The Forwarding Index of Antisymmetric Routings

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Abstract: Given a connected graph G with n vertices, a routing R is a collection of $n(n-1)$ paths, one path $R(x,y)$ for each ordered pair x, y of vertices. A routing is said to be *vertex/edge-antisymmetric*, if for every pair x, y of vertices, the paths $R(x,y)$ and $R(y,x)$ are internally vertex/edge-disjoint. Compared to other types of routing found in the literature, antisymmetric routing is interesting from a practical point of view because it ensures greater network reliability. For a given graph G and routing R, the vertex/edge load of G with respect to R is the maximum number of paths passing through any vertex/edge of G. The *vertex/edge-forwarding-index* of a graph is the minimum vertex/edge load taken over all routings. If routing R is vertex/edge-antisymmetric we talk about *antisymmetric-indices*. Several results exist in the literature for the forwarding-indices of graphs. In this paper, we derive upper and lower bounds for the antisymmetric-indices of graphs in terms of their connectivity or minimum degree. These bounds are often the best possible. Whenever this is the case, a network that meets the corresponding bound is described. Several related conjectures are proposed throughout the paper.

1. Introduction

The vertex-forwarding-index was introduced by Chung et al. in [1] to formalize problems arising in the forwarding of information in networks of processors. Since then, it has attracted considerable attention and several variations of it emerged. Bounds on the forwarding-index of general routings were derived in [2], [3], [6], [7]. The forwarding-index of directed networks was dealt with in [8]. The special type of consistent routing was considered in [5]. Constructions of networks with high degree of symmetry and relatively small vertex-forwarding-index were shown in [9]. In [10] it was shown that the problem of deciding whether the forwarding-index of a network is smaller than a specific number is NP-complete. Given a connected graph G with n vertices, a routing R is a collection of $n(n-1)$ paths, one path $R(x,y)$ for each ordered pair x, y of vertices. In this paper, we introduce a new type of routing called antisymmetric routing. The idea is that for each pair x, y of vertices, the paths $R(x,y)$ and $R(y,x)$ are vertex- or edge-disjoint. A *vertex-antisymmetric routing* of a graph G, denoted

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by R_{va}, is a set of *n(n-1)* paths, one path $R_{va}(x,y)$ for each ordered pair x and y of vertices, such that paths $R_{va}(x,y)$ and $R_{va}(y,x)$ are *internally* vertex-disjoint. Similarly, in an *edge-antisymmetric routing*, denoted by R_{ea}, paths $R_{ea}(x,y)$ and $R_{ea}(y,x)$ are edge-disjoint for each ordered pair x, y of vertices. Obviously, each edge *(x,y)* gives rise to two internally vertex-disjoint paths: (x,y) and (y,x) inducing no charge. Therefore, whenever vertex-antisymmetric routings are considered, we will not be concerned with routing the edges of the graph. Compared with other routings found in the literature, antisymmetric routing is interesting from a practical point of view, because it ensures greater network reliability. From a theoretical point of view, we notice that 2-connectivity/2-edge-connectivity is a trivial necessary condition for the existence of vertex/edge-antisymmetric routings in a graph. Furthermore, we know that defining antisymmetric routings in directed graphs is NP-complete, since deciding whether or not any two specified vertices of a digraph belong to the same cycle is an NP-complete problem [4]. The same problem is polynomial for undirected graphs, since it is equivalent to constructing a cycle/closed-walk that contains each pair x, y of vertices. The load $\xi(G,R,x)$ or $\pi(G,R,e)$ of a vertex x or an edge e, with respect to routing R, is the number of paths of R containing vertex x (internally) or edge e, respectively. If R is a routing for which the sum of the path lengths of $R(x,y)$ and $R(y,x)$ is the minimum possible for each pair x, y of vertices, then the corresponding vertex, edge load is denoted by $\xi_m(G,R,x)$, and $\pi_m(G,R,e)$, respectively. The *vertex-forwarding-index*, denoted by $\xi(G)$, and the *edge-forwarding-index*, denoted by $\pi(G)$, of G are defined as the minimum, taken over all possible routings, of the maximum vertex, edge load, respectively. The forwarding-indices of minimum-length routings are denoted by $\xi_m(G)$ and $\pi_m(G)$. Thus:

$\xi(G)=\min_R \max_x \xi(G,R,x)$;

$\xi_m(G)=\min_R \max_x \xi_m(G,R,x)$; $\pi(G)=\min_R \max_e \pi(G,R,e)$;

$\pi_m(G)=\min_R \max_e \pi_m(G,R,e)$.

If we consider vertex/edge-antisymmetric routings, we talk about vertex/edge-antisymmetric-indices, and we denote them by $\xi_{va}(G)$, $\pi_{va}(G)$, $\xi_{ea}(G)$, and $\pi_{ea}(G)$. The antisymmetric-indices of minimum length antisymmetric routings are denoted by $\xi_{va,m}(G)$, $\pi_{va,m}(G)$, $\xi_{ea,m}(G)$, and $\pi_{ea,m}(G)$.

All the upper bounds that hold for antisymmetric-indices hold true for forwarding-indices as well.

In the sequel, *G(V,E)* denotes a simple (i.e., without multiple edges) undirected graph on n vertices with vertex set *V(G)* and edge set *E(G)*. Throughout, we let n denote the number $|V(G)|$ of vertices. By *c(x,y)*, *w(x,y)* we denote the length of the shortest-cycle, shortest-closed-walk, respectively, containing vertices x and y. We let *N(x)* denote the set of neighbors of vertex x. We say that a graph G is k-connected, if the removal of less than k vertices always results in a connected graph. The definition of *k-edge-connectivity* is analogous. For a graph G and a vertex x of G, G-x will denote the graph obtained from G by removing vertex x. Extensive use is made in our proofs of the well known result stated in the following lemma (whose proof is omitted):

Lemma 1.1 *Every 2-connected graph G of minimum degree $\delta\geq 3$ has a vertex x such that G-x is 2-connected.*

The remainder of this paper proceeds as follows: In Sections 2, 3, 4, 5, we derive upper and lower bounds on the antisymmetric-indices for 2-connected graphs, 2-edge-connected graphs, k-connected graphs and graphs with given minimum degree, respectively. Related conjectures are proposed throughout the paper.

2. Bounds for 2-connected graphs

In this section, we establish lower bounds on the antisymmetric-indices of 2-connected graphs.

Proposition 2.1 *Let G be a 2-connected graph of order $n\geq 3$. Then*

$$\frac{1}{2n}\sum_{x\in V}\sum_{y\in V, y\neq x}(c(x,y)-2)\leq \xi_{va}(G)\leq \xi_{va,m}(G)\leq \frac{(n-1)(n-2)}{2}.$$

Proof: Since the minimum over all antisymmetric routings is less than or equal to the minimum over minimum-length antisymmetric routings only, we have obviously: $\xi_{va}(G)\leq \xi_{va,m}(G)$. The maximum number of routes passing through a vertex x in a vertex-antisymmetric routing is at most half the total number of ordered pairs, excluding the pairs containing x. Thus, $\xi_{va}(G)\leq \frac{n(n-1)-2(n-1)}{2}=\frac{(n-1)(n-2)}{2}$. Concerning the lower bound, we notice that for every ordered pair x, y of vertices, *c(x,y)-2* is the minimum load induced on all vertices of G from the paths $R_{va}(x,y)$ and $R_{va}(y,x)$. So the total load induced on G by any antisymmetric routing is at least $\sum_{x\in V}\sum_{y\in V, y\neq x}(c(x,y)-2)$. In the best of cases this load is equally distributed among the vertices of G, so that it is greater than or equal to $\frac{1}{2n}\sum_{x\in V}\sum_{y\in V, y\neq x}(c(x,y)-2)$.

Equality to this lower bound is attained if and only if there exists an antisymmetric routing of shortest cycles that induces a uniform load on the vertices of G.

Q.E.D

The following proposition is the edge-counterpart of Proposition 2.1. We omit its proof since it is identical to that of Proposition 2.1.

Proposition 2.2: *Let G be a 2-connected graph of order n≥3. Then*
$$\frac{1}{2|E(G)|}\sum_{x\in V}\sum_{y\in V, y\neq x} c(x,y)\le \pi_{va}(G)\le \pi_{va,m}(G)\le \frac{n(n-1)}{2}.$$

We now improve the upper bound given in Proposition 2.1 for the vertex-antisymmetric index of 2-connected graphs. The upper bound given below is attained for the cycle C_n.

Theorem 2.1: *Let G be a 2-connected graph of order* $n\ge 3$. *Then* $\xi_{va}(G)\le \frac{(n-2)(n-3)}{2}$.

Proof: The proof proceeds by induction on n. The base of the induction is trivially satisfied for all 2-connected graphs of order 3 and 4, which are basically the cycles C_3 and C_4, and the graph obtained from C_4 by inserting a chord. In order to verify the induction step we distinguish two cases depending on the minimum degree of the graph:

1. Suppose that the minimum degree of G is $\delta\ge 3$. In this case, from Lemma 1.1, there is a vertex x such that $G'=G-x$ is 2-connected. From the

induction hypothesis, there is a routing R'_{va} of G' satisfying

$$\xi_{va}(G',R'_{va}) \le \frac{(n-3)(n-4)}{2} \quad (*)$$

From routing R'_{va} of G', we construct a routing R_{va} of G as follows:

For all pairs of vertices in G, excluding pairs with vertex x, the path in R'_{va} is used. The routes from vertex x to all other vertices are chosen as follows: From vertex x to every vertex v of $G-x-N(x)$, take two vertex-disjoint paths of G for (antisymmetric) routes. From the remark of the introduction, we need not be concerned with the routes from x to its neighbors. Thus, the extra load introduced on a vertex of G due to the routes with endpoint in x is at most $n-3$. Thus,

$$\xi_{va}(G) \le \xi_{va}(G',R'_{va}) + (n-3) \le \frac{(n-3)(n-4)}{2} + n - 3 = \frac{(n-2)(n-3)}{2}$$

.

There is no path passing through vertex x, thus $\xi_{va}(G,R_{va},x) = 0$.

2. Suppose now that the minimum degree is $\delta = 2$. Let x be a vertex of G with degree 2 and set of neighbors $N(x) = \{y,z\}$. Consider the graph G' obtained from G by removing vertex x, and inserting edge (y,z), if that edge does not already exist. Clearly, G' remains 2-connected, and from the assumption of the induction, there is a routing R'_{va} on G' such that (*) is true. From routing R'_{va} on G', we obtain a routing R_{va} on G as above, except that if a path of R'_{va} passes through edge (y,z) of G' (which may not exist in G), it is forwarded through vertex x. The extra load introduced on a vertex of $G-x$ is obviously $n-3$, as in the preceding case. The load introduced on vertex x is no more than the load of one of its neighbors, say y, plus the number of paths with endpoint at y that pass through edge *(y,z)*. The number of paths with endpoint at y that pass through *(y,z)* is $n-3$ at most (we exclude the paths from y to itself and to its neighbors in G, of which there are two, at least).

Thus, $\xi_{va}(G,R,x) \le \xi_{va}(G',R') + n - 3 \le \frac{(n-2)(n-3)}{2}$. Q.E.D

3. Bounds for 2-edge connected graphs

Proposition 3.1 *Let G be a 2-edge-connected graph of order* $n \geq 3$. *Then*

$$\frac{1}{2|E(G)|}\sum_{x\in V}\sum_{y\in V, y\neq x} w(x,y) \leq \pi_{ea}(G) \leq \pi_{ea,m}(G) \leq \frac{n(n-1)}{2}.$$

Proof: Similar to that of Proposition 2.1.

Theorem 3.1: *Let G be a 2-edge-connected graph of order n≥3. Then*

$$\xi_{ea}(G) \leq (n-1)(n-2) - \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof: To prove the bound, consider the following antisymmetric routing:
For each ordered pair *x, y* of vertices, if $(x,y) \notin E(G)$, then the paths $R_{ea}(x,y)$ and $R_{ea}(y,x)$ are defined arbitrarily. Otherwise, we set $R_{ea}(x,y) = (x,y)$ and we define $R_{ea}(y,x)$ arbitrarily. Let *u* be any vertex of *G*. Furthermore, let $G_1, G_2, \ldots, G_p$, be the connected components of *G-u*. Since *G* is simple and 2-edge-connected, each component G_i has at least two vertices, hence, $p \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Since the paths between adjacent vertices of *G-u* do not introduce extra load on *u*, it follows that

$$\xi_{ea}(G, R_{ea}, u) \leq (n-1)(n-2) - \sum_{i=1}^{p} |E(G_i)|$$

$\leq (n-1)(n-2) - (n-1-p)$. Using the upper bound on p, we obtain the bound of the theorem.
Q.E.D

In fact the only property used in the proof of the theorem is that for every edge (x,y) of G, either the ordered pair (x,y) is routed along the edge (x,y) (that is, $R_{ea}(x,y) = (x,y)$), or (y,x) is. This property is clearly satisfied for every minimum length antisymmetric routing. Therefore, the inequality $\xi_{ea,m}(G) \leq (n-1)(n-2) - \left\lfloor \frac{n-1}{2} \right\rfloor$ holds as well.

The upper bound in Theorem 3.1 is attained for the graph consisting of $\frac{n-1}{2}$ disjoint edges, another vertex x and all edges between x and the endpoints of the initial $\frac{n-1}{2}$ disjoint edges.

In all the propositions and theorems of sections 2 and 3 equalities with the lower bounds are attained if and only if there exist antisymmetric routings of shortest-cycles/shortest-closed-walks that load all vertices/edges of the graph equally.

4. Bounds for graphs of given connectivity

In this subsection, we give upper bounds on antisymmetric-indices of k-connected graphs. If G is 2-connected, then by Theorem 2.1, $\xi_{va}(G) \leq \frac{(n-2)(n-3)}{2}$, and this bound is attained for the cycle C_n. For $k \geq 3$, we propose the following conjecture.

Conjecture 4.1 *Let G be a k-connected graph of order $n \geq k+1 \geq 4$. Then $\xi_{va}(G) \leq \left\lceil \frac{(n-k)(n-k-1)}{k} \right\rceil$.*

This bound is attained for the complete bipartite graph $K_{k,n-k}$. Some support to the above conjecture may be found in the following theorem.

Theorem 4.1: *Let G be a k-connected graph of order $n \geq k+1 \geq 3$. Then*

$$\xi_{va}(G) \leq \frac{n(n-1)}{2\left\lfloor \frac{k}{2} \right\rfloor} + n - \frac{n}{\left\lfloor \frac{k}{2} \right\rfloor}.$$

Proof: Set $m = \left\lfloor \frac{k}{2} \right\rfloor$. Since G is k-connected, for every vertex x and every set of m vertices $y_1, y_2, \ldots, y_m$ there exist m cycles $C_1, C_2, \ldots, C_m$ such that for

every $1 \leq i \leq m$, C_i contains x and y_i and for every $j \neq i$, $V(C_i) \cap V(C_j) = \{x\}$. To see this, it suffices to replace each vertex y_i of *S* by two new vertices y_i^1, y_i^2 and add the edges (y_i^1, v) and (y_i^2, v) for all edges $(y_i, v) \in E(G)$. Clearly, the resulting graph is still k-connected. Then by Menger's theorem, there exist *2m* internally vertex-disjoint paths connecting *x* to $y_1^1, y_1^2, y_2^1, y_2^2, \dots, y_m^1, y_m^2$. These paths define the cycles $C_1, C_2, \dots, C_m$ in *G*. An antisymmetric routing can now be defined on *G* as follows:
The vertices of *G* are labeled $v_1, v_2, \dots, v_n$. For *i:=2,…,n* , we apply the following step: partition the set of vertices $v_1, v_2, \dots, v_{i-1}$ into arbitrary subsets of cardinality *m* each (except possibly for one of them of cardinality less than *m*) and for each such subset, construct *m* cycles linking v_i to the subset as above. Clearly, this defines an antisymmetric routing on *G*.
Let us denote by ξ^i the maximum load induced on any vertex of *G* at step *i* of the above procedure. Clearly,

$\xi^i \leq \xi^{i-1} + \dfrac{i-1}{m} + f(i-1)$, where $f(i-1) = 0$ if *i-1* is a multiple of *m* else *f(i-1)=1.*

Calculating ξ^n recursively, we get the bound of the theorem. Q.E.D

Using arguments very similar to those in the proof of Theorem 4.1, we derive the following theorem.

Theorem 4.2: *Let G be a k-connected graph of order n,* $n \geq k+1 \geq 3$. *Then*

$$\pi_{ea}(G) \leq \frac{n(n-1)}{2\left\lfloor \frac{k}{2} \right\rfloor} + n.$$

As every vertex-antisymmetric routing is also edge-antisymmetric, we get the following corollary.

Corollary 4.1 *Let G be a k-connected graph of order* $n \geq k+1 \geq 3$. *Then*

$$\xi_{ea}(G) \leq \frac{n(n-1)}{2\left\lfloor \frac{k}{2} \right\rfloor} + n - \left\lfloor \frac{k}{2} \right\rfloor, \text{ and } \pi_{ea}(G) \leq \frac{n(n-1)}{2\left\lfloor \frac{k}{2} \right\rfloor} + n.$$

In the following conjecture we propose upper bounds on $\pi_{va}(G)$ and $\pi_{ea}(G)$ for k-connected graphs G with $k \geq 3$. Some support to this conjecture can be obtained from the above results. Recall that if G is 2-connected, then, from Proposition 2.2, we know that $\pi_{va}(G) \leq \frac{n(n-1)}{2}$.

Conjecture 4.2: *Let G be a k-connected graph of order n,* $n \geq k+1 \geq 3$. *Then*

$$\pi_{va}(G) \leq \left\lceil \frac{n^2}{2k} \right\rceil, \text{ and } \pi_{ea}(G) \leq \left\lceil \frac{n^2}{2k} \right\rceil.$$

The bounds of this conjecture are attained for the graph consisting of two complete subgraphs of equal orders and a matching of k edges between them.

We conclude this section with the following conjecture for $\xi_{va,m}$ and $\xi_{ea,m}$.

Conjecture 4.3: *Let G be a k-connected graph of order* $n \geq 2k+2$ *and* $k \geq 3$. *Then:*

$$\xi_{va,m}(G) \leq \frac{(n-1)(n-2)-2(k-2)(n-1)}{2}, \text{ and}$$

$$\xi_{ea,m}(G) \leq \frac{(n-1)(n-2)-2(k-2)(n-1)}{2}.$$

If true, this conjecture would be the best possible as shown by the generalized wheel W_r^p of order *p+1*. This graph consists of a cycle of length p, an extra vertex called the center joined to all the vertices of the cycle, and each vertex of the cycle is linked to all the other vertices of the cycle at distance less than *r*. Clearly, W_r^p is k-connected, with $k = 2r+1$.

5. Graphs with given minimum degree

In this section, we prove upper bounds on the antisymmetric-indices of graphs with minimum degree δ for the particular case when $\delta = 3$. A conjecture is formulated for the general case, which, if true, would yield be the best upper bound. The following straightforward lemma turns out to be helpful in our proof.

It mainly states that, in any attempt to prove the conjecture below, we need not be concerned with the vertices of degree δ.

Lemma 5.1 *Let G be 2-connected graph of minimum degree* δ *, and x a vertex of degree* δ *. Then, for every antisymmetric routing R of G, we have:* $\xi(G,R,x) \le \left\lceil \frac{(n-2)(n-\delta-1)}{2} \right\rceil$.

Proof: Clearly, $\xi(G,R,x) \le e(\overline{G})-(n-1-\delta)$, where $e(\overline{G})$ stands for the number of non-adjacent pairs of vertices in G. Bounding $e(\overline{G})$ from above by $\frac{n(n-\delta-1)}{2}$ and substituting appropriately, we get the bound of the lemma.
Q.E..D

Theorem 5.1 *Let G be a 2-connected graph of order n with minimum degree 3 and* $n \ge 4$ *. Then* $\xi_{va}(G) \le \left\lceil \frac{(n-2)(n-4)}{2} \right\rceil$.

Proof: To begin with, let us introduce some more definitions for the purposes of the proof. Let us say that an edge (i, j) is essential if its removal from G either destroys the 2-connectivity of G, or causes the minimum degree to decrease by a unit. Let us say that a graph is critical if all its edges are essential. The proof proceeds by induction on *n*. Let then G be a graph as in the theorem. Since removing edges from G can only deteriorate its antisymmetric forwarding index, we may suppose without loss of generality that G is critical. Let then *x* be a vertex of G such that *G-x* is still 2-connected (the existence of such a vertex is granted by Lemma 1.1). Since the case when *x* has degree *n-2* or more is easily seen to imply that the antisymmetric forwarding index is of G is equal (up to a unit) to that of *G-x*, we may suppose that two vertices at least are not neigbors of *x*. This latter assumption, combined with the 2-connectivity of G, clearly implies that *x* has two vertex-disjoint paths of length two, *xyu* and *xy'v*, from *x* to two vertices *u* and *v*, none of which is a neighbor of *x*. Throughout this proof, vertices *y* and *y'* will be referred to as the "attaching points of *x*".
With these general assumptions in mind, we distinguish now two cases depending on whether or not the degree of *x* is greater than three.

Case 1: $d(x) \ge 4$

As the connectivity of G is not affected by the removal of any edge xy and the of edges of G are all essential, the neighbors of x must all have degree 3. Let y be an attaching point of x as above. Now, as y has three neighbors, two of which (x and u) are not neighbors of x, vertices x and y have no more than one common neighbor, say w (if any). Now, delete vertex x and add all edges zy, for all neighbors z of x other than w. Moreover, if w exists, we create the edge uw as well. Clearly, the resulting graph G_x has order $n-1$ and still has minimum degree 3. From the assumption of the induction, there is an antisymmetric routing R_x of G_x such that:

$\xi(G_x, R_x) \leq \left\lceil \frac{(n-3)(n-5)}{2} \right\rceil$. Now, R_x is extended to a routing R of G in three steps, as follows:

Step 1:

(1) Delete all artificial edges (That is, the edges of G_x that are not in G)

(2) For every route P of R_x that uses an artificial edge of type (y,t), substitute the path yxt for edge (y,t) in P

(3) For every route that uses artificial edge (u,w),substitute the path uyw for edge uw in P

All that remains to be seen now is how to define the routes of G corresponding to the artificial edges of G_x, along with the routes from x to all other vertices. The purpose of the next two steps is to deal with each of these types of routes separately.

Step 2:

(1) Partition the set of $d(x)-2$ artificial edges of type (y,t) into pairs of cardinality two each (except possibly for one subset of cardinality one, if $d(x)-2$ is odd)

(2) For every pair $\{(y,t_1),(y,t_2)\}$ of artificial edges in the partition, find two (internally) vertex-disjoint paths P_1 and P_2 in $G-x$ linking y to t_1 and t_2. Such a pair of paths exists, of course, because $G-x$ is 2-connected. Next, extend P_1 (respectively P_2) into a cycle C_1 of G (respectively C_2) by linking y to t_1 (resp. t_2) along the path yxt_1 (resp. yxt_2). Observe that the two cycles have only vertex y in common (which happens to be of degree 3) and both define antisymmetric routes from y to t_1 and t_2 respectively.

(3) To route artificial edge (u,w) in G in case that w exists, find any path P from u to w which does not pass through y in $G-x$. Recall that $G-x$ is 2-connected, so

P exists. Define then *P* as a route from *u* to *w* in *G* and the path *wyxu* as the route from *w* to *u* in *G*.

Step 3:
For every vertex *t* that is not a neighbor of *x*, find a cycle *C* in *G* containing *x* and *t*, and define *C* as an antisymmetric route from *x* to *t*. Observe that we have a total of *n-d(x)-1* antisymmetric routes in this step.

At termination of the third step, we get an antisymmetric routing *R* of *G* whose load on some vertex *t* are counted as follows. Observe first that since the vertices of degree 3 are taken care of by Lemma 5.1, the neighbors of *x* need not be considered in this case. So, let *t* be a 'non-neighbor' of *x*. Then, in the worst case, *t* is charged the load $\left\lceil \frac{d(x)-2}{2} \right\rceil + 1$ from Step 2. As for Step 3, it induces an additional load of $n-d(x)-2$ on *t* in the worst case (recall that *t* is not a neighbor of *x*, so one of the routes of Step 3 does not charge *t* : the route from *x* to *t*). Step 1 does not create any charge whatsoever. In conclusion the total charge on *t* is no more than:

$$\xi(G,R,t) \le \left\lceil \frac{(n-3)(n-5)}{2} \right\rceil + n-2-d(x) + \left\lceil \frac{d(x)-2}{2} \right\rceil + 1,$$ which can be written as:

$$\xi(G,R,t) \le \left\lceil \frac{(n-3)(n-5)}{2} \right\rceil + n-2-\left\lfloor \frac{d(x)}{2} \right\rfloor \qquad (**)$$

Now, bounding *d(x)* from below by 4 and substituting in (**), we get the bound of the theorem.

The other case and subcases being fairly similar to Case 1, their proofs are only sketched in some cases.

Case 2: d(x)=3

Let x $_2$ x $_3$ be the set of neighbors of *x*. We distinguish now several subcases depending on the structure of *N(x)*.

Subcase a: No neighbor of x has degree 3

Then G-x has minimum degree 3. Applying the assumption of the induction and using Step 3 as in Case 1 gives us the bound of the theorem.

Subcase b: Two pairs of neighbors of x are not edges of G

Suppose without loss of generality that both [illegible] and R' are not edges of G. Denote by (x_2, x_3) the graph obtained from G-x by adding the missing edges X and (x_1, x_3). Clearly, G_x is 2-connected with minimum degree 3. From the induction assumption, there is a routing R_x for G_x such that $\xi(G_x, R_x) \leq \left\lceil \frac{(n-3)(n-5)}{2} \right\rceil$. Now, as in the previous case, R_x is extended to a routing R of G following the same steps 1-3. The count of the loads on vertex t runs as follows:

If t is not a neighbor of x, it receives a load of n-5 from Step 3 and one extra charge from Step 2, in the worst case. Hence, the total load on t would not exceed $\left\lceil \frac{(n-3)(n-5)}{2} \right\rceil + n - 4$, which falls within the bound of our theorem. Unlike the previous case however, we have to consider the case when t is a neighbor of x. In that case precisely, Step 2 induces no charge on t at all, while Step 3 may charge t up to n-4. Thus, once again, the total load does not exceed the bound of the theorem.

Subcase c: All the neighbors of x have degree 3

Following Subcase a, let us suppose that no two pairs of neighbors of x are not edges of G. Let us suppose then, without loss of generality, that both pairs x_1x_2 and x_1x_3 are edges of G (the reader should convince himself that this is indeed the negation of Subcase a, since x has only three neighbors). Notice that x_1 being of degree 3, it has no neighbor outside of $\{x, x_2, x_3\}$. So, x_2 and x_3 are the two attaching points of x, as mentioned above. Therefore, the pair x_2x_3 is not an edge of G. The rest is similar to the proof of Case 1: vertex x is deleted; edges (x_2, x_3) and (x_1, u) are added to G-x to form a new graph G_x, to which the induction assumption is applied. Next, the same steps are executed as in Case 1, except for the following modification of Step 2:

<u>(Modified) Step 2</u>:

(1) Route the pair x_2x_3 in G using the cycle $x_2xx_3x_1x_2$.

(2) To route the artificial edge (x_1, u) in G, find a path P in G-x that does not pass through x_3 and then form the return route from u to x_1 along the path ux_2x_1. The existence of P is granted from the fact that $G-x$ is 2-connected.

Thus modified, Step 2 induces only a unit load on vertex t, whenever t is not a neighbor of x. Therefore, whenever t is not a neighbor of x, the total load on t is no more than $\left\lceil \frac{(n-3)(n-5)}{2} \right\rceil + n - 4$, just as in the preceding case. On the other hand, all the neighbors of x being of degree 3, the load of any of them should not exceed the bound of the theorem, as stated in Lemma 5.1, which proves the case.

Subcase d: Two neighbors of x have degree 3

Let x_2 and x_3 be the attaching points of x, as mentioned above. Recall that the attaching points of x are neighbors of x. Now, as x has two neighbors of degree 3, one of its attaching points (say x_2) must have degree 3. It follows that x_2x_3 is not an edge of G. Let u be a neighbor of x_2 other than x, x_1, x_3, as follows from our definition of an attaching point. Now, the rest of the proof for this case is identical to the previous: vertex x is deleted; edge (x_2, x_3) is added. If x_1 has degree 3 in G, edge (x_1, u) is added to G-x as well, and again, we go through the same steps as in Subcase b. A careful analysis of the inequalities involved in the previous shows that they carry over to this case as well.

Subcase e: Exactly one neighbor of x has degree 3

Suppose that x_1 is the unique neighbor of x of degree 3. If either pair of x_1x_2 or x_1x_3 is not an edge of G, then deleting x and adding the appropriate edge (whichever is missing in G among the two pairs), yields a 2-connected graph of minmum degree 3. Applying induction as in the previous cases, would lead us to the same conclusion again. So, let us suppose that both pairs x_1x_2 and x_1x_3 are edges of G. Now, a close look into the case shows that the pair x_2x_3 should not be an edge of G (otherwise, it would be non-essential, contradicting the fact that G is critical). On the other hand, observe that if we remove both x and x_1 from G and create edge (x_2, x_3), we obtain a 2-connected graph G' of minimum degree 3. From the assumption of the induction, there is an antisymmetric routing R' of G' such that $\xi(G', R') \leq \left\lceil \frac{(n-4)(n-6)}{2} \right\rceil$. Again, we apply the same steps as in Case 1, namely:

Step 1:
(1) Delete all artificial edges
(2) For every route P of R' that uses artificial edge (x_2, x_3), substitute the path $x_2 x x_3$ for edge (x_2, x_3) in P
Step 2:
To route the pair $x_2 x_3$ in G, find a path from x_2 to x_3 in $G - \{x, x_1\}$, and use path $x_3 x x_2$ as a return route.
Step 3:
For every vertex t that is not a neighbor of x, find two cycles C and C' respectively in G (which need not necessarily be vertex-disjoint in any way) containing x and t (respectively x_1 and t). Define C and C' as the antisymmetric routes from x to t and from x_1 to t, respectively.

Now let us count the loads induced by the so-constructed routing R on any vertex t. If t is not a neighbor of x, Step 2 induces a unit charge on t at most. Step 3 induces a charge of $2(n-5)$ in the worst case. Hence, t is charged no more than $\left\lceil \frac{(n-4)(n-6)}{2} \right\rceil + 2(n-5) + 1$, which satisfies the bound of the theorem. On the other hand, if t is a neighbor of x other than x_1, then Step 2 induces no charge on t at all, whereas Step 3 induces a charge of $2(n-4)$. Therefore, the total load of t does not exceed

$$\left\lceil \frac{(n-4)(n-6)}{2} \right\rceil + 2(n-4) = \left\lceil \frac{(n-2)(n-4)}{2} \right\rceil.$$

The other vertices (x_1 and x) all have degree 3, and their charge should therefore not exceed the bound of the theorem, which completes the proof.
Q.E.D

Now, we can formulate our conjecture.

Conjecture 5.1: *For every 2-connected graph of minimum degree* δ, $\xi_{va}(G) \le \left\lceil \frac{(n-2)(n-\delta-1)}{2} \right\rceil$.

If true, the conjecture would yield the best possible upper bound in terms of the minimum degree. Indeed, let *n*, δ and *m* be three non-zero integers satisfying $n = m(\delta-1)+2$. Now, let *G* be the graph consisting of *m* copies of the complete graph $K_{\delta-1}$, two extra vertices *x,y* and all edges between each of *x*, *y* and each $K_{\delta-1}$. In the resulting graph *G*, the load on either of *x* or *y* is equal to $\frac{m(\delta-1)(n-(\delta-1)-2}{2} = \frac{(n-2)(n-\delta-1)}{2}$.

This extremal graph also shows that our theorem is the best possible when the minimum degree is 3.

The remainder of this section deals with edge-antisymmetric forwarding indices of 2-connected graphs with minimum degree δ. Obviously, edge-antisymmetric routings are likely to be more congested than their vertex-antisymmetric counterparts, because, for one thing, we have the additional burden of routing the edges of *G*. So, the number of edges of *G* is expected to contribute substantially toward the count of charges on any given vertex. Still, our intuition here is that, on the average, an edge should contribute only half a charge on any vertex, as suggested by the forthcoming theorem. On the other hand, we feel that the requirement that our routes be only edge-disjoint is not of much help when it comes to minimizing the load on any vertex. In fact, we expect a good edge-antisymmetric routing to be vertex-antisymmetric as well (or close to being so), as in the proof of the following:

Theorem 5.1 *Let G be a 2-connected graph of order n. Then* $\xi_{ea}(G) \le \frac{n(n-1)}{2} - \frac{|E(G)|}{2}$.

Proof: The proof proceeds by induction on *n*, along the same lines as in the foregoing. The base of the induction is trivially verified. Let δ be the minimum degree of *G*. Let us first suppose that $\delta = 2$, and set $N(x) = \{y, z\}$, for some vertex *x* of degree 2. Consider the graph *G'* obtained from *G* by removing *x* and adding the edge *e=(y,z)*, if it is missing in *G*. As *G'* is 2-connected, let *R'* be an edge-antisymmetric of *G'* such that

$\xi_{ea}(G',R') \le \frac{(n-1)(n-2)}{2} - \frac{|E(G')|}{2}$. Then *R'* is extended to a routing *R* of *G* in much the same way as we did before:

(1) Delete the artificial edge *(y,z)* when applicable,

(2) For every route *P* of *R'* that uses the artificial edge *e* (if any), substitute the path *yxz* for *e*

(3) Use a cycle *C* containing *y* and *z* as an antisymmetric route from *y* to *z*

(4) For every vertex *u* other than *x*, find an arbitrary cycle containing *x* and *u*, and use it as an atisymmetric route from *x* to *u*.

Now, let *t* be any vertex. If *t* is not a neighbor of *x*, then step 3 induces the charge *n-2* (at most) on *t*, while step 4 may introduce one extra charge on *t*. On the other hand, if *t* is a aneighbor of *x*, step 4 induces the charge *n-1* (at most) on *t*, and step 4 introduces no charge at all on *t*. Therefore, the total charge on any vertex of *G* is no more than:

$$\xi_{ea}(G,R) \le \frac{(n-1)(n-2)}{2} - \frac{|E(G')|}{2} + n - 1$$

Using inequality $|E(G')| \le |E(G)|$, we get the bound of the theorem.

Supppose now that $\delta \ge 3$, and let *x* be a vertex such that *G-x* is 2-connected, as stated in Lemma 1.1. Then *G-x* has an edge-antisymmetric routing *R"* satisfying the assumption of the induction. We extend *R"* into an edge-antisymmetric routing *R* of *G* as follows:

(a) For every vertex $t \notin N(x) \cup \{x\}$, find a cycle *C(t)* containing *x* and *t* and use it as an antisymmetric route from *x* to *t*

(b) Pick a particular cycle $C(t_0)$ from the cycles of (a), and let *y* and *z* be the two neighbors of *x* in $C(t_0)$. Next, partition the set $N(x)$ into (ordered) pairs of vertices $w_1w_2, w_3w_4, \ldots$, etc, with $w_1 = y$ and $w_2 = z$ (if some vertex *w* of *N(x)* is left over because of $|N(x)|$ being odd, we include singleton $\{w\}$ in the partition as well).

(c) For every pair $S_i = \{w_{2i-1}w_{2i}\}$ of the partition described in (b), consider any vertex $q \in N(x)$ such that $q \notin S_i$. Such a *q* exists, since *x* has degree at least 3. Then, in *G-x* , find two internally vertex-disjoint paths P_1 and P_2 from w_{2i-1} to *q* and from *q* to w_{2i}, respectively. Such a pair exists because *G-x* is 2-connected.

Now, we define the edge-antisymmetric routes for the edges (x, w_{2i-1}) and (x, w_{2i}) as follows:

$R(x, w_{2i-1}) = xw_{2i-1}$ (the edge itself is used as a route from x to w_{2i-1}, $R(w_{2i-1}, x) = P_1x$, $R(x, w_{2i}) = xw_{2i}$, and $R(w_{2i}, x) = P_2x$

(d) If $\{w\}$ is the unique singleton of the partition, find a path *P'* in *G-x* from *w* to w_1 that does not contain w_2 (which is clearly possible from the 2-connectivity of *G-x*).

Now, le us count the charges on a vertex *t*. If *t* is not a neighbor of *x*, the charge induced by (a) is *n-d(x)-1* at most, since the antisymmetric route from *x* to *t* does not charge *t*. The cumulated charge induced by (c) and (d) is no more than $\left\lceil \frac{d(x)}{2} \right\rceil$, since each member of the partition, including the singleton, may introduce no more than a unit charge on any vertex. Thus, the cumulated charge of (a), (c) and (d) is less than or equal to $n-d(x)-1+\left\lceil \frac{d(x)}{2} \right\rceil \leq n-d(x)+\left\lfloor \frac{d(x)}{2} \right\rfloor$. Therefore, the total charge on *t* does not exceed: $\frac{(n-1)(n-2)}{2}-\frac{|E(G-x)|}{2}+n-d(x)+\left\lfloor \frac{d(x)}{2} \right\rfloor$. Using equality $|E(G)| = |E(G-x)| + d(x)$, we get the bound of the theorem. On the other hand, if *t* is neighbor of *x* other than w_1 or w_2, then one antisymmetric route from (a) at least (namely, $C(t_0)$)does not charge *t*. Therefore step (a) induces charge $n-d(x)-1$ at most on *t*. All the other charges being equal, we get the upper bound of the theorem. Now, if $t = w_1$, it is easily seen that the ordered pair w_1w_2 in the above partition does not charge w_1 (although it charges w_2). Therefore, the charge of (c) and (d) on w_1 does not exceed $\left\lfloor \frac{d(x)}{2} \right\rfloor$, while (a) induces the load $n-d(x)$, and the same bound follows. Finally, if $t = w_2$, step (d) does not induce a charge on t, and therefore, (c) and (d) induce no more charge on t than $\left\lfloor \frac{d(x)}{2} \right\rfloor$, and the bound of the theorem follows again.

Q.E.D

We conclude this section with the following corollary:

Corollary 5.1: *For any 2-connected graph G with minimum degree* δ*, we have* $\xi_{ea}(G) \leq \frac{n(n-1-\frac{\delta}{2})}{2}$.

Consistent with our feeling that in edge-antisymmetric routings edges do contribute a substantial charge on any vertex (to the tune of half a charge per edge), we suspect that the best upper bound should take the form: $\frac{n^2}{2}-\frac{\delta n}{4}-O(n)$ where the non-negative constants involved in $O(n)$ are absolute. If true, the bound of our corollary would be nearly optimal.

References

[1] F. Chung, E. Coffman, M. Reiman, and B. Simon, The Forwarding Index of Communication Networks,
IEEE Transactions on Information Theory 33 (1987) 224-232.

[2] M.El Haddad, Y. Manoussakis, and R. Saad Upper Bounds on the Forwarding Indices of Interconnection Networks, *Discrete Math* 286 (2004) pp 233-240.

[3] W. Fernandez de la Vega and Y. Manoussakis, The forwarding index of communication networks with given connectivity, *Discrete Applied Mathematics* 37/38 (1992) 147-155.

[4] S.Fortune, J.Hopcroft, and J. Wyllie, The directed subgraph Homeomorphism Problem, *Theoretical Computer Science* 10 (1980) 111-121.

[5] M.C. Heydemann, J.C. Meyer, J. Opatrny, and D. Sotteau, Forwarding indices of consistent routings and their complexity, *Networks* 24 (1994) 75-82.

[6] M.C. Heydemann, J.C.Meyer, J. Opatrny, and D. Sotteau, On the Forwarding Indices of k-connected Graphs, *Discrete Applied Mathematics* 37/38 (1992) 287-296.

[7] M.C.Heydemann, J.C. Meyer, and D. Sotteau, On the Forwarding Indices of Networks,
Discrete Applied Mathematics 23 (1989) 103-123.

[8] Y.Manoussakis and Z.Tuza, The Forwarding Index of Directed Networks, *Discrete Applied Mathematics* 68 (1996) 279-291.

[9] Y.Manoussakis and Z.Tuza, Optimal Routing in Communication Networks with linearly bounded Forwarding index, *Networks* 28 (1996) 177-180.

[10] R. Saad, Complexity of the forwarding index problem, *SIAM Journal on Discrete Mathematics* 6(3) (1993) 418-427.