansions*

$$(n!)^{2-\mu}\frac{(1-x)_n^{\mu}}{(x)_{n+1}^2} = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^{\mu}\Big\{\frac{1}{(x+k)^2} + \frac{1}{x+k}\Big[(2+\mu)H_k - 2H_{n-k} - \mu H_{n+k}\Big]\Big\}$$

and the three corresponding harmonic number identities:

$$0 = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^{\mu}\Big\{(2+\mu)H_k - 2H_{n-k} - \mu H_{n+k}\Big\}.$$

For $\mu = 0$, the corresponding partial fraction expansion has been given by (4).

Example 3 ($\lambda = 3$). *For $\mu = 0, 1, 2, 3$, there hold partial fraction expansions*

$$(n!)^{3-\mu}\frac{(1-x)_n^{\mu}}{(x)_{n+1}^{3} } = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\binom{n+k}{k}^{\mu}\left\{\frac{1}{(x+k)^3} + \frac{1}{(x+k)^2}\Big[(3+\mu)H_k - 3H_{n-k} - \mu H_{n+k}\Big] + \frac{1/2}{x+k}\left[\begin{array}{l}\{(3+\mu)H_k - 3H_{n-k} - \mu H_{n+k}\}^2\\ +\{(3+\mu)H_k^{\langle 2\rangle} + 3H_{n-k}^{\langle 2\rangle} - \mu H_{n+k}^{\langle 2\rangle}\}\end{array}\right]\right\}$$

and the four corresponding harmonic number identities:

$$0 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\binom{n+k}{k}^{\mu}\left\{\begin{array}{l}[(3+\mu)H_k - 3H_{n-k} - \mu H_{n+k}]^2\\ + [(3+\mu)H_k^{\langle 2\rangle} + 3H_{n-k}^{\langle 2\rangle} - \mu H_{n+k}^{\langle 2\rangle}]\end{array}\right\}.$$

When $\mu = 0$, the corresponding partial fraction decomposition and harmonic number identity have been exhibited respectively in (5a-5b) and (6).

Example 4 ($\lambda = 4$). *For $\mu = 0, 1, 2, 3, 4$, there hold partial fraction expansions*

$$(n!)^{4-\mu}\frac{(1-x)_n^{\mu}}{(x)_{n+1}^{4}} = \sum_{k=0}^{n}\binom{n}{k}^4\binom{n+k}{k}^{\mu}\left\{\frac{1}{(x+k)^4} + \frac{(4+\mu)H_k - 4H_{n-k} - \mu H_{n+k}}{(x+k)^3} + \frac{1/2}{(x+k)^2}\Big[\Big\{(4+\mu)H_k - 4H_{n-k} - \mu H_{n+k}\Big\}^2 + \Big\{(4+\mu)H_k^{\langle 2\rangle} + 4H_{n-k}^{\langle 2\rangle} - \mu H_{n+k}^{\langle 2\rangle}\Big\}\Big] + \frac{1/6}{x+k}\left[\begin{array}{l}\Big\{(4+\mu)H_k - 4H_{n-k} - \mu H_{n+k}\Big\}^3 + 2\Big\{(4+\mu)H_k^{\langle 3\rangle} - 4H_{n-k}^{\langle 3\rangle} - \mu H_{n+k}^{\langle 3\rangle}\Big\}\\ +3\Big\{(4+\mu)H_k - 4H_{n-k} - \mu H_{n+k}\Big\}\times\Big\{(4+\mu)H_k^{\langle 2\rangle} + 4H_{n-k}^{\langle 2\rangle} - \mu H_{n+k}^{\langle 2\rangle}\Big\}\end{array}\right]\right\}$$

and the five corresponding harmonic number identities:

$$0 = \sum_{k=0}^{n}\binom{n}{k}^4\binom{n+k}{k}^{\mu} \times \left\{\begin{array}{l}\Big[(4+\mu)H_k - 4H_{n-k} - \mu H_{n+k}\Big]^3 + 2\Big[(4+\mu)H_k^{\langle 3\rangle} - 4H_{n-k}^{\langle 3\rangle} - \mu H_{n+k}^{\langle 3\rangle}\Big]\\ +3\Big[(4+\mu)H_k - 4H_{n-k} - \mu H_{n+k}\Big]\times\Big[(4+\mu)H_k^{\langle 2\rangle} + 4H_{n-k}^{\langle 2\rangle} - \mu H_{n+k}^{\langle 2\rangle}\Big]\end{array}\right\}.$$

Example 5 ($\lambda = 5$). *For $\mu = 0,1,2,3,4,5$, there hold partial fraction expansions*

$$(n!)^{5-\mu}\frac{(1-x)_n^\mu}{(x)_{n+1}^5} = \sum_{k=0}^n (-1)^k\binom{n}{k}^5\binom{n+k}{k}^\mu\Bigg\{\frac{1}{(x+k)^5} + \frac{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}}{(x+k)^4}$$
$$+\frac{1/2}{(x+k)^3}\Big[\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}^2+\Big\{(5+\mu)H_k^{\langle 2\rangle}+5H_{n-k}^{\langle 2\rangle}-\mu H_{n+k}^{\langle 2\rangle}\Big\}\Big]$$
$$+\frac{1/6}{(x+k)^2}\begin{bmatrix}\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}^3+2\Big\{(5+\mu)H_k^{\langle 3\rangle}-5H_{n-k}^{\langle 3\rangle}-\mu H_{n+k}^{\langle 3\rangle}\Big\}\\ +3\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}\times\Big\{(5+\mu)H_k^{\langle 2\rangle}+5H_{n-k}^{\langle 2\rangle}-\mu H_{n+k}^{\langle 2\rangle}\Big\}\end{bmatrix}$$
$$+\frac{1/24}{x+k}\begin{bmatrix}\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}^4+3\Big\{(5+\mu)H_k^{\langle 2\rangle}+5H_{n-k}^{\langle 2\rangle}-\mu H_{n+k}^{\langle 2\rangle}\Big\}^2+6(5+\mu)H_k^{\langle 4\rangle}\\ +6\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}^2\times\Big\{(5+\mu)H_k^{\langle 2\rangle}+5H_{n-k}^{\langle 2\rangle}-\mu H_{n+k}^{\langle 2\rangle}\Big\}+30H_{n-k}^{\langle 4\rangle}\\ +8\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}\times\Big\{(5+\mu)H_k^{\langle 3\rangle}-5H_{n-k}^{\langle 3\rangle}-\mu H_{n+k}^{\langle 3\rangle}\Big\}-6\mu H_{n+k}^{\langle 4\rangle}\end{bmatrix}\Bigg\}$$

and the six corresponding harmonic number identities:

$$0 = \sum_{k=0}^n (-1)^k\binom{n}{k}^5\binom{n+k}{k}^\mu$$
$$\times\begin{bmatrix}\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}^4+3\Big\{(5+\mu)H_k^{\langle 2\rangle}+5H_{n-k}^{\langle 2\rangle}-\mu H_{n+k}^{\langle 2\rangle}\Big\}^2+6(5+\mu)H_k^{\langle 4\rangle}\\ +6\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}^2\times\Big\{(5+\mu)H_k^{\langle 2\rangle}+5H_{n-k}^{\langle 2\rangle}-\mu H_{n+k}^{\langle 2\rangle}\Big\}+30H_{n-k}^{\langle 4\rangle}\\ +8\Big\{(5+\mu)H_k-5H_{n-k}-\mu H_{n+k}\Big\}\times\Big\{(5+\mu)H_k^{\langle 3\rangle}-5H_{n-k}^{\langle 3\rangle}-\mu H_{n+k}^{\langle 3\rangle}\Big\}-6\mu H_{n+k}^{\langle 4\rangle}\end{bmatrix}.$$

By means of the standard partial fraction method, we can also derive the following algebraic identities and the corresponding harmonic number formulae, even though they are not consequences of the theorem and the corollary proved in the present paper.

Example 6. *Partial fraction decomposition formula*

$$\frac{x(1-x)_n^2}{(x)_{n+1}^2} = \frac{1}{x}+\sum_{k=1}^n\binom{n}{k}^2\binom{n+k}{k}^2 \times \left\{\frac{-k}{(x+k)^2}+\frac{1+2kH_{n+k}+2kH_{n-k}-4H_k}{x+k}\right\}$$

and the corresponding harmonic number identity associated with Beukers' conjecture (cf. [1] and [2])*:*

$$0 = \sum_{k=1}^n\binom{n}{k}^2\binom{n+k}{k}^2\Big\{1+2kH_{n+k}+2kH_{n-k}-4H_k\Big\}.$$

Example 7. *Partial fraction decomposition formula*

$$\begin{aligned}\frac{(1-x)_n^2}{(1+x)_n^2} &= 1+\sum_{k=1}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\\ &\times \left\{\frac{k^2}{(x+k)^2}-\frac{2k^2}{x+k}\Big(\frac{1}{k}+H_{n+k}+H_{n-k}-2H_k\Big)\right\}\end{aligned}$$

and the corresponding harmonic number identity:

$$n(n+1) = \sum_{k=1}^{n} k^2\binom{n}{k}^2\binom{n+k}{k}^2\left\{\frac{1}{k}+H_{n+k}+H_{n-k}-2H_k\right\}.$$

Example 8. *Partial fraction expansion with denominator polynomial of type "$2+1$"*

$$\begin{aligned}\frac{n!\times(2n)!}{(x)_{n+1}^2(1-x)_n} &= \sum_{k=1}^{n}\frac{\binom{2n}{n+k}}{\binom{n+k}{n+1}}\frac{(-1)^k}{(1+n)(x-k)}\\ &+\sum_{k=0}^{n}\binom{n}{k}\binom{2n}{n+k}\left\{\frac{1}{(x+k)^2}+\frac{H_k+H_{n+k}-2H_{n-k}}{x+k}\right\}\end{aligned}$$

and the corresponding harmonic number identity:

$$\sum_{k=1}^{n}\frac{(-1)^k}{1+n}\frac{\binom{2n}{n+k}}{\binom{n+k}{n+1}} = \sum_{k=0}^{n}\binom{n}{k}\binom{2n}{n+k}(2H_{n-k}-H_k-H_{n+k}).$$

Example 9. *Partial fraction expansion with denominator polynomial of type "$3+1$"*

$$\begin{aligned}\frac{(n!)^2\times(2n)!}{(x)_{n+1}^3(1-x)_n} &= \frac{1}{(1+n)^2}\sum_{k=1}^{n}\frac{\binom{2n}{n+k}}{\binom{n+k}{n+1}^2}\frac{(-1)^k}{x-k}+\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\binom{2n}{n+k}\\ &\times\left\{\frac{1}{(x+k)^3}+\frac{2H_k+H_{n+k}-3H_{n-k}}{(x+k)^2}+\frac{1/2}{x+k}\left[\begin{array}{l}\big(2H_k+H_{n+k}-3H_{n-k}\big)^2\\ +\big(2H_k^{\langle 2\rangle}+H_{n+k}^{\langle 2\rangle}+3H_{n-k}^{\langle 2\rangle}\big)\end{array}\right]\right\}\end{aligned}$$

and the corresponding harmonic number identity:

$$\sum_{k=1}^{n}\frac{2(-1)^{k-1}}{(1+n)^2}\frac{\binom{2n}{n+k}}{\binom{n+k}{n+1}^2} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2\binom{2n}{n+k}\left\{\begin{array}{l}\big(2H_k+H_{n+k}-3H_{n-k}\big)^2\\ +\big(2H_k^{\langle 2\rangle}+H_{n+k}^{\langle 2\rangle}+3H_{n-k}^{\langle 2\rangle}\big)\end{array}\right\}.$$

Example 10. *Partial fraction decomposition formula with denominator polynomial of type "3 + 2"*

$$\begin{aligned}\frac{n!\times\big\{(2n)!\big\}^2}{(x)_{n+1}^3(1-x)_n^2} &= \frac{1}{1+n}\sum_{k=1}^{n}\frac{\binom{2n}{n+k}^2}{\binom{n+k}{n+1}}\left\{\frac{1}{(x-k)^2}+\frac{H_{k-1}+2H_{n-k}-3H_{n+k}}{x-k}\right\}\\ &+\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{n+k}^2\left\{\frac{1}{(x+k)^3}+\frac{H_k+2H_{n+k}-3H_{n-k}}{(x+k)^2}\right.\\ &\left.+\frac{1}{2(x+k)}\left[\big(H_k+2H_{n+k}-3H_{n-k}\big)^2+\big(H_k^{(2)}+2H_{n+k}^{(2)}+3H_{n-k}^{(2)}\big)\right]\right\}\end{aligned}$$

and the corresponding harmonic number identity:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2n}{n+k}^2\left\{\begin{matrix}\big(H_k+2H_{n+k}-3H_{n-k}\big)^2\\+\big(H_k^{(2)}+2H_{n+k}^{(2)}+3H_{n-k}^{(2)}\big)\end{matrix}\right\}\\ &= \frac{2}{1+n}\sum_{k=1}^{n}\frac{\binom{2n}{n+k}^2}{\binom{n+k}{n+1}}\big(3H_{n+k}-H_{k-1}-2H_{n-k}\big).\end{aligned}$$

Example 11. *For nonnegative integer θ with $0\le\theta<4+4n$, there hold partial fraction decomposition formulae:*

$$\frac{(n!)^4x^\theta}{(x)_{n+1}^4}=\sum_{k=0}^{n}\binom{n}{k}^4\left\{\frac{(-k)^\theta}{(x+k)^4}+\frac{(-k)^{\theta-1}}{(x+k)^3}\Big[\theta-4k(H_k-H_{n-k})\Big]\right. \tag{21a}$$

$$+\frac{(-k)^{\theta-2}}{2(x+k)^2}\Big[\big\{\theta-4k(H_k-H_{n-k})\big\}^2-\big\{\theta-4k^2(H_k^{(2)}+H_{n-k}^{(2)})\big\}\Big] \tag{21b}$$

$$\left.+\frac{(-k)^{\theta-3}}{6(x+k)}\left[\begin{matrix}\big\{\theta-4k(H_k-H_{n-k})\big\}^3+2\big\{\theta-4k^3(H_k^{(3)}-H_{n-k}^{(3)})\big\}\\-3\big\{\theta-4k(H_k-H_{n-k})\big\}\big\{\theta-4k^2(H_k^{(2)}+H_{n-k}^{(2)})\big\}\end{matrix}\right]\right\}. \tag{21c}$$

The corresponding harmonic identities read as

$$\sum_{k=0}^{n}k^{\theta-3}\binom{n}{k}^4\left[\begin{matrix}\big\{\theta-4k(H_k-H_{n-k})\big\}^3+2\big\{\theta-4k^3(H_k^{(3)}-H_{n-k}^{(3)})\big\}\\-3\big\{\theta-4k(H_k-H_{n-k})\big\}\big\{\theta-4k^2(H_k^{(2)}+H_{n-k}^{(2)})\big\}\end{matrix}\right] \tag{22a}$$

$$=\begin{cases}0, & 0\le\theta\le2+4n\\ 6(n!)^4, & \theta=3+4n.\end{cases} \tag{22b}$$

For $\theta=0,1,2$, the corresponding results to this identity have been conjectured by Weideman [10, Eq 21] and confirmed by Driver et al [4, Eq 20]. In particular, we recover, with the case $\theta=1$, the identity found by Driver et al [5, Eq 21].

The list can be endless. However, we are not bothered to extend it further. The interested reader can do that for enjoyment.

References

[1] S. Ahlgren - S. B. Ekhad - K. Ono - D. Zeilberger, *A binomial coefficient identity associated to a conjecture of Beukers,* The Electronic J. Combinatorics 5 (1998), #R10.

[2] W. Chu, *A Binomial Coefficient Identity Associated with Beukers' Conjecture on Apéry numbers,* The electronic journal of combinatorics 11 (2004), N15.

[3] W. Chu - L. De Donno, *Hypergeometric series and harmonic number identities,* Advances in Applied Math. 34 (2005), 123-137.

[4] K. Driver - H. Prodinger - C. Schneider - J. A. C. Weideman, *Padé Approximations to the logarithm II: Identities, recurrences and symbolic computation,* To appear in "The Ramanujan Journal".

[5] K. Driver - H. Prodinger - C. Schneider - J. A. C. Weideman, *Padé Approximations to the logarithm III: Alternative methods and additional results,* To appear in "The Ramanujan Journal".

[6] R. L. Graham - D. E. Knuth - O. Patashnik, *Concrete Mathematics,* Addison-Wesley Publ. Company, Reading, Massachusetts, 1989.

[7] P. J. Larcombe - E. J. Fennessey - W. A. Koepf, *Integral proofs of two alternating sign binomial coefficients identities,* Utilitas mathematica 66 (2004), 93-103.

[8] P. Paule - C. Schneider, *Computer proofs of a new family of harmonic number identities,* Adv. in Appl. Math. 31 (2003), 359-378.

[9] A. J. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$,* Math. Intelligencer 1 (1979), 195-203.

[10] J. A. C. Weideman, *Padé Approximations to the logarithm I: Derivation via differential equations,* Preprint, (2003).

Appendix A. The Ω-Coefficients Computed via (11) and (19)

$$\Omega_0(\lambda,\mu,-k) \equiv 1. \tag{A0}$$

$$\Omega_1(\lambda,\mu,-k) = \lambda\Big\{H_k - H_{n-k}\Big\} + \mu\Big\{H_k - H_{n+k}\Big\}. \tag{A1}$$

$$\Omega_2(\lambda,\mu,-k) = \Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^2 \tag{A2a}$$

$$+\lambda\Big\{H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\Big\} + \mu\Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\}. \tag{A2b}$$

$$\Omega_3(\lambda,\mu,-k) = \Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^3 \tag{A3a}$$

$$+2\Big\{\lambda\big(H_k^{\langle 3\rangle} - H_{n-k}^{\langle 3\rangle}\big) + \mu\big(H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\big)\Big\} \tag{A3b}$$

$$+3\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\} \tag{A3c}$$

$$\times\Big\{\lambda\big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) + \mu\big(H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\big)\Big\}. \tag{A3d}$$

$$\Omega_4(\lambda,\mu,-k) = \Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^4 \tag{A4a}$$

$$+ 6\Big\{\lambda\big(H_k^{\langle 4\rangle} + H_{n-k}^{\langle 4\rangle}\big) + \mu\big(H_k^{\langle 4\rangle} - H_{n+k}^{\langle 4\rangle}\big)\Big\} \tag{A4b}$$

$$+ 8\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\} \tag{A4c}$$

$$\times \Big\{\lambda\big(H_k^{\langle 3\rangle} - H_{n-k}^{\langle 3\rangle}\big) + \mu\big(H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\big)\Big\} \tag{A4d}$$

$$+ 6\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^2 \tag{A4e}$$

$$\times \Big\{\lambda\big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) + \mu\big(H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\big)\Big\} \tag{A4f}$$

$$+ 3\Big\{\lambda\big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) + \mu\big(H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\big)\Big\}^2. \tag{A4g}$$

$$\Omega_5(\lambda,\mu,-k) = \Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^5 \tag{A5a}$$

$$+ 24\Big\{\lambda\big(H_k^{\langle 5\rangle} - H_{n-k}^{\langle 5\rangle}\big) + \mu\big(H_k^{\langle 5\rangle} - H_{n+k}^{\langle 5\rangle}\big)\Big\} \tag{A5b}$$

$$+ 10\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^3 \tag{A5c}$$

$$\times \Big\{\lambda\big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) + \mu\big(H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\big)\Big\} \tag{A5d}$$

$$+ 20\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\}^2 \tag{A5e}$$

$$\times \Big\{\lambda\big(H_k^{\langle 3\rangle} - H_{n-k}^{\langle 3\rangle}\big) + \mu\big(H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\big)\Big\} \tag{A5f}$$

$$+ 15\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\} \tag{A5g}$$

$$\times \Big\{\lambda\big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) + \mu\big(H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\big)\Big\}^2 \tag{A5h}$$

$$+ 30\Big\{\lambda\big(H_k - H_{n-k}\big) + \mu\big(H_k - H_{n+k}\big)\Big\} \tag{A5i}$$

$$\times \Big\{\lambda\big(H_k^{\langle 4\rangle} + H_{n-k}^{\langle 4\rangle}\big) + \mu\big(H_k^{\langle 4\rangle} - H_{n+k}^{\langle 4\rangle}\big)\Big\} \tag{A5j}$$

$$+ 20\Big\{\lambda\big(H_k^{\langle 2\rangle} + H_{n-k}^{\langle 2\rangle}\big) + \mu\big(H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\big)\Big\} \tag{A5k}$$

$$\times \Big\{\lambda\big(H_k^{\langle 3\rangle} - H_{n-k}^{\langle 3\rangle}\big) + \mu\big(H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\big)\Big\}. \tag{A5l}$$

Appendix B. The ϖ-Coefficients Computed via (20a)

$$\varpi_0(\lambda,-k) \equiv 1. \tag{B0}$$

$$\varpi_1(\lambda,-k) = \lambda\Big\{H_k - H_{n-k}\Big\}. \tag{B1}$$

$$\varpi_2(\lambda,-k) = \lambda^2\Big\{H_k - H_{n-k}\Big\}^2 + \lambda\Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\}. \tag{B2}$$

$$\varpi_3(\lambda,-k) = \lambda^3\Big\{H_k - H_{n-k}\Big\}^3 + 2\lambda\Big\{H_k^{(3)} - H_{n-k}^{(3)}\Big\} \tag{B3a}$$

$$+ 3\lambda^2\Big\{H_k - H_{n-k}\Big\} \times \Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\}. \tag{B3b}$$

$$\varpi_4(\lambda,-k) = \lambda^4\Big\{H_k - H_{n-k}\Big\}^4 + 6\lambda\Big\{H_k^{(4)} + H_{n-k}^{(4)}\Big\} \tag{B4a}$$

$$+ 8\lambda^2\Big\{H_k - H_{n-k}\Big\} \times \Big\{H_k^{(3)} - H_{n-k}^{(3)}\Big\} \tag{B4b}$$

$$+ 6\lambda^3\Big\{H_k - H_{n-k}\Big\}^2 \times \Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\} \tag{B4c}$$

$$+ 3\lambda^2\Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\}^2. \tag{B4d}$$

$$\varpi_5(\lambda,-k) = \lambda^5\Big\{H_k - H_{n-k}\Big\}^5 + 24\lambda\Big\{H_k^{(5)} - H_{n-k}^{(5)}\Big\} \tag{B5a}$$

$$+ 10\lambda^4\Big\{H_k - H_{n-k}\Big\}^3 \times \Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\} \tag{B5b}$$

$$+ 20\lambda^3\Big\{H_k - H_{n-k}\Big\}^2 \times \Big\{H_k^{(3)} - H_{n-k}^{(3)}\Big\} \tag{B5c}$$

$$+ 15\lambda^3\Big\{H_k - H_{n-k}\Big\} \times \Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\}^2 \tag{B5d}$$

$$+ 30\lambda^2\Big\{H_k - H_{n-k}\Big\} \times \Big\{H_k^{(4)} + H_{n-k}^{(4)}\Big\} \tag{B5e}$$

$$+ 20\lambda^2\Big\{H_k^{(2)} + H_{n-k}^{(2)}\Big\} \times \Big\{H_k^{(3)} - H_{n-k}^{(3)}\Big\}. \tag{B5f}$$

Appendix C. The ω-Coefficients Computed via (20b)

$$\omega_0(\mu,-k) \equiv 1. \tag{C0}$$

$$\omega_1(\mu,-k) = \mu\Big\{H_k - H_{n+k}\Big\}. \tag{C1}$$

$$\omega_2(\mu,-k) = \mu^2\Big\{H_k - H_{n+k}\Big\}^2 + \mu\Big\{H_k^{(2)} - H_{n+k}^{(2)}\Big\}. \tag{C2}$$

$$\omega_3(\mu,-k) = \mu^3\Big\{H_k - H_{n+k}\Big\}^3 + 2\mu\Big\{H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\Big\} \tag{C3a}$$

$$+ 3\mu^2\Big\{H_k - H_{n+k}\Big\} \times \Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\}. \tag{C3b}$$

$$\omega_4(\mu,-k) = \mu^4\Big\{H_k - H_{n+k}\Big\}^4 + 6\mu\Big\{H_k^{\langle 4\rangle} - H_{n+k}^{\langle 4\rangle}\Big\} \tag{C4a}$$

$$+ 8\mu^2\Big\{H_k - H_{n+k}\Big\} \times \Big\{H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\Big\} \tag{C4b}$$

$$+ 6\mu^3\Big\{H_k - H_{n+k}\Big\}^2 \times \Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\} \tag{C4c}$$

$$+ 3\mu^2\Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\}^2. \tag{C4d}$$

$$\omega_5(\mu,-k) = \mu^5\Big\{H_k - H_{n+k}\Big\}^5 + 24\mu\Big\{H_k^{\langle 5\rangle} - H_{n+k}^{\langle 5\rangle}\Big\} \tag{C5a}$$

$$+ 10\mu^4\Big\{H_k - H_{n+k}\Big\}^3 \times \Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\} \tag{C5b}$$

$$+ 20\mu^3\Big\{H_k - H_{n+k}\Big\}^2 \times \Big\{H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\Big\} \tag{C5c}$$

$$+ 15\mu^3\Big\{H_k - H_{n+k}\Big\} \times \Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\}^2 \tag{C5d}$$

$$+ 30\mu^2\Big\{H_k - H_{n+k}\Big\} \times \Big\{H_k^{\langle 4\rangle} - H_{n+k}^{\langle 4\rangle}\Big\} \tag{C5e}$$

$$+ 20\mu^2\Big\{H_k^{\langle 2\rangle} - H_{n+k}^{\langle 2\rangle}\Big\} \times \Big\{H_k^{\langle 3\rangle} - H_{n+k}^{\langle 3\rangle}\Big\}. \tag{C5f}$$