FACTORIZATIONS OF COMPLETE GRAPHS INTO $[R, S, T, 2]$-CATERPILLARS OF DIAMETER 5

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ABSTRACT. A tree R such that after deleting all leaves we obtain a path P is called a *caterpillar*. The path P is called the *spine* of the caterpillar R. If the spine has length 3 and R on $2n$ vertices contains vertices of degrees $r, s, t, 2$, where $2 < r, s, t < n$, then we say that R is an $[r, s, t, 2]$-*caterpillar* of diameter 5. We completely characterize $[r, s, t, 2]$-caterpillars of diameter 5 on $4k + 2$ vertices that factorize the complete graph K_{4k+2}.

1. INTRODUCTION

Let G be a simple graph with at most n vertices. A graph H with n vertices has a *decomposition* into subgraphs $G_0, G_1, G_2, \ldots, G_s$ if each edge of H belongs to exactly one G_i. When all subgraphs $G_i, 0 \leq i \leq s$, are isomorphic to a graph G, we say that H has a G-*decomposition*. If G has exactly n vertices and none of them is isolated, then G is called a *factor* and the decomposition is called a G-*factorization* of H.

Graph factorizations have been extensively studied for many years. Special attention has been paid to isomorphic factorizations. Among graphs whose G-factorizations have been sought, the most popular ones are the obvious candidates—complete graphs and complete bipartite graphs (see, e.g., [3,12]). In this paper we concentrate on isomorphic factorizations of complete graphs into spanning trees and in particular into spanning caterpillars of diameter 5.

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A simple arithmetic condition shows that only complete graphs with an even number of vertices can be factorized into spanning trees. Moreover, every spanning tree, which factorizes K_{2n}, satisfies the *maximum degree condition*, which means that for each vertex v in such a tree on $2n$ vertices it holds that $deg(v) \leq n$.

It is a part of graph theory folklore that each graph K_{2n} can be factorized into hamiltonian paths P_{2n}. On the other hand, it is easy to observe that each K_{2n} can be also factorized into double stars; that is, two stars $K_{1,n-1}$ joined by an edge with the endvertices in the centers of both stars. The first attempt to fill the gap between these two extremal cases was P. Eldergill's thesis [2], where he dealt with symmetric trees. Some classes of non-symmetric trees were examined by Fronček [4,5], Fronček and the author [7], and by the author [8]. Other papers on caterpillars of diameter 5 of types not included in this paper are under preparation. In [6] Fronček proves that certain classes of caterpillars of diameter 4 and 5 do not factorize complete graphs of order $2n$.

Results in this paper give a complete characterization of certain class of caterpillars of order $4k+2$ and diameter 5, $[r,s,t,2]$-caterpillars, that factorize K_{4k+2}. An exact definition of this class of graphs is given in Section 2.

The labeling used in constructions in this paper exists only for graphs with $4k+2$ vertices. Therefore, we examine just a special class of caterpillars of diameter 5, namely the caterpillars of order $4k+2$ with exactly one vertex of degree 2 and without a vertex of degree greater than $2k$. The reason why we do not present here a more general class is that the other caterpillars with one vertex of degree 2 or the caterpillars with two or none vertices of degree 2 require many different and usually very long constructions. The results for the remaining classes are already in preparation.

2. Definitions and notation

A *labeling* of G with at most $2n+1$ vertices is an injection $\lambda : V(G) \to S$, where S is often a subset of the set $\{0, 1, ..., 2n\}$—however, in this paper we have $S = \{0_0, 1_0, \ldots, (n-1)_0, 0_1, 1_1, \ldots, (n-1)_1\}$. The labels of vertices u, v, denoted $\lambda(u) = i, \lambda(v) = j$, respectively, where $i, j \in S$, induce uniquely the *length* $\ell(e)$ of the edge $e = (u,v)$ with endvertices u, v. All labelings used here are generalizations of labelings introduced by A. Rosa [10,11].

The following definition was introduced in [5,1].

Let T be a tree with $2n = 4k+2$ vertices, $V(T) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2k+1$. Notice that the sets V_0 and V_1 are *not* the partite sets of T. Because we are factorizing the complete graph into isomorphic spanning trees, every vertex of the complete graph appears in every factor. Therefore, the labeling is a bijection from $V(T)$ to $\{0_0, 1_0, \ldots, (n-1)_0, 0_1, 1_1, \ldots, (n-1)_1\}$ and V_0 is the set of vertices labeled $0_0, 1_0, \ldots, (n-1)_0$, and V_1 is the set of vertices labeled $0_1, 1_1, \ldots, (n-1)_1$. Let λ be a bijection, $\lambda : V_i \rightarrow \{0_i, 1_i, 2_i, \ldots, (2k)_i\}, i = 0, 1$. The *pure length* of an edge (x_i, y_i) with $x_i, y_i \in V_i, i \in \{0, 1\}$ is defined as follows: If $\lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$, then $\ell_{ii}(x_i, y_i) = \min\{|a-b|, 2k+1-|a-b|\}$ for $i = 0, 1$. The *mixed length* of an edge (x_0, y_1) with $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$, is defined as $\ell_{01}(x_0, y_1) = b - a \mod (2k+1)$ for $x_0 \in V_0, y_1 \in V_1$. We say that T has a *blended ρ-labeling* or just *blended labeling* if

(1) $\{\ell_{ii}(x_i, y_i) | (x_i, y_i) \in E(T)\} = \{1, 2, \ldots, k\}$ for $i = 0, 1$,

(2) $\{\ell_{01}(x_0, y_1) | (x_0, y_1) \in E(T)\} = \{0, 1, 2, \ldots, 2k\}$.

To simplify our notation, we often unify vertices with their respective labels. We will say "a vertex a_i" rather than "a vertex x with $\lambda(x) = a_i$". Similarly, we will say "an edge (a_i, b_j)" rather than "an edge xy, where $\lambda(x) = a_i$ and $\lambda(y) = b_j$".

Notice that the lengths of pure and mixed edges are computed differently. Suppose we have the complete graph K_{14} with the vertex labels $0_0, 1_0, \ldots, 6_0, 0_1, 1_1, \ldots 6_1$. Then both the edges $(1_0, 3_0)$ and $(1_0, 6_0)$ have the pure length 2. On the other hand, the edge $(1_0, 3_1)$ has the mixed length 2 while the edge $(1_1, 3_0)$ has the mixed length 5. Similarly, the edge $(1_0, 6_1)$ has the mixed length 5 while the edge $(6_0, 1_1)$ has the mixed length 2.

It was proved in [5] that a tree T of order $4k+2$ with a blended labeling allows a T-factorization of K_{4k+2}.

We want to characterize some classes of trees on $4k+2$ vertices of diameter 5, which allow a blended ρ-labeling. Since the factorization into hamiltonian paths P_{4k+2} is well-known, we start our work with the caterpillars. From now on we will only consider caterpillars with $4k+2$ vertices.

A tree R such that after deleting all leaves we obtain a path P or a trivial graph is called a *caterpillar*. The path P is called the *spine* of the caterpillar R.

It is clear that the caterpillars on $4k+2$ vertices of diameter 2 are the stars $K_{1,4k+1}$, which clearly do not satisfy the maximum degree condition. The caterpillars of order $4k+2$ with diameter 3 are the double stars mentioned above. Therefore, the first interesting case is the class of caterpillars of diameter 4. The results obtained in [6] and [8] give the complete characterization of the caterpillars of order $4k+2$ with diameter 4, which factorize the complete graphs K_{4k+2}. Hence, we continue with the caterpillars on $4k+2$ vertices of diameter 5. Recall that if R is a *caterpillar of diameter 5* then the spine of R has four vertices.

Let the spine of a caterpillar R of diameter 5 have vertices A, a, b, B and edges Aa, ab, bB. Then we see that the endvertices of the spine of R

of diameter 5 are denoted by A, B and the internal vertices are denoted by a, b. If $\deg(A) = d_1, \deg(a) = d_2, \deg(b) = d_3, \deg(B) = d_4$, then such a caterpillar will be called a (d_1, d_2, d_3, d_4)-*caterpillar*. If we specify just the degrees of the vertices, say as $r_1 \geq r_2 \geq r_3 \geq r_4$, without specifying their location on the spine, then we will denote R as an $[r_1, r_2, r_3, r_4]$-*caterpillar.*

If $\deg(x) < 2k + 1$, where $x \in \{A, a, b, B\}$, and R has $4k + 2$ vertices and exactly one vertex of degree 2, then we call this caterpillar R an $[r, s, t, 2]$-*caterpillar.*

Notice that we deal only with trees with $4k + 2$ vertices, since trees with $4k$ vertices do not allow a blended labeling (see [7]). A complete characterization of $[r, s, 2, 2]$-caterpillars of order $4k + 2$ and diameter 5, where $3 \leq r, s \leq 2k + 1$, was given in [6] and [9]. Recall that we know that every caterpillar with $2n$ vertices and diameter 5 that factorizes K_{2n} and has exactly one vertex of degree 2 must contain a vertex of degree at least $n - 1$ (see [6]). Further, recall that we do not present here a more general class of the caterpillars of order $4k+2$ and diameter 5 that factorize K_{4k+2} because the proofs require many different and usually very long constructions. The results for the remaining classes are already in preparation.

We conclude this section with the main result of this paper that will be proved in Section 3.

Theorem 2.1. *Let R be an $[r, s, t, 2]$-caterpillar of order $4k + 2$ and diameter 5, where $k \geq 2$ and $2k \geq r \geq s \geq t \geq 3$. Then R factorizes K_{4k+2} if and only if $r = 2k$.*

3. $[r, s, t, 2]$-CATERPILLARS OF ORDER $4k + 2$ AND DIAMETER 5

We will use the following results to prove Theorem 2.1.

Theorem 3.1. (Fronček [6]) *For every $[r, s, t, 2]$-caterpillar of order $2n$ with $r \geq s \geq t$ that factorizes K_{2n} it holds that $r \geq n - 1$.*

Lemma 3.2. [8] *Let T be a tree with a blended ρ-labeling λ and x, y be arbitrary vertices of T such that $x \in V_0$ and $y \in V_1$. Then there exists a blended ρ-labeling λ' such that $\lambda'(x) = 0_0$ and $\lambda'(y) = 0_1$.*

The proof is straightforward and can be found in [8].

Lemma 3.3. [9] *Let T be a tree on $4k+2$ vertices, which allows a blended ρ-labeling. Then $\sum_{i \in V_0} deg(i) = \sum_{j \in V_1} deg(j) = 4k + 1$.*

The proof can be found in [9].

From Theorem 3.1 it follows that we can further consider just $[2k, r, s, 2]$-caterpillars of order $4k + 2$ and diameter 5 for $2 < r, s < 2k$. Moreover, it is obvious that such caterpillars exist only for $k \geq 2$.

Recall that every tree T with a blended labeling has vertices labeled so that $V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $V(T) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$. Therefore in all following constructions we assume that the vertices are already labeled and then join them by edges, keeping in mind that we need to construct the $[2k, r, s, 2]$-caterpillar while obtaining exactly one edge of each mixed length from 0 to $2k$ and exactly one edge of every pure length from 1 to k in each set V_i for $i = 0, 1$.

Lemma 3.4. *A $(2k, r, 2, s)$-caterpillar on $4k+2$ vertices and with diameter 5 allows a blended ρ-labeling for every k, r, s, where $2 < r, s < 2k$ and $k \geq 2$.*

Proof. By constructions. Let $r = n + 2, s = 2k - n$ for $1 \leq n \leq 2k - 3$. Notice that for some values of n in the following constructions it can happen that we get an edge sequence of type $(x_i, a_j), (x_i, (a+1)_j), (x_i, (a+2)_j), ..., (x_i, b_j)$, where $a > b$ and $i, j \in \{0, 1\}$. In this case this sequence is indeed empty.

Case 1. Let k be even, $k = 2q$.

Subcase 1.1 Let R be a $(2k, n+2, 2, 2k-n)$-caterpillar on $4k+2$ vertices, where n is odd, $n = 2p+1$, and $1 \leq n \leq k-1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_1, b = (k+1)_0, B = 0_1$. We see that the spine of R contains mixed edges of lengths $k, 0$ and $k+1$. First we attach each vertex from the sets $\{k_0, (k+2)_0, (k+3)_0, ..., (2k)_0\}$ and $\{1_1, 2_1, ..., (k-1)_1\}$ by an edge to 0_0. We obtain pure 00-edges of lengths $k, k-1, k-2, ..., 1$ and mixed edges of lengths $1, 2, ..., k-1$. Then we join every vertex from the sets $\{1_0, 2_0, ..., (k-1)_0\}$, $\{(k+2)_1, (k+3)_1, ..., (k+q-p)_1\}$, and $\{(k+q+p+2)_1, (k+q+p+3)_1, ..., (2k)_1\}$, and the vertex k_1 by an edge to 0_1. This way we obtain mixed edges of lengths $2k, 2k-1, ..., k+2$ and the pure 11-edges of lengths $k-1, k-2, ..., q+p+1$ and $q-p-1, q-p-2, ..., 1$, and k. Finally we attach each vertex from the set $(k+q-p+1)_1, (k+q-p+2)_1, ..., (k+q+p+1)_1$ by an edge to $(k+1)_1$. We obtain 11-edges of lengths $q-p, q-p+1, ..., q+p$.

If we replace in the previous construction the pure 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the edge $(0_1, (k+q+1)_1)$ of length q, then we obtain the proof for every n even, $0 \leq n \leq k-2$.

Subcase 1.2 Let R be a $(2k, n+2, 2, 2k-n)$-caterpillar on $4k+2$ vertices, where n is even, $k \leq n \leq 2k-2$, $n - (k-1) = 2p+1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_1, b = (k+1)_0, B = 0_1$. Again, the spine of R contains mixed edges of lengths $k+1, 0$ and k. First we attach each vertex from the sets $\{(q-p)_0, (q-p+1)_0, ..., (q+p)_0\}$, $\{k_0, (k+2)_0, (k+3)_0, ..., (k+q-p)_0\}$, $\{(k+q+p+2)_0, (k+q+p+3)_0, ..., (2k)_0\}$, and $\{1_1, 2_1, ..., (k-1)_1\}$ by an edge to 0_0. We obtain pure 00-edges of lengths $q-p, q-p+1, ..., q+p$ and $k, k-1, k-2, ..., q+p+1$, and $q-p-1, q-p-2, ..., 1$ and mixed edges of lengths

$1, 2, \ldots, k-1$. Then we join every vertex from the sets $\{1_0, 2_0, \ldots, (q-p-1)_0\}$ and $\{(q+p+1)_0, (q+p+2)_0, \ldots, (k-1)_0\}$, and the vertex k_1 by an edge to 0_1. This way we obtain mixed edges of lengths $2k, 2k-1, \ldots, 3q+p+2$ and $3q-p, 3q-p-1, \ldots, k+2$, and a pure 11-edge of length k. Finally we attach every vertex from the sets $\{(k+q-p+1)_0, (k+q-p+2)_0, \ldots, (k+q+p+1)_0\}$ and $\{(k+2)_1, (k+3)_1, \ldots, (2k)_1\}$ by an edge to $(k+1)_1$. We obtain mixed edges of lengths $3q+p+1, 3q+p, \ldots, 3q-p+1$ and the pure 11-edges of length $1, 2, \ldots, k-1$. Note that R is $(2k, 2k, 2, 2)$-caterpillar for $n = 2k-2$, but we need it for the construction of $(2k, 2k-1, 2, 3)$-caterpillar.

If we replace in the previous construction the 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the 11-edge $(0_1, (k+q+1)_1)$ of length q, then we obtain the construction for every n odd, $k-1 \leq n \leq 2k-3$.

Case 2. Let k be odd, $k = 2q+1$.

Subcase 2.1 Let R be $(2k, n+2, 2, 2k-n)$-caterpillar on $4k+2$ vertices, where n is odd, $n = 2p+1, 1 \leq n \leq k$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, \ldots, (2k)_0\}, V_1 = \{0_1, 1_1, \ldots, (2k)_1\}$ and $A = 0_0, a = k_1, b = k_0, B = 0_1$. We see that the spine of R contains mixed edges of lengths $k, 0$ and $k+1$. First we attach each vertex from the sets $\{(k+1)_0, (k+2)_0, \ldots, (2k)_0\}$ and $\{1_1, 2_1, \ldots, (k-1)_1\}$ by an edge to 0_0. We obtain pure 00-edges of lengths $k, k-1, k-2, \ldots, 1$ and mixed edges of lengths $1, 2, \ldots, k-1$. Then we join every vertex from the sets $\{1_0, 2_0, \ldots, (k-1)_0\}$, $\{(k+1)_1, (k+2)_1, \ldots, (k+q-p)_1\}$, and $\{(k+q+p+2)_1, (k+q+p+3)_1, \ldots, (2k)_1\}$ by an edge to 0_1. This way we obtain mixed edges of lengths $2k, 2k-1, \ldots, k+2$ and the pure 11-edges of lengths $k, k-1, \ldots, q+p+2$ and $q-p, q-p-1, \ldots, 1$. And finally we attach each vertex from the set $(k+q-p+1)_1, (k+q-p+2)_1, \ldots, (k+q+p+1)_1$ by an edge to k_1. We obtain 11-edges of lengths $q-p+1, q-p+2, \ldots, q+p+1$.

If we replace in the previous construction the pure 11-edge $(k_1, (k+q+1)_1)$ of length $q+1$ by the edge $(0_1, (k+q+1)_1)$ of length $q+1$, then we obtain the proof for every n even, $0 \leq n \leq k-1$.

Subcase 2.2 Let R be a $(2k, n+2, 2, 2k-n)$-caterpillar on $4k+2$ vertices, where n is even, $k+1 \leq n \leq 2k-2$, $n-k = 2p+1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, \ldots, (2k)_0\}, V_1 = \{0_1, 1_1, \ldots, (2k)_1\}$ and $A = 0_0, a = k_1, b = k_0, B = 0_1$.

Again, the spine of R contains mixed edges of lengths $k, 0$ and $k+1$. First we attach each vertex from the sets $\{(q-p+1)_0, (q-p+2)_0, \ldots, (q+p+1)_0\}$, $\{(k+1)_0, (k+2)_0, \ldots, (k+q-p)_0\}$, $\{(k+q+p+2)_0, (k+q+p+3)_0, \ldots, (2k)_0\}$, and $\{1_1, 2_1, \ldots, (k-1)_1\}$ by an edge to 0_0. We obtain pure 00-edges of lengths $q-p+1, q-p+2, \ldots, q+p+1$ and $k, k-1, \ldots, q+p+2$, and $q-p, q-p-1, \ldots, 1$ and mixed edges of lengths $1, 2, \ldots, k-1$. Then we join every vertex from the sets $\{1_0, 2_0, \ldots, (q-p)_0\}$ and $\{(q+p+2)_0, (q+p+3)_0, \ldots, (k-1)_0\}$ by an edge to 0_1. This way we obtain mixed edges of lengths

$2k, 2k-1, ..., 3q+p+3$ and $3q-p+1, 3q-p, ..., k+2$. Finally we attach every vertex from the sets $\{(k+q-p+1)_0, (k+q-p+2)_0, ..., (k+q+p+1)_0\}$ and $\{(k+1)_1, (k+2)_1, ..., (2k)_1\}$ by an edge to k_1. We obtain mixed edges of lengths $3q+p+2, 3q+p+1, ..., 3q-p+2$ and the pure 11-edges of length $1, 2, ..., k$.

If we replace in the previous construction the 11-edge $(k_1, (k+q+1)_1)$ by the 11-edge $(0_1, (k+q+1)_1)$, then we obtain construction for every n odd, $k \leq n \leq 2k-3$. □

Lemma 3.5. *A $(2k, 2, r, s)$-caterpillar on $4k+2$ vertices and with diameter 5 allows a blended ρ-labeling for every k, r, s, where $2 < r, s < 2k$ and $k \geq 2$.*

Proof. By constructions. Let $r = n+2, s = 2k-n$ for $1 \leq n \leq 2k-3$.

Case 1. Let k be even, $k = 2q$.

Subcase 1.1 Let R be a $(2k, 2, n+2, 2k-n)$-caterpillar on $4k+2$ vertices, where n is odd, $n = 2p+1, 1 \leq n \leq k-1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_0, b = (k+1)_1, B = 0_1$.

Then R contains

(i) pure 00-edges $(0_0, (k+1)_0), (0_0, (k+2)_0), ..., (0_0, (2k)_0)$ of lengths $k, k-1, ..., 1$, and

(ii) pure 11-edges $(0_1, (k+1)_1), (0_1, (k+2)_1), ..., (0_1, (k+q-p)_1)$ of lengths $k, k-1, ..., q+p+1$ and $(0_1, (k+q+p+2)_1), (0_1, (k+q+p+3)_1), ..., (0_1, (2k)_1)$ of lengths $q-p-1, q-p-2, ..., 1$, and $((k+1)_1, (k+q-p+1)_1), ((k+1)_1, (k+q-p+2)_1), ..., ((k+1)_1, (k+q+p+1)_1)$ of lengths $q-p, q-p+1, ..., q+p$, and

(iii) mixed edges $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1)$ of lengths $1, 2, ..., k$ and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, k_0)$ of lengths $2k, 2k-1, ..., k+1$, and finally the edge $((k+1)_0, (k+1)_1)$ of length 0.

If we replace in the previous construction the pure 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the edge $(0_1, (k+q+1)_1)$ of length q, then we obtain the proof for every n even, $0 \leq n \leq k-2$.

Subcase 1.2 Let R be a $(2k, 2, n+2, 2k-n)$-caterpillar on $4k+2$ vertices, where n is even, $k \leq n \leq 2k-2$, $n-(k-1) = 2p+1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_0, b = (k+1)_1, B = 0_1$.

Then R contains

(iv) pure 00-edges $(0_0, (q-p)_0), (0_0, (q-p+1)_0), ..., (0_0, (q+p)_0)$ of lengths $q-p, q-p+1, ..., q+p$ and $(0_0, (k+1)_0), (0_0, (k+2)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k, k-1, ..., q+p+1$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k)_0)$ of lengths $q-p-1, q-p-2, ..., 1$, and

(v) pure 11-edges $(0_1,(k+1)_1)$ of length k and $((k+1)_1,(k+2)_1),((k+1)_1,(k+3)_1),...,((k+1)_1,(2k)_1)$ of lengths $1,2,...,k-1$, and

(vi) mixed edges $(0_0,1_1),(0_0,2_1),...,(0_0,k_1)$ of lengths $1,2,...,k$ and $(0_1,1_0),(0_1,2_0),...,(0_1,(q-p-1)_0)$ of lengths $2k,2k-1,...,3q+p+2$, and $(0_1,(q+p+1)_0),(0_1,(q+p+2)_0),...,(0_1,k_0)$ of lengths $3q-p,3q-p-1,...,k+1$, and $((k+1)_1,(k+q-p+1)_0),((k+1)_1,(k+q-p+2)_0),...,((k+1)_1,(k+q+p+1)_0)$ of lengths $3q+p+1,3q+p,...,3q-p+1$.

If we replace in the previous construction the 11-edge $((k+1)_1,(k+q+1)_1)$ of length q by the 00-edge $(0_1,(k+q+1)_1)$ of length q, then we obtain the construction for every n odd, $k-1\le n\le 2k-3$.

Case 2. Let k be odd, $k=2q+1$.

Subcase 2.1 Let R be a $(2k,2,n+2,2k-n)$-caterpillar on $4k+2$ vertices, where n is odd, $n=2p+1$ and $1\le n\le k-2$. Furthermore, let $V(R)=V_0\cup V_1, V_0=\{0_0,1_0,...,(2k)_0\}, V_1=\{0_1,1_1,...,(2k)_1\}$ and $A=0_0, a=k_0, b=k_1, B=0_1$.

Then R contains

(vii) pure 00-edges $(0_0,k_0)$ of length k and $(0_0,(k+2)_0),(0_0,(k+3)_0),...,(0_0,(2k)_0)$ of lengths $k-1,k-2,...,1$, and

(viii) pure 11-edges $(0_1,k_1)$ of length k and $(0_1,(k+2))_1,(0_1,(k+3)_1),...,(0_1,(k+q-p)_1)$ of lengths $k-1,k-2,...,q+p+2$, and $(0_1,(k+q+p+2)_1),(0_1,(k+q+p+3)_1),...,(0_1,(2k)_1)$ of lengths $q-p,q-p-1,...,1$, and $(k_1,(k+q-p+1)_1),(k_1,(k+q-p+2)_1),...,(k_1,(k+q+p+1)_1)$ of lengths $q-p+1,q-p+2,...,q+p+1$, and

(ix) mixed edges $(0_0,(k+1)_1),(0_1,(k+1)_0)$ and (k_0,k_1) of lengths $k+1,k$ and 0, and $(0_0,1_1),(0_0,2_1),...,(0_0,(k-1)_1)$ of lengths $1,2,...,k-1$, and $(0_1,1_0),(0_1,2_0),...,(0_1,(k-1)_0)$ of lengths $2k,2k-1,...,k+2$.

If we replace in the previous construction the 11-edge $(k_1,(k+q+1)_1)$ of length $q+1$ by the edge $(0_1,(k+q+1)_1)$ of length $q+1$, then we obtain the construction for every n even, $0\le n\le k-3$.

Subcase 2.2 Let R be a $(2k,2,n+2,2k-n)$-caterpillar on $4k+2$ vertices, where n is even, $n-(k-2)=2p+1$ and $k-1\le n\le 2k-4$. Furthermore, let $V(R)=V_0\cup V_1, V_0=\{0_0,1_0,...,(2k)_0\}, V_1=\{0_1,1_1,...,(2k)_1\}$ and $A=0_0, a=k_0, b=k_1, B=0_1$.

Then R contains

(x) pure 00-edges $(0_0,k_0)$ of length k and $(0_0,(q-p+1)_0),(0_0,(q-p+2)_0),...,(0_0,(q+p+1)_0)$ of lengths $q-p+1,q-p+2,...,q+p+1$, and $(0_0,(k+2)_0),(0_0,(k+3)_0),...,(0_0,(k+q-p)_0)$ of lengths $k-1,k-2,...,q+p+2$, and $(0_0,(k+q+p+2)_0),(0_0,(k+q+p+3)_0),...,(0_0,(2k)_0)$ of lengths $q-p,q-p-1,...,1$, and

(xi) pure 11-edges $(0_1, k_1), (0_1, (2k)_1)$ of lengths k and 1, and $(k_1, (k+2)_1), (k_1, (k+3)_1), ..., (k_1, (2k-1)_1)$ of lengths $2, 3, ..., k-1$, and

(xii) mixed edges $(0_0, (k+1)_1), (0_1, (k+1)_0)$ and (k_0, k_1) of lengths $k+1, k$ and 0, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (q-p)_0)$ of lengths $2k, 2k-1, ..., 3q+p+3$, and $(0_1, (q+p+2)_0), (0_1, (q+p+3)_0), ..., (0_1, (k-1)_0)$ of lengths $3q-p+1, 3q-p, ..., k+2$, and $(k_1, (k+q-p+1)_0), (k_1, (k+q-p+2)_0), ..., (k_1, (k+q+p+1)_0)$ of lengths $3q+p+2, 3q+p+1, ..., 3q-p+2$, and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, (k-1)_1)$ of lengths $1, 2, ..., k-1$.

If we replace in the previous construction the 11-edge $(k_1, (k+q+1)_1)$ of length $q+1$ by the edge $(0_1, (k+q+1)_1)$ of length $q+1$, then we obtain the construction for every n odd, $k-2 \leq n \leq 2k-5$.

Subcase 2.3 Let R be a $(2k, 2, n+2, 2k-n)$-caterpillar on $4k+2$ vertices, where $n = 2k-3$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}$, $V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_0, b = 0_1, B = k_1$.

Then R contains

(xiii) pure 00-edges $(0_0, (k+1)_0), (0_0, (k+2)_0), ..., (0_0, (2k)_0)$ of lengths $k, k-1, ..., 1$, and

(xiv) pure 11-edges $(0_1, k_1), (k_1, (k+q+1)_1)$ of lengths k and $q+1$, and $(0_1, (k+2)_1), (0_1, (k+3)_1), ..., (0_1, (k+q)_1)$ of lengths $k-1, k-2, ..., q+2$, and $(0_1, (k+q+2)_1), (0_1, (k+q+3)_1), ..., (0_1, (2k)_1)$ of lengths $q, q-1, ..., 1$, and

(xv) mixed edges $(0_0, (k+1)_1), (0_1, (k+1)_0)$ and (k_0, k_1) of lengths $k+1, k$ and 0, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$, and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, (k-1)_1)$ of lengths $1, 2, ..., k-1$. □

Lemma 3.6. *A $(2k, r, s, 2)$-caterpillar on $4k+2$ vertices and with diameter 5 allows a blended ρ-labeling for every $2 < r, s < 2k$ and every k even, $k \geq 2$.*

Proof. By constructions. Let $r = n+2$ and $s = 2k-n$ for $1 \leq n \leq 2k-3$ and let R be a $(2k, n+2, 2k-n, 2)$-caterpillar of order $4k+2$ and diameter 5 and $k = 2q$.

Case 1. Let n be even, $n-1 = 2p+1$ and $2 \leq n \leq k$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_1, b = 0_1, B = 1_0$.

Then R contains

(i) pure 00-edges $(0_0, (k+1)_0), (0_0, (k+2)_0), ..., (0_0, (2k)_0)$ of lengths $k, k-1, ..., 1$, and

(ii) pure 11-edges $(0_1, (k+1)_1)$ of length k, and $(0_1, (k+2)_1), (0_1, (k+3)_1), ..., (0_1, (k+q-p)_1)$ of lengths $k-1, k-2, ..., q+p+1$, and $(0_1, (k+q+p+2)_1), (0_1, (k+q+p+3)_1), ..., (0_1, (2k)_1)$ of lengths $q-p-1, q-p-2, ..., 1$, and $((k+1)_1, (k+q-p+1)_1)((k+1)_1, (k+q-$

$p+2)_1), ..., ((k+1)_1, (k+q+p+1)_1)$ of lengths $q-p, q-p+1, ..., q+p$, and

(iii) mixed edges $(k_0, (k+1)_1)$ and $(1_0, 1_1)$ of length 1 and 0, and $(0_1, 1_0)$, $(0_1, 2_0), ..., (0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$, and $(0_0, 2_1)$, $(0_0, 3_1), ..., (0_0, k_1), (0_0, (k+1)_1)$ of lengths $2, 3, ..., k, k+1$.

Case 2. Let n be odd, $n-k = 2p+1$ and $k+1 \le n \le 2k-3$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = (k+1)_1, b = 0_1, B = 1_0$.

Then R contains

(iv) pure 00-edges $(0_0, (2k)_0)$ of length 1, and $(0_0, (q-p)_0), (0_0, (q-p+1)_0), ..., (0_0, (q+p)_0)$ of lengths $q-p, q-p+2, ..., q+p$, and $(0_0, (k+1)_0), (0_0, (k+2)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k, k-1, ..., q+p+1$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k-1)_0)$ of lengths $q-p-1, q-p-2, ..., 2$, and

(v) pure 11-edges $(0_1, (k+1)_1)$ of length k and $((k+1)_1, (k+2)_1), ((k+1)_1, (k+3)_1), ..., ((k+1)_1, (2k)_1)$ of lengths $1, 2, ..., k-1$, and

(vi) mixed edges $(k_0, (k+1)_1)$ and $(1_0, 1_1)$ of lengths 1 and 0, and $(0_1, 1_0)$, $(0_1, 2_0), ..., (0_1, (q-p-1)_0)$ of lengths $2k, 2k-1, ..., 3q+p+2$, and $(0_1, (q+p+1)_0), (0_1, (q+p+2)_0), ..., (0_1, (k-1)_0)$ of lengths $3q-p, 3q-p-1, ..., k+2$, and $((k+1)_1, (k+q-p+1)_0), ((k+1)_1, (k+q-p+2)_0), ..., ((k+1)_1, (k+q+p+1)_0)$ of lengths $3q+p+1, 3q+p, ..., 3q-p+1$, and $(0_0, 2_1), (0_0, 3_1), ..., (0_0, k_1), (0_0, (k+1)_1)$ of lengths $2, 3, ..., k, k+1$.

If we replace in both previous constructions the pure 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the edge $(0_1, (k+q+1)_1)$ of length q, then we obtain constructions for every n odd if $1 \le n \le k-1$ and for every n even if $k \le n \le 2k-4$ □

Lemma 3.7. *A $(2k, r, s, 2)$-caterpillar on $4k+2$ vertices and with diameter 5 allows a blended ρ-labeling for every $2 < r, s < 2k$ and every k odd, $k \ge 3$.*

Proof. By constructions. Let $r = n+2$ and $s = 2k-n$ for $1 \le n \le 2k-3$ and let R be a $(2k, n+2, 2k-n, 2)$-caterpillar of order $4k+2$ and diameter 5 and $k = 2q+1$.

Case 1. Let n be even, $n-1 = 2p+1$ and $2 \le n \le k-1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = k_1, b = 0_1, B = 1_0$.

Then R contains

(i) pure 00-edges $(0_0, (2k)_0)$ and $(1_0, (k+1)_0)$ of lengths 1 and k, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (2k-1)_0)$ of lengths $k-1, k-2, ..., 2$, and

(ii) pure 11-edges $(0_1, k_1)$ of length k, and $(0_1, (k+2)_1), (0_1, (k+3)_1), ...,$ $(0_1, (k+q-p)_1)$ of lengths $k-1, k-2, ..., q+p+2$, and $(0_1, (k+q+p+$

$2)_1), (0_1, (k+q+p+3)_1), ..., (0_1, (2k)_1)$ of lengths $q-p, q-p-1, ..., 1$, and $(k_1, (k+q-p+1)_1)(k_1, (k+q-p+2)_1), ..., (k_1, (k+q+p+1)_1)$ of lengths $q-p+1, q-p+2, ..., q+p+1$, and

(iii) mixed edges (k_0, k_1) of length 0 and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$, and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1), (0_0, (k+1)_1)$ of lengths $1, 2, ..., k, k+1$.

Case 2. Let n be odd, $n-(k-1) = 2p+1$ and $k \le n \le 2k-3$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = k_1, b = 0_1, B = 1_0$.

Then R contains

(iv) pure 00-edges $(0_0, (2k)_0)$ and $(1_0, (k+1)_0)$ of lengths 1 and k, and $(0_0, (q-p+1)_0), (0_0, (q-p+2)_0), ..., (0_0, (q+p+1)_0)$ of lengths $q-p+1, q-p+2, ..., q+p+1$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k-1, k-2, ..., q+p+2$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k-1)_0)$ of lengths $q-p, q-p-1, ..., 2$, and

(v) pure 11-edges $(0_1, k_1)$ and $(0_1, (2k)_1)$ of lengths k and 1 $(k_1, (k+2)_1), (k_1, (k+3)_1), ..., (k_1, (2k-1)_1)$ of lengths $2, 3, ..., k-1$, and

(vi) mixed edges (k_0, k_1) of length 0 and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (q-p)_0)$ of lengths $2k, 2k-1, ..., 3q+p+3$, and $(0_1, (q+p+2)_0), (0_1, (q+p+3)_0), ..., (0_1, (k-1)_0)$ of lengths $3q-p+1, 3q-p, ..., k+2$, and $(k_1, (k+q-p+1)_0), (k_1, (k+q-p+2)_0), ..., (k_1, (k+q+p+1)_0)$ of lengths $3q+p+2, 3q+p+1, ..., 3q-p+2$, and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1), (0_0, (k+1)_1)$ of lengths $1, 2, ..., k, k+1$.

If we replace in both previous constructions the pure 11-edge $(k_1, (k+q+1)_1)$ of length $q+1$ by the edge $(0_1, (k+q+1)_1)$ of length $q+1$, then we obtain constructions for every n odd if $1 \le n \le k-2$ and for every n even if $k-1 \le n \le 2k-4$ □

Lemma 3.8. *An $(r, 2k, 2, s)$-caterpillar of order $4k+2$ and diameter 5 allows a blended ρ-labeling for every $3 \le r, s \le 2k$ and every k even, $k \ge 2$.*

Proof. By constructions. Let R be a $(r, 2k, 2, s)$-caterpillar of order $4k+2$ and diameter 5 and let $r = n+1$ and $s = 2k-n+1$ for $2 \le n \le 2k-2$ and $k = 2q$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = (k+1)_1, a = 0_0, b = 1_0, B = 0_1$.

Case 1. Let n be odd, $n-2 = 2p+1$, and $3 \le n \le k+1$. Then R contains

(i) pure 00-edges $(0_0, 1_0)$ and $(0_0, k_0)$ of lengths 1 and k, and further 00-edges $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (2k-1)_0)$ of lengths $k-1, k-2, ..., 2$,

(ii) pure 11-edges $((k+1)_1, 1_1)$ of length k and $(0_1, (k+2)_1), (0_1, (k+3)_1), ..., (0_1, (k+q-p)_1)$ of lengths $k-1, k-2, ..., q+p+1$, and $(0_1, (k+q+p+2)_1), (0_1, (k+q+p+3)_1), ..., (0_1, (2k)_1)$ of lengths

$q-p-1, q-p-2, \ldots, 1$, and finally 11-edges $((k+1)_1, (k+q-p+1)_1), ((k+1)_1, (k+q-p+2)_1), \ldots, ((k+1)_1, (k+q+p+1)_1)$ of lengths $q-p, q-p+1, \ldots, q+p$, and

(iii) mixed 01-edges $(0_1, (2k)_0)$ and $((k+1)_0, (k+1)_1)$ of lengths 1 and 0, and 01-edges $(0_1, 1_0), (0_1, 2_0), \ldots (0_1, (k-1)_0)$ of lengths $2k, 2k-1, \ldots, k+2$, and $(0_0, 2_1), (0_0, 3_1), \ldots, (0_0, (k+1)_1)$ of lengths $2, 3, \ldots, k+1$.

Case 2. Let n be even, $n-(k+1) = 2p+1$, and $k+2 \leq n \leq 2k-2$. Then R contains

(iv) pure 00-edges $(0_0, 1_0)$ and $(0_0, k_0)$ of lengths 1 and k, and further 00-edges $(0_0, (q-p)_0), (0_0, (q-p+1)_0), \ldots, (0_0, (q+p)_0)$ of lengths $q-p, q-p+1, \ldots, q+p$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), \ldots, (0_0, (k+q-p)_0)$ of lengths $k-1, k-2, \ldots, q+p+1$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), \ldots, (0_0, (2k-1)_0)$ of lengths $q-p-1, q-p-2, \ldots, 2$,

(v) pure 11-edges $((k+1)_1, 1_1)$ of length k and 11-edges $((k+1)_1, (k+2)_1), ((k+1)_1, (k+3)_1), \ldots, ((k+1)_1, (2k)_1)$ of lengths $1, 2, \ldots, k-1$,

(vi) mixed 01-edges $((k+1)_1, (k+1)_0)$ and $(0_1, (2k)_0)$ of lengths 0 and 1, and $(0_1, 1_0), (0_1, 2_0), \ldots, (0_1, (q-p-1)_0)$ of lengths $2k, 2k-1, \ldots, 3q+p+2$, and $(0_1, (q+p+1)_0), (0_1, (q+p+2)_0), \ldots, (0_1, (k-1)_0)$ of lengths $3q-p, 3q-p-1, \ldots, k+2$, and $((k+1)_1, (k+q-p+1)_0), ((k+1)_1, (k+q-p+2)_0), \ldots, ((k+1)_1, (k+q+p+1)_0)$ of lengths $3q+p+1, 3q+p, \ldots, 3q-p+1$, and finally 01-edges $(0_0, 2_1), (0_0, 3_1), \ldots, (0_0, (k+1)_1)$ of lengths $2, 3, \ldots, k+1$.

If we replace in both previous cases the pure 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the edge $(0_1, (k+q+1)_1)$ of length q, then we obtain the construction for every n even, when $2 \leq n \leq k$, and for every n odd, when $k+1 \leq n \leq 2k-3$. □

Lemma 3.9. *An $(r, 2k, 2, s)$-caterpillar of order $4k+2$ and diameter 5 allows a blended ρ-labeling for every $3 \leq r, s \leq 2k$ and every k odd, $k \geq 3$.*

Proof. By constructions. Let R be a $(r, 2k, 2, s)$-caterpillar of order $4k+2$ and diameter 5 and let $r = n+1$ and $s = 2k-n+1$ for $2 \leq n \leq 2k-2$ and $k = 2q+1$. Furthermore, let $V(R) = V_0 \cup V_1$, $V_0 = \{0_0, 1_0, \ldots, (2k)_0\}$, $V_1 = \{0_1, 1_1, \ldots, (2k)_1\}$ and $A = k_1, a = 0_0, b = 1_0, B = 0_1$.

Case 1. Let n be odd, $n-2 = 2p+1$, and $3 \leq n \leq k+2$. Then R contains

(i) pure 00-edges $(0_0, 1_0)$ of length 1, and further 00-edges $(0_0, (k+1)_0), (0_0, (k+2)_0), \ldots, (0_0, (2k-1)_0)$ of lengths $k, k-1, \ldots, 2$,

(ii) pure 11-edges $(0_1, (k+1)_1), (0_1, (k+2)_1), \ldots, (0_1, (k+q-p)_1)$ of lengths $k-1, k-2, \ldots, q+p+2$, and $(0_1, (k+q+p+2)_1), (0_1, (k+q+p+3)_1), \ldots, (0_1, (2k)_1)$ of lengths $q-p, q-p-1, \ldots, 1$, and finally 11-edges

$(k_1,(k+q-p+1)_1),(k_1,(k+q-p+2)_1),...,(k_1,(k+q+p+1)_1)$ of lengths $q-p+1,q-p+2,...,q+p+1$, and

(iii) mixed 01-edges $(k_1,(2k)_0)$ and (k_0,k_1) of lengths $k+1$ and 0, and 01-edges $(0_1,1_0),(0_1,2_0),...(0_1,(k-1)_0)$ of lengths $2k,2k-1,...,k+2$, and $(0_0,1_1),(0_0,2_1),...,(0_0,k_1)$ of lengths $1,2,...,k$.

Case 2. Let n be even, $n-(k+2)=2p+1$, and $k+2\leq n\leq 2k-2$. Then R contains

(iv) pure 00-edges $(0_0,1_0)$ of length 1 and further 00-edges $(0_0,(q-p+1)_0),(0_0,(q-p+2)_0),...,(0_0,(q+p+1)_0)$ of lengths $q-p+1,q-p+2,...,q+p+1$, and $(0_0,(k+1)_0),(0_0,(k+2)_0),...,(0_0,(k+q-p)_0)$ of lengths $k,k-1,...,q+p+2$, and $(0_0,(k+q+p+2)_0),(0_0,(k+q+p+3)_0),...,(0_0,(2k-1)_0)$ of lengths $q-p,q-p-1,...,2$,

(v) pure 11-edges $(k_1,(k+1)_1),(k_1,(k+2)_1),...,(k_1,(2k)_1)$ of lengths $1,2,...,k$,

(vi) mixed 01-edges (k_1,k_0) and $(k_1,(2k)_0)$ of lengths 0 and $k+1$, and $(0_1,1_0),(0_1,2_0),...,(0_1,(q-p)_0)$ of lengths $2k,2k-1,...,3q+p+3$, and $(0_1,(q+p+2)_0),(0_1,(q+p+3)_0),...,(0_1,(k-1)_0)$ of lengths $3q-p+1,3q-p,...,k+2$, and $(k_1,(k+q-p+1)_0),(k_1,(k+q-p+2)_0),...,(k_1,(k+q+p+1)_0)$ of lengths $3q+p+2,3q+p+1,...,3q-p+2$, and finally 01-edges $(0_0,1_1),(0_0,2_1),...,(0_0,k_1)$ of lengths $1,2,...,k$.

If we replace in both previous cases the pure 11-edge $(k_1,(k+q+1)_1)$ of length $q+1$ by the edge $(0_1,(k+q+1)_1)$ of length $q+1$, then we obtain the construction for every n even, when $2\leq n\leq k+1$, and for every n odd, when $k+1\leq n\leq 2k-3$. □

Lemma 3.10. *An $(r,2k,s,2)$-caterpillar of order $4k+2$ and diameter 5 allows a blended ρ-labeling for every $3\leq r,s\leq 2k$ and every k even, $k\geq 2$.*

Proof. By constructions. Let R be a $(r,2k,s,2)$-caterpillar of order $4k+2$ and diameter 5 and let $r=n+1$ and $s=2k-n+1$ for $2\leq n\leq 2k-2$ and $k=2q$. Furthermore, let $V(R)=V_0\cup V_1$, $V_0=\{0_0,1_0,...,(2k)_0\}$, $V_1=\{0_1,1_1,...,(2k)_1\}$ and $A=(k+1)_1$, $a=0_0$, $b=0_1$, $B=(k+1)_0$.

Case 1. Let n be odd, $n=2p+1$, and $1\leq n\leq k-1$. Then R contains

(i) pure 00-edges $(0_0,k_0)$ and $((k+1)_0,(k+q+1)_0)$ of lengths k and q, and further 00-edges $(0_0,(k+2)_0),(0_0,(k+3)_0),...,(0_0,(k+q)_0)$ of lengths $k-1,k-2,...,q+1$, and $(0_0,(k+q+2)_0),(0_0,(k+q+3)_0),...,(0_0,(2k)_0)$ of lengths $q-1,q-2,...,1$,

(ii) pure 11-edges $(0_1,k_1)$ of length k and $(0_1,(k+2)_1),(0_1,(k+3)_1),...,(0_1,(k+q-p)_1)$ of lengths $k-1,k-2,...,q+p+1$, and $(0_1,(k+q+p+2)_1),(0_1,(k+q+p+3)_1),...,(0_1,(2k)_1)$ of lengths $q-p-1,q-p-2,...,1$, and finally 11-edges $((k+1)_1,(k+q-p+1)_1),((k+1)_1,(k+q-$

$p+2)_1), ..., ((k+1)_1, (k+q+p+1)_1)$ of lengths $q-p, q-p+1, ..., q+p$, and

(iii) mixed 01-edges $(0_0, 0_1)$, $(0_0, (k+1)_1)$ and $(0_1, (k+1)_0)$ of lengths $0, k+1$ and k, and 01-edges $(0_1, 1_0), (0_1, 2_0), ...(0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$, and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, (k-1)_1)$ of lengths $1, 2, ..., k-1$.

Case 2. Let n be odd, $n-(k-1)=2p$, and $k+1 \le n \le 2k-3$. Then R contains

(iv) pure 00-edges $((k+1)_0, (k+q+1)_0)$ and $(0_0, k_0)$ of lengths q and k, and further 00-edges $(0_0, (q-p)_0), (0_0, (q-p+1)_0), ..., (0_0, (q-1)_0)$ of lengths$q-p, q-p+1, ..., q-1$, and $(0_0, (q+1)_0), (0_0, (q+2)_0), ..., (0_0, (q+p)_0)$ of lengths $q+1, q+2, ..., q+p$ and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k-1, k-2, ..., q+p+1$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k)_0)$ of lengths $q-p-1, q-p-2, ..., 2, 1$,

(v) pure 11-edges $(0_1, k_1)$ of length k and 11-edges $((k+1)_1, (k+2)_1)$, $((k+1)_1, (k+3)_1), ..., ((k+1)_1, (2k)_1)$ of lengths $1, 2, ..., k-1$,

(vi) mixed 01-edges $(0_1, 0_0)$ and $(0_1, q_0)$ of lengths 0 and $3q+1$ and $(0_0, (k+1)_1)$ and $(0_1, (k+1)_0)$ of lengths $k+1$ and k. Further $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (q-p-1)_0)$ of lengths $2k, 2k-1, ..., 3q+p+2$, and $(0_1, (q+p+1)_0), (0_1, (q+p+2)_0), ..., (0_1, (k-1)_0)$ of lengths $3q-p, 3q-p-1, ..., k+2$, and $((k+1)_1, (k+q-p+1)_0), ((k+1)_1, (k+q-p+2)_0), ..., ((k+1)_1, (k+q)_0)$ of lengths $3q+p+1, 3q+p, ..., 3q+2$, and $((k+1)_1, (k+q+2)_1), ((k+1)_1, (k+q+3)_1), ..., ((k+1)_1, (k+q+p+1)_1)$ of lengths $3q, 3q-1, ..., 3q-p+1$, and finally 01-edges $(0_0, 1_1), (0_0, 2_1), ..., (0_0, (k-1)_1)$ of lengths $1, 2, ..., k-1$.

If we replace in both previous cases the pure 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the edge $(0_1, (k+q+1)_1)$ of length q, then we obtain the construction for every n even, when $0 \le n \le 2k-4$.

Case 3. Let $n = 2k-2$ and $A = 0_1, a = 0_0, b = k_1, B = (k+1)_0$. Then R contains

(vii) pure 00-edges $(0_0, 1_0), (0_0, k_0)$ and $((k+1)_0, (k+q+1)_0)$ of lengths $1, k$ and q, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+1$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k-1)_0)$ of lengths $q-1, q-2, ..., 2$,

(viii) pure 11-edges $(0_1, (k+1)_1,), (0_1, (k+2)_1), ..., (0_1, (2k)_1)$ of lengths $k, k-1, ..., 1$, and

(ix) mixed 01-edges $((k+1)_0, k_1)$ and $((2k)_0, k_1)$ of length $2k$ and $k+1$, and $(0_1, 2_0), (0_1, 3_0), ..., (0_1, (k-1)_0)$ of lengths $2k-1, 2k-2, ..., k+2$, and $(0_0, 0_1), (0_0, 1_1), ..., (0_0, k_1)$ of lengths $1, 2, ..., k$. □

Lemma 3.11. *An $(r, 2k, s, 2)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended ρ-labeling for every $3 \le r, s \le 2k$ and every k odd, $k \ge 3$.*

Proof. By constructions. Let R be a $(r, 2k, s, 2)$-caterpillar of order $4k + 2$ and diameter 5 and let $r = n+1$ and $s = 2k - n + 1$ for $2 \le n \le 2k - 2$ and $k = 2q + 1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = k_1, a = 0_0, b = 0_1, B = k_0$.

Case 1. Let n be odd, $n = 2p + 1$, and $1 \le n \le k$. Then R contains

(i) pure 00-edges $(k_0, (k+q+1)_0)$ of length $q+1$, and 00-edges $(0_0, (k+1)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+2$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k)_0)$ of lengths $q, q-1, ..., 1$,

(ii) pure 11-edges $(0_1, (k+1)_1), (0_1, (k+2)_1), ..., (0_1, (k+q-p)_1)$ of lengths $k, k-1, ..., q+p+2$, and $(0_1, (k+q+p+2)_1), (0_1, (k+q+p+3)_1), ..., (0_1, (2k)_1)$ of lengths $q-p, q-p-1, ..., 1$, and finally 11-edges $(k_1, (k+q-p+1)_1), (k_1, (k+q-p+2)_1), ..., (k_1, (k+q+p+1)_1)$ of lengths $q-p+1, q-p+2, ..., q+p+1$, and

(iii) mixed 01-edges $(0_1, 1_0), (0_1, 2_0), ..., (0_1, k_0)$ of lengths $2k, 2k-1, ..., k+1$, and $(0_0, 0_1), (0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1)$ of lengths $0, 1, 2, ..., k$.

Case 2. Let n be odd, $n - k = 2p$, and $k+2 \le n \le 2k-3$. Then R contains

(iv) pure 00-edges $(k_0, (k+q+1)_0)$ of length $q+1$ and $(0_0, (q-p+1)_0), (0_0, (q-p+2)_0), ..., (0_0, q_0)$ of lengths $q-p+1, q-p+2, ..., q$, and $(0_0, (q+2)_0), (0_0, (q+3)_0), ..., (0_0, (q+p+1)_0)$ of lengths $q+2, q+3, ..., q+p+1$ and $(0_0, (k+1)_0), (0_0, (k+2)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k, k-1, ..., q+p+2$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k)_0)$ of lengths $q-p, q-p-1, ..., 1$,

(v) pure 11-edges $(k_1, (k+1)_1), (k_1, (k+2)_1), ..., (k_1, (2k)_1)$ of lengths $1, 2, ..., k$,

(vi) mixed 01-edges $(0_1, 0_0)$ and $(0_1, (q+1)_0)$ of lengths 0 and $3q+2$. Further $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (q-p)_0)$ of lengths $2k, 2k-1, ..., 3q+p+3$, and $(0_1, (q+p+2)_0), (0_1, (q+p+3)_0), ..., (0_1, k_0)$ of lengths $3q-p+1, 3q-p, ..., k+1$, and $(k_1, (k+q-p+1)_0), (k_1, (k+q-p+2)_0), ..., (k_1, (k+q)_0)$ of lengths $3q+p+2, 3q+p+1, ..., 3q+3$, and $(k_1, (k+q+2)_1), (k_1, (k+q+3)_1), ..., (k_1, (k+q+p+1)_1)$ of lengths $3q+1, 3q, ..., 3q-p+2$, and finally 01-edges $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1)$ of lengths $1, 2, ..., k$.

If we replace in both previous cases the pure 11-edge $(k_1, (k+q+1)_1)$ of length $q+1$ by the edge $(0_1, (k+q+1)_1)$ of length $q+1$, then we obtain the construction for every n even, when $0 \le n \le 2k-4$.

Case 3. Let $n = 2k - 2$ and $A = 0_1, a = 0_0, b = k_1, B = (k+1)_0$. Then R contains

(vii) pure 00-edges $((k+1)_0, 1_0)$ of length k and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (2k)_0)$ of lengths $k-1, k-2, ..., 1$,

(viii) pure 11-edges $(k_1, (k+q+1)_1)$ of length $q+1$ and $(0_1, (k+1)_1,), (0_1, (k+2)_1), ..., (0_1, (k+q)_1)$ of lengths $k, k-1, ..., q+2$, and $(0_1, (k+q+2)_1), (0_1, (k+q+3)_1), ..., (0_1, (2k)_1)$ of lengths $q, q-1, ..., 1$, and

(ix) mixed 01-edges $((k+1)_0, k_1)$ of length $2k$ and $(0_1, 2_0), (0_1, 3_0), ..., (0_1, k_0)$ of lengths $2k-1, 2k-2, ..., k+1$, and $(0_0, 0_1), (0_0, 1_1), ..., (0_0, k_1)$ of lengths $1, 2, ..., k$. □

Lemma 3.12. *A $(2, 2k, r, s)$-caterpillar of order $4k+2$ and diameter 5 allows a blended ρ-labeling for every $2 < r, s < 2k-1$ and every k even, $k \geq 2$.*

Proof. By constructions. Let $r = n+2$ and $s = 2k-n$ for $1 \leq n \leq 2k-3$. Furthermore, let R be a $(2, 2k, n+2, 2k-n)$-caterpillar of order $4k+2$ and diameter 5 and $k = 2q$.

Case 1. Let n be odd, $k+1 \leq n \leq 2k-1$, and $n-k = 2p+1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = (k+1)_0, a = 0_0, b = 0_1, B = (k+1)_1$.

Then R contains

(i) pure 00-edges $(0_0, (k+1)_0)$ and $((k+1)_0, (k+q+1)_0)$ of lengths k and q, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+1$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k)_0)$ of lengths $q-1, q-2, ..., 1$, and

(ii) pure 11-edges $(0_1, (k+1)_1)$ of length k and $(0_1, (k+q-p+1)_1), (0_1, (k+q-p+2)_1), ..., (0_1, (k+q+p+1)_1)$ of lengths $q+p, q+p-1, ..., q-p$, and $((k+1)_1, (k+2)_1), ((k+1)_1, (k+3)_1), ..., ((k+1)_1, (k+q-p)_1)$ of lengths $1, 2, ..., q-p-1$, and $((k+1)_1, (k+q+p+2)_1), ((k+1)_1, (k+q+p+3)_1), ..., ((k+1)_1, (2k)_1)$ of lengths $q+p+1, q+p+2, ..., k-1$, and

(iii) mixed edges $(0_0, 0_1), (0_0, 1_1), ..., (0_0, k_1)$ of lengths $0, 1, ..., k$ and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, k_0)$ of lengths $2k, 2k-1, ..., k+1$.

If we replace the pure 11-edge $(0_1, (k+q+1)_1)$ of length q by the edge $((k+1)_1, (k+q+1)_1)$ of length q, then we obtain the constructions for every n even if $k \leq n \leq 2k-2$.

Case 2. Let n be even, $2 \leq n \leq k$, and $n+2 = 2p$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = (k+1)_0, a = 0_0, b = 0_1, B = (k+1)_1$.

Then R contains

(iv) pure 00-edges $(0_0, (k+1)_0)$ and $((k+1)_0, (k+q+1)_0)$ of lengths k and q, and $(0_0, (q-p)_0), (0_0, (q-p+1)_0), ..., (0_0, (q-1)_0)$ of lengths $q-p, q-p+1, ..., q-1$, and $(0_0, (q+1)_0), (0_0, (q+2)_0), ..., (0_0, (q+p)_0)$ of lengths $q+1, q+2, ..., q+p$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k-1, k-2, ..., q+p+1$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k)_0)$ of lengths $q-p-1, q-p-2, ..., 1$, and

(v) pure 11-edges $(0_1, (k+1)_1)$ of length k and $((k+1)_1, (k+2)_1), ((k+1)_1, (k+3)_1), ..., ((k+1)_1, (2k)_1)$ of lengths $1, 2, ..., k-1$, and

(vi) mixed edges $(0_1, q_0)$ and $(0_1, k_0)$ of lengths $3q+1$ and $k+1$, and $(0_0, 0_1), (0_0, 1_1), ..., (0_0, k_1)$ of lengths $0, 1, ..., k$, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (q-p-1)_0)$ of lengths $2k, 2k-1, ..., 3q+p+2$, and $(0_1, (q+p+1)_0), (0_1, (q+p+2)_0), ..., (0_1, (k-1)_0)$ of lengths $3q-p, 3q-p-1, ..., k+2$, and $((k+1)_1, (k+q-p+1)_0)((k+1)_1, (k+q-p+2)_0), ..., ((k+1)_1, (k+q)_0)$ of lengths $3q+p+1, 3q+p+2, ..., 3q+2$, and $((k+1)_1, (k+q+2)_0)((k+1)_1, (k+q+3)_0), ..., ((k+1)_1, (k+q+p+1)_0)$ of lengths $3q, 3q-1, ..., 3q-p+1$.

If we replace the pure 11-edge $((k+1)_1, (k+q+1)_1)$ of length q by the edge $(0_1, (k+q+1)_1)$ of length q, then we obtain the constructions for every n odd if $3 \leq n \leq k+1$.

Case 3. Let $n = 1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = (k+1)_0, a = 0_0, b = k_1, B = 0_1$.
Then R contains

(vii) pure 00-edges $(0_0, (k+1)_0)$ and $((k+1)_0, (k+q+1)_0)$ of lengths k and q, $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+1$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k)_0)$ of lengths $q-1, q-2, ..., 1$, and

(viii) pure 11-edges $(0_1, k_1)$ of length k and $(0_1, (k+2)_1), ((0_1, (k+3)_1), ..., (0_1, (2k)_1)$ of lengths $k-1, k-2, ..., 1$, and

(ix) mixed edges (k_1, k_0) of length 0 and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1), (0_0, (k+1)_1))$ of lengths $1, 2, ..., k, k+1$, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$. □

Lemma 3.13. *A $(2, 2k, r, s)$-caterpillar of order $4k+2$ and diameter 5 allows a blended ρ-labeling for every $2 < r, s < 2k-1$ and every k odd, $k \geq 3$.*

Proof. By constructions. Let $r = n+2$ and $s = 2k-n$ for $1 \leq n \leq 2k-3$. Furthermore, let R be a $(2, 2k, n+2, 2k-n)$-caterpillar of order $4k+2$ and diameter 5 and $k = 2q+1$.

Case 1. Let n be even, $k+3 \leq n \leq 2k$, and $n-(k+2) = 2p+1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = k_0, a = 0_0, b = 0_1, B = k_1$.

Then R contains

(i) pure 00-edges $(0_0, k_0)$ and $(k_0, (k+q+1)_0)$ of lengths k and $q+1$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+2$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k)_0)$ of lengths $q, q-1, ..., 1$, and

(ii) pure 11-edges $(0_1, k_1)$ and $(0_1, (2k)_1)$ of lengths k and 1, and $(0_1, (k+q-p+1)_1), (0_1, (k+q-p+2)_1), ..., (0_1, (k+q+p+1)_1)$ of lengths $q+p+1, q+p, ..., q-p+1$, and $(k_1, (k+2)_1), (k_1, (k+3)_1), ..., (k_1, (k+q-p)_1)$ of lengths $2, 3, ..., q-p$, and $(k_1, (k+q+p+2)_1), (k_1, (k+q+p+3)_1), ..., (k_1, (2k-1)_1)$ of lengths $q+p+2, q+p+3, ..., k-1$, and

(iii) mixed edges $(0_0, 0_1), (0_0, 1_1), ..., (0_0, (k-1)_1), (0_0, (k+1)_1)$ of lengths $0, 1, ..., k-1, k+1$ and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (k-1)_0), (0_1, (k+1)_0)$ of lengths $2k, 2k-1, ..., k+2, k$.

If we replace the pure 11-edge $(0_1, (k+q+1)_1)$ of length $q+1$ by the edge $(k_1, (k+q+1)_1)$ of length $q+1$, then we obtain the constructions for every n odd if $k+2 \le n \le 2k-1$.

Case 2. Let n be even, $4 \le n \le k+1$, and $n = 2p+1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = k_0, a = 0_0, b = 0_1, B = k_1$.

Then R contains

(iv) pure 00-edges $(0_0, k_0), (k_0, (k+q+1)_0)$ and $(0_0, (2k)_0)$ of lengths $k, q+1$ and 1, and $(0_0, (q-p+1)_0), (0_0, (q-p+2)_0), ..., (0_0, q_0)$ of lengths $q-p+1, q-p+2, ..., q$, and $(0_0, (q+2)_0), (0_0, (q+3)_0), ..., (0_0, (q+p+1)_0)$ of lengths $q+2, q+3, ..., q+p+1$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q-p)_0)$ of lengths $k-1, k-2, ..., q+p+2$, and $(0_0, (k+q+p+2)_0), (0_0, (k+q+p+3)_0), ..., (0_0, (2k-1)_0)$ of lengths $q-p, q-p-1, ..., 2$, and

(v) pure 11-edges $(0_1, k_1)$ and $(0_1, (2k)_1)$ of lengths k and 1, and $(k_1, (k+2)_1), (k_1, (k+3)_1), ..., (k_1, (2k-1)_1)$ of lengths $2, 3, ..., k-1$, and

(vi) mixed edges $(0_1, (q+1)_0)$ and $(0_1, (k+1)_0)$ of lengths $3q+2$ and k, and $(0_0, 0_1), (0_0, 1_1), ..., (0_0, (k-1)_1), (0_0, (k+1)_1)$ of lengths $0, 1, ..., k-1, k+1$, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (q-p)_0)$ of lengths $2k, 2k-1, ..., 3q+p+3$, and $(0_1, (q+p+2)_0), (0_1, (q+p+3)_0), ..., (0_1, (k-1)_0)$ of lengths $3q-p+1, 3q-p, ..., k+2$, and $(k_1, (k+q-p+1)_0), (k_1, (k+q-p+2)_0), ..., (k_1, (k+q)_0)$ of lengths $3q+p+2, 3q+p+1, ..., 3q+3$, and $(k_1, (k+q+2)_0)(k_1, (k+q+3)_0), ..., (k_1, (k+q+p+1)_0)$ of lengths $3q+1, 3q, ..., 3q-p+2$.

If we replace the pure 11-edge $(k_1, (k+q+1)_1)$ of length $q+1$ by the edge $(0_1, (k+q+1)_1)$ of length $q+1$, then we obtain the constructions for every n odd if $5 \le n \le k+2$.

Case 3. Let $n = 1$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = k_0, a = 0_0, b = (k+1)_1, B = 0_1$.

Then R contains

(vii) pure 00-edges $(0_0, k_0)$ and $(k_0, (k+q+1)_0)$ of lengths k and $q+1$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+2$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k)_0)$ of lengths $q, q-1, ..., 1$, and

(viii) pure 11-edges $(0_1, (k+1)_1), ((0_1, (k+2)_1), ..., (0_1, (2k)_1)$ of lengths $k, k-1, ..., 1$, and

(ix) mixed edges $((k+1)_1, (k+1)_0)$ of length 0 and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1), (0_0, (k+1)_1))$ of lengths $1, 2, ..., k, k+1$, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$.

Case 4. Let $n = 3$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = k_0, a = 0_0, b = (k+1)_1, B = 0_1$.

Then R contains

(x) pure 00-edges $(0_0, k_0)$ and $(k_0, (k+q+1)_0)$ of lengths k and $q+1$, and $(0_0, (k+2)_0), (0_0, (k+3)_0), ..., (0_0, (k+q)_0)$ of lengths $k-1, k-2, ..., q+2$, and $(0_0, (k+q+2)_0), (0_0, (k+q+3)_0), ..., (0_0, (2k)_0)$ of lengths $q, q-1, ..., 1$, and

(xi) pure 11-edges $((k+1)_1, (k+q+1)_1)$ and $((k+1)_1, (k+q+2)_1)$ of lengths q and $q+1$, and $(0_1, (k+1)_1), ((0_1, (k+2)_1), ..., (0_1, (k+q)_1)$ of lengths $k, k-1, ..., q+2$, and $(0_1, (k+q+3)_1), ((0_1, (k+q+4)_1), ..., (0_1, (2k)_1)$ of lengths $q-1, q-2, ..., 1$

(xii) mixed edges $((k+1)_1, (k+1)_0)$ of length 0 and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, k_1), (0_0, (k+1)_1))$ of lengths $1, 2, ..., k, k+1$, and $(0_1, 1_0), (0_1, 2_0), ..., (0_1, (k-1)_0)$ of lengths $2k, 2k-1, ..., k+2$.

Case 5. Let $n = 2$. Furthermore, let $V(R) = V_0 \cup V_1, V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = k_0, a = 0_0, b = 0_1, B = k_1$.

Then R contains

(xiii) pure 00-edges $(0_0, 1_0), (0_0, 2_0), ..., (0_0, k_0)$ of lengths $1, 2, ..., k$, and

(xiv) pure 11-edges $(0_1, k_1), (0_1, (k+q)_1)(0_1, (k+q+2)_1)$ and $(k_1, (k+q+1)_1)$ of lengths $k, q+2, q$ and $q+1$, and $(k_1, (k+1)_1), ((k_1, (k+2)_1), ..., (k_1, (k+q-1)_1)$ of lengths $1, 2, ..., q-1$, and $(k_1, (k+q+3)_1), ((k_1, (k+q+4)_1), ..., (k_1, (2k-1)_1)$ of lengths $q+3, q+4, ..., k-1$, and

(xv) mixed edges $(k_0, (2k)_1)$ of length k and $(0_0, 1_1), (0_0, 2_1), ..., (0_0, (k-1)_1)$ of lengths $1, 2, ..., k-1$, and $(k_1, (k+1)_0), (k_1, (k+2)_0), ..., (k_1, (2k)_0)$ of lengths $2k, 2k-1, ..., k+1$. □

By now we have in fact proved Theorem 2.1, as we have covered all cases. We state the proof formally below.

Proof of Theorem 2.1.

(1) An $(r, s, t, 2)$-caterpillar of order $4k + 2$ and diameter 5 does not factorize K_{4k+2} for every $r < 2k$ if $k \geq 2$. It follows from Theorem 3.1.

(2) A $(2k, 2, r, s)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended labeling and therefore it factorizes K_{4k+2} for every $2 < r, s < 2k$ and $k \geq 2$. It follows from Lemma 3.5.

(3) A $(2k, r, 2, s)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended labeling and therefore it factorizes K_{4k+2} for every $2 < r, s < 2k$ and $k \geq 2$. It follows from Lemma 3.4.

(4) A $(2k, r, s, 2)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended labeling and therefore it factorizes K_{4k+2} for every $2 < r, s < 2k - 1$ and $k \geq 2$. It follows from Lemmas 3.6 and 3.7.

(5) A $(2, 2k, r, s)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended labeling and therefore it factorizes K_{4k+2} for every $2 < r, s < 2k$ and $k \geq 2$. It follows from Lemmas 3.12 and 3.13.

(6) An $(r, 2k, 2, s)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended labeling and therefore it factorizes K_{4k+2} for every $2 < r, s < 2k - 1$ and $k \geq 2$. It follows from Lemmas 3.8 and 3.9.

(7) An $(r, 2k, s, 2)$-caterpillar of order $4k + 2$ and diameter 5 allows a blended labeling and therefore it factorizes K_{4k+2} for every $2 < r, s < 2k - 1$ and $k \geq 2$. It follows from Lemmas 3.10 and 3.11.

□

References

1. S. Brandt and M. Wozniak, *On cyclic packing of a tree*, Graphs and Combinatorics **20** (4) (2004), 435–442.
2. P. Eldergill, *Decompositions of the complete graph with an even number of vertices* (1997), M.Sc. thesis, McMaster University, Hamilton.
3. S. El-Zanati and C. Vanden Eynden, *Factorizations of $K_{m,n}$ into spanning trees*, Graphs and Combinatorics **15** (1999), 287–293.
4. D. Fronček, *Cyclic decompositions of complete graphs into spanning trees*, Discussiones Mathematicae Graph Theory **24** (2004), 345–353.
5. D. Fronček, *Bi-cyclic decompositions of complete graphs into spanning trees*, submitted for publication.
6. D. Fronček, *Note on factorization of complete graphs into caterpillars with small diameters*, preprint.
7. D. Fronček and M. Kubesa, *Factorizations of complete graphs into spanning trees*, Congressus Numerantium **154** (2002), 125–134.
8. M. Kubesa, *Spanning tree factorizations of complete graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, accepted for publication.
9. M. Kubesa, *Factorizations of complete graphs into $[r, s, 2, 2]$-caterpillars of diameter 5*, submitted.
10. A. Rosa, *Cyclic decompositions of complete graphs* (1965), Ph.D. thesis, Slovak Academy of Science, Bratislava.

11. A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris, 1967, pp. 349–355.
12. Y. Shibata and Y. Seki, *The isomorphic factorization of complete bipartite graphs into trees*, Ars Combinatoria **33** (1992), 3–25.

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