4_t-Critical Graphs with Maximum Diameter

Lucas van der Merwe and Marc Loizeaux
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403

Abstract

Let $\gamma_t(G)$ denote the total domination number of the graph G. A graph G is said to be total domination edge critical, or simply γ_t-critical, if $\gamma_t(G+e) < \gamma_t(G)$ for each edge $e \in E(\overline{G})$. We show that, for 4_t-critical graphs G, that is, γ_t-critical graphs with $\gamma_t(G) = 4$, the diameter of G is either 2, 3 or 4. Further, we characterize structurally the 4_t-critical graphs G with *diam* $G = 4$.

Keywords: diameter, total domination, edge addition, edge critical, extremal graphs
AMS subject classification: 05C35, 05C69, 05C75

1 Introduction

A set $S \subseteq V(G)$ of a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of all dominating sets. A *total dominating set* in a graph G is a subset S of $V(G)$ such that every vertex in $V(G)$ is adjacent to a vertex of S. Every graph G without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The *total dominating number* $\gamma_t(G)$ is the minimum cardinality of all total dominating sets. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*, while a total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*. For sets S, $X \subseteq V$, if S dominates X, then we write $S \succ X$, while if S totally dominates X, we write written $S \succ_t X$. If $S = \{s\}$ or $X = \{x\}$, we also write $s \succ_t X$, $S \succ x$, etc. Domination-related concepts not defined here can be found in [1].

The *open neighborhood* of a vertex v is the set of vertices adjacent to v, that is, $N(v) = \{w \mid vw \in E(G)\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. We denote the subgraph induced by a set $S \subseteq V(G)$ as $\langle S \rangle$.

Denote the distance from x to y as $d(x, y)$. If G is a graph with *diam* $G = k$ and $d(u, v) = k$, then we say that u and v are *diametrical vertices.* A shortest u-v path in G is a *diametrical path.* The *eccentricity* of a vertex x of a connected graph G is the number $e(x) = \max_{y \in V(G)} d(x, y)$, the distance between x and a vertex furthest from x. Finally, a *leaf* is a vertex with degree one, and a *support vertex* is a vertex which is adjacent to a leaf.

A graph G is *total domination edge critical*, or just γ_t *-critical*, if $\gamma_t(G + e) < \gamma_t(G)$ for any edge $e \in E(\overline{G}) \neq \emptyset$. Van der Merwe, Mynhardt, and Haynes [4] studied total domination edge critical graphs G where $\gamma_t(G) = 3$. In this paper, we restrict our attention to 4_t-critical graphs G, that is, γ_t-critical graphs G with $\gamma_t(G) = 4$.

It is shown in [4], and we restate it here for emphasis, that the addition of an edge to a graph can change the total domination number by at most two.

Proposition 1 [4] *For any edge* $e \in E(\overline{G})$,

$$\gamma_t(G) - 2 \leq \gamma_t(G + e) \leq \gamma_t(G).$$

Graphs G with the property $\gamma_t(G + e) = \gamma_t(G) - 2$ for any $e \in E(\overline{G})$ are called *supercritical* and are characterized in [3]. For an example of a 4_t-critical graph which is supercritical, consider the cycle on six vertices. It was also shown in [3] that the addition of an edge to vertices at distance two apart can reduce the total domination number by at most one.

This paper is organized as follows: Section 2 identifies several properties of 4_t-critical graphs. Section 3 studies 4_t-critical graphs with endvertices and cutvertices. In section 4 we charaterize the 4_t-critical graphs with diameter four.

2 Some properties of 4_t-critical graphs.

In this section we prove some fundamental properties of 4_t-critical graphs. For 3_t-critical graphs, a property similar to Proposition 2 below can be found in [5].

Proposition 2 *For any 4_t-critical graph G and non-adjacent vertices u and v, either*

1. $\{u,v\} \succ G$, *or*

2. *for either u or v, without loss of generality, say u, $\{w,u,v\} \succ G$, for some $w \in N(u)$ and $w \notin N(v)$, in which case we write $[uw,v] \succ G$, or*

3. *for either u or v, without loss of generality, say u, $\{x,y,u\} \succ G - v$ (but not v), and $\langle\{x,y,u\}\rangle$ is connected. In this case we write $xyu \mapsto v$.*

Proof:

Let u and v be non-adjacent vertices. Then $G+\{uv\}$ is totally dominated by a set S of cardinality 2 or 3, which includes at least one of u or v. If both u and v are in S, then either $S = \{u,v\}$, and we have Case 1, or $S = \{w,u,v\}$, and we have Case 2. (Note that if $w \in N(v)$ as well, then $\{w,u,v\} \succ_t G$, contradicting that $\gamma_t(G) = 4$.)

If only one of u or v is in S, say u, then $S = \{x,y,u\}$ totally dominates $G + \{uv\}$. (Note that if $|S| = 2$, then $S \cup \{w\} \succ_t G$, where $w \in N(v)$, contradicting that $\gamma_t(G) = 4$.) Thus $\{x,y,u\} \succ G - v$, but not v, and $\langle\{x,y,u\}\rangle$ is connected. □

We now determine bounds on the diameter of connected 4_t-critical graphs.

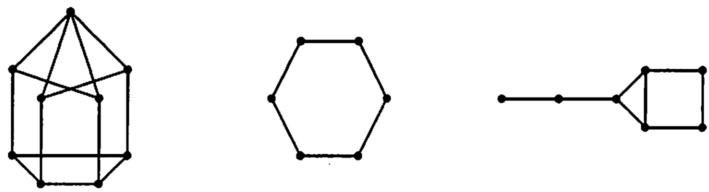
Proposition 3 *If G is a 4_t-critical graph, then*

$$2 \leq diam\ G \leq 4.$$

Proof. Let G be a 4_t-critical graph, and suppose *diam* $G = 5$. Let u_0 and u_5 be diametrical vertices on a diametrical path. Let $S = \{x,y,z\}$ be a γ_t-set of $G + u_0u_5$. If $u_0 = x$ and $u_5 = y$, then z dominates u_2 and u_3. Since S is a total dominating set we may assume without loss of generality that $zu_0 \in E(G)$. But then $dist(u_0,u_3) \leq 2$, contradicting the choice of the $u_0 - u_5$ diametrical path. Thus we may assume that only one of u_0 and u_5 is in S. Without loss of generality, let $u_0 = x$ and let z be the vertex that dominates u_4. Since $\langle\{x,y,z\}\rangle$ is connected, the distance from x to v_5 is at most four again contradicting the choice of a diametrical path. □

Figure 1 gives examples of 4_t-critical graphs of diameter 2, 3, and 4.

The following observation characterizes the disconnected 4_t-critical graphs.



a *b* *c*

Figure 1: 4_t-critical graphs with diameters 2, 3, and 4 respectively.

Observation 4 *If G is a disconnected 4_t-critical graph, then G is the union of two nontrivial complete graphs.*

We now show that 4_t-critical graphs have no forbidden subgraph characterization, that is, any graph G is an induced subgraph of a 4_t-critical graph H. Take two copies of $G \neq K_1$, label them G_1 and G_2, with corresponding vertices $u_1, u_2, \ldots, u_n \in G_1$ and $v_1, v_2, \ldots, v_n \in G_2$. For $i \neq j$, add edge u_iv_j if and only if $u_iu_j \notin E(G_1)$. See Figure 2.

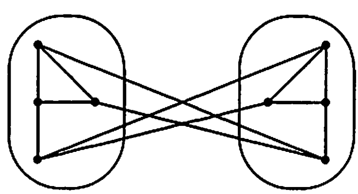


Figure 2: A 4_t-critical graph $H \in \mathcal{F}$.

Theorem 5 *Let $\mathcal{F}$ be the family of graphs constructed as described above. If $H \in \mathcal{F}$, then H is 4_t-critical.*

Proof: If $G = K_n$, then H is the union of two complete graphs, and H is not only 4_t-critical, but supercritical, see [3].

It is easily seen that $\gamma_t(H) \neq 2$. Now suppose that $\gamma_t(H) = 3$. Then, without loss of generality, any $\gamma_t(H)$- set must be of the form $S = \{u_i, v_j, u_k\}$ or $S' = \{u_i, u_j, u_k\}$, with $i \neq j, j \neq k$, and $i \neq k$. In the first case, note that S does not dominate u_j. In the second case, with u_j adjacent to both u_i and u_k, S' does not dominate v_j. Thus we must have $\gamma_t(H) \geq 4$.

Now we will find a total dominating set of cardinality 4. Consider u_i and v_i, and note that u_i dominates $V(G_2) - N[v_i]$ and v_i dominates $V(G_1) - N[u_i]$. Let $u_k \in N(u_i)$ and $v_j \in N(v_i)$. Then $\{u_i, u_k, v_i, v_j\}$ is a total dominating set of cardinality 4. This implies that $\gamma_t(H) \leq 4$, and hence, $\gamma_t(H) = 4$.

Finally, we show that H is 4_t-critical. If the edge u_iu_j is added to H, then $\{u_i, u_j, v_i\}$ dominates $H + u_iu_j$. If we add the edge u_iv_i, then these two vertices form a total dominating set. And if we add the edge u_iv_j, then $\{u_i, v_j, v_i\}$ is a total dominating set. By the symmetry of H, it follows that H is 4_t-critical. □

If $H \in \mathcal{F}$, H is disconnected if and only if G is complete, or G is the union of two complete graphs. If H is connected, then $\text{diam}(H) = 3$, which we state and prove as a lemma below. If $G = \overline{K}_n$, then H is $K_{n,n}$ minus a perfect matching. In particular, if $G = \overline{K}_3$, then H is C_6.

Lemma 6 *If $H \in \mathcal{F}$ and H is connected, then diam$(H) = 3$.*

Proof: First we show that, for any $u_i, u_j \in G_1$, $\text{dist}(u_iu_j) \leq 3$. If there is a shortest path $u_iu_1u_2u_3 \ldots u_j$ in G_1 from u_i to u_j. Then $u_iv_2v_1u_j$ is a path in H from u_i to u_j of length 3. If there is no path from u_i to u_j in G_1, then G_1 consists of two or more components. Suppose G_1 consists of at least three components, say U_1, U_2, $U_3, \ldots$, U_n, with $u_i \in U_1$, $u_j \in U_2$. Then for $u_k \in U_3$, the path $u_iv_ku_j$ is of length 2.

If G_1 consists of two components then at least one of them, say U_2 is not complete. Without loss of generality, suppose $u_i \in U_1$ and $u_j \in U_2$. If u_j is not adjacent to $u_k \in U_2$, then $u_iv_ku_j$ is a path in H from u_i to u_j. If u_j is adjacent to every vertex in U_2, then since U_2 is not complete, there exist $u_r \in U_2$, $u_s \in U_2$, such that $\text{dist}(u_r, u_s) = 2$. Then $u_iv_ru_su_j$ is a path of length 3 from u_i to u_j.

For any $u_i, v_j \in H$, $i \neq j$, $\text{dist}(u_iv_j) \leq 3$: If $u_i \notin N(u_j)$ then $\text{dist}(u_iv_j) = 1$. Now suppose $u_i \in N(u_j)$. Without loss of generality, if there is a $u_k \in N(u_j)$ such that $\text{dist}(u_i, u_k) = 2$, then u_k is adjacent to v_i and there is a path of length 3 from u_i to v_i. If no such u_k exists, find $u_r \in N(u_i)$, $u_s \in N(u_j)$, with $u_r \notin N[u_s]$. (Two such points exist, else G_1 is complete, and H is not connected.) Then $u_iu_rv_sv_j$ is a path of length 3.

Finally, $\text{dist}(u_i, v_i) = 3$: If there is a u_k such that $\text{dist}(u_i, u_k) = 2$, then u_k is adjacent to v_i and there is a path of length 3 from u_i to v_i. If no such u_k exists, find $u_r \in N(u_i)$, $u_s \in N(u_i)$, with $u_r \notin N[u_s]$. Then $u_iu_rv_sv_j$ is a path of length 3. Note that a path of length 2, say $u_iu_rv_i$, implies that

u_r is adjacent to both u_i and v_i, contradicting the construction of H. □

3 Endvertices and Cutvertices

For the remainder of this paper, we restrict our attention to connected graphs. In this section we discuss endvertices and cutvertices in 4_t-critical graphs. It is shown in Haynes, et. al. [2] that no tree is γ_t-critical. Van der Merwe, et. al. [4] show that any 3_t-critical graph has at most one endvertex. They also show that if a 3_t-critical graph G has a cutvertex x, then $G - x$ has exactly two components. We show similar results for the 4_t-critical graphs.

Lemma 7 *If G is a 4_t-critical graph, then G has at most one endvertex.*

Proof: Let u and v be endvertices of a 4_t-critical graph G. By Proposition 11 in [4], u and v do not have a common support vertex. Let r be the support of u and s the support of v, and consider the graph $G + \{uv\}$. Let S be a total dominating set of $G + \{uv\}$ with $|S| = 3$. By Observation 1 (again in [4]) at least one of u or v is in S. Suppose S contains exactly one of u or v. Without loss of generality, let $u \in S$. Then $r \in S$, and for some $k \in G$, $\{r, k\} \succ_t G - v$. But then $\{r, k, s\} \succ_t G$, contradicting the fact that $\gamma_t(G) = 4$.

Now suppose both $u \in S$ and $v \in S$. Then, without loss of generality, $r \in S$, and $r \succ G - \{s, v\}$. Since G is connected, for some $k \in N(s)$, $k \neq v$, $\{r, k\} \succ_t G - v$. But then $\{r, k, s\} \succ_t G$, again a contradiction. □

Figure 3 shows that it is possible for a γ_t-critical graph to have more than one endvertex if $\gamma_t > 4$.

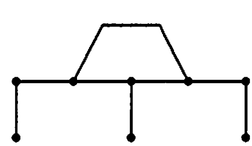


Figure 3: A 5_t-critical graph with more than one endvertex.

Consider now a 4_t-critical graph G with a cutvertex x. This implies that $\text{diam}(G) = 3$ or 4. (If $\text{diam}(G) = 2$, then x dominates G and $\gamma_t(G) < 4$, a contradiction.) We will show that $G - x$ consists of exactly two components.

If $\text{diam}(G) = 3$, one of these components must be K_1, that is, x is adjacent to an endvertex. If $\text{diam}(G) = 4$, then depending on $e(x)$, the eccentricity of x, one of these components will be either K_1 or K_t, with $t \geq 2$. See [4] for parallel results when $\gamma_t = 3$.

Lemma 8 *Let G be a 4_t-critical graph with a cutvertex x. Then $G - x$ consists of exactly two components.*

Proof: Suppose $G - x$ consists of more than two components. Choose 3 components and label them C_1, C_2, and C_3. Since G is connected, we can select vertices $u \in C_1$ and $v \in C_2$, with u and v both adjacent to x. Now u and v are not adjacent, and $\{u, v\} \not\succ G$. Also, $[uw, v] \succ G$ with $w \notin N(v)$ is a contradiction, since w must dominate C_3, implying $w = x$. Finally, $wyu \mapsto v$ implies again (without loss of generality) that $w = x$. But then $\{x, y, u\} \succ_t G$, contradicting the fact that $\gamma_t(G) = 4$. Thus by Proposition 2, $G - x$ consists of two components. □

Lemma 9 *Let G be a 4_t-critical graph with diameter 3 and a cutvertex x. Then one of the components of $G - x$ is K_1.*

Proof: Without loss of generality, every vertex in one component, say C_1, is dominated by x. It is easily seen that C_1 is complete. Suppose $|V(C_1)| > 1$, and let u_1 and u_2 be vertices of C_1. Set $v \in V(C_2)$, with $v \in N(x)$. If $\{u_1, v\} \succ G$ or $[u_1w, v] \succ G$, then $\{u_1, x, v\} \succ_t G$ and $\gamma_t(G) < 4$, a contradiction. If $wyu_1 \mapsto v$, then either $w = x$ or $y = x$, a contradiction since x is adjacent to v. If $wyv \mapsto u_1$, then in order to dominate u_2 we must again have $w = x$ or $y = x$, a contradiction. Hence by Proposition 2, $C_1 = K_1$. □

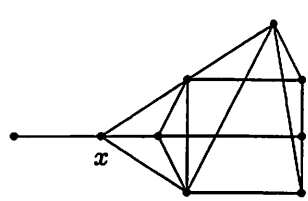


Figure 4: A 4_t-critical graph with diameter 3 and a cutvertex x.

Lemma 10 *Let G be a 4_t-critical graph with diameter 4 and a cutvertex x. Then one of the components of $G - x$ is complete.*

Proof: Since $\text{diam}(G) = 4$, the eccentricity of x is 2 or 3. If $e(x) = 3$, then the proof given in Lemma 9 shows that $C_1 = K_1$.

Now suppose $e(x) = 2$. Then x does not dominate C_1 or C_2. Let $U = V(C_1) \cap N(x)$, and let $S = V(C_2) \cap N(x)$. Also, let $W = V(C_1) - U$ and $T = V(C_2) - S$. It follows easily from Proposition 2 that each of $\langle U \rangle$, $\langle W \rangle$, $\langle S \rangle$, and $\langle T \rangle$ is complete.

Now suppose that for some $u \in U$ and some $w \in W$, $uw \in E(\overline{G})$. In addition, suppose that for some $s \in S$ and some $t \in T$, $st \in E(\overline{G})$. Then a $\gamma_t(G + uv)$-set is $\{u, x, v\}$, where v must be in S and $v \succ T$. Similarly, a $\gamma_t(G + st)$-set is $\{s, x, z\}$, where z must be in U and $z \succ V$. But then $\{z, x, v\} \succ_t G$, contradicting the fact that $\gamma_t(G) = 4$. It follows that C_1 or C_2 is complete. □

The graph in Figure 1(c) illustrates a 4_t-critical graph with diameter four and two cutvertices x_1 and x_2, where $e(x_1) = 3$ and $e(x_2) = 2$. It is clear that (borrowing notation from the proof above) C_1 and C_2 cannot both be complete. In the case where $e(x) = 2$, these 4_t-critical graphs follow a very specific structure.

Lemma 11 *Let G be a 4_t-critical graph with diameter 4 and a cutvertex x with eccentricity two. If C_1 and C_2 are the components of $G - x$, with C_1 complete, and W, U, S, and T are as described previously, then $\langle W \rangle = K_1$, $|S| \geq 2$, $|T| \geq 2$, every vertex in T is adjacent to some vertex in S, and every vertex in S dominates $|T| - 1$ vertices in T.*

Proof: Let $w \in W$, and consider the vertices w and x. Since C_1 is complete, by Proposition 2 we must have $rsx \mapsto w$ for some r and s in C_2, impling that x is adjacent to every vertex in $C_1 - w$. Hence $|W| = 1$.

Now suppose $S = \{s\}$. Then since $\text{diam}(G) = 4$, s is adjacent to every $t \in T$, and C_2 is complete, a contradiction. Hence $|S| \geq 2$. Also, if $|T| = 1$ then since $\langle S \rangle$ is complete there is an $s \in S$ and $c_1 \in V(C_1)$ such that $\{c_1, x, s\} \succ_t G$, contradicting the fact that $\gamma_t(G) = 4$. Hence $|T| \geq 2$.

That every vertex in T is adjacent to some vertex in S follows from the fact that $\text{diam}(G) = 4$. Now let $s \in S$. If s dominates every vertex in T, then $\{c_1, x, s\} \succ_t (G)$, with $c_1 \in V(C_1)$, contradicting the fact that $\gamma_t(G) = 4$. Now suppose there are vertices t_1 and t_2 in T which are not dominated by s. Then there are no vertices r, y such that $rys \mapsto t_1$, contradicting Proposition 2, and hence the fact that $\gamma_t(G) = 4$. Thus every vertex in S dominates $|T| - 1$ vertices in T. □

Lemma 11 leads us to the following characterization: Let $\mathcal{G}$ be the family of diameter four graphs with a cutvertex x such that the removal of x leaves two components, C_1 and C_2, with $|V(C_1)| \geq 2$ and $|V(C_2)| \geq 4$. The cutvertex x dominates $C_1 - w$ for some $w \in V(C_1)$, and C_1 is complete. Set $V(C_2) = S \cup T$, with $|S| \geq 2$ and $|T| \geq 2$, where $S = V(C_2) \cap N(x)$ and $T = V(C_2) - S$. In addition, $\langle S \rangle$ and $\langle T \rangle$ are both complete. Finally, every vertex in T is adjacent to some vertex in S and every vertex in S dominates $|T| - 1$ vertices in T. Figure 5 depicts the family $\mathcal{G}$.

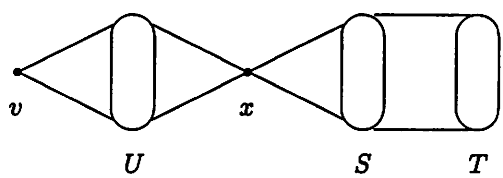


Figure 5: The 4_t-critical family $\mathcal{G}$

Theorem 12 *A graph G with diameter four and a cutvertex with eccentricity two is 4_t-critical if and only if $G \in \mathcal{G}$.*

Proof: Sufficiency follows from Lemma 11.

Now suppose $G \in \mathcal{G}$. By the structure of G it is easy to see that $\gamma_t(G) = 4$. Since every vertex in T has a neighbor in S, it follows that for any $s \in S$ or $t \in T$, each of the graphs $G + vx$, $G + vs$, or $G + vt$ is totally dominated by a set $\{x, s', t'\}$, where $s' = s$ or $t' = t$, and s' is adjacent to t'. Similarly, for $u \in U$, the graphs $G + us$ and $G + ut$ are dominated by $\{u, s', t'\}$.

For any $t \in T$, $\{u, x, t\} \succ_t G + xt$. And finally, if $st \in E(\overline{G})$, then $\{u, x, s\} \succ_t G + st$. Thus G is 4_t-critical. □

4 A Characterization of 4_t-critical Graphs with Diameter Four

The following definition will be helpful in our characterization of 4_t-critical graphs with diameter four. If two adjacent vertices x and y dominate a set S, we say that edge xy *dominates* S, or that xy is a *dominating edge*.

To aid in our characterization, we partition the vertices of the graph in terms of their distances to a selected vertex, as illustrated in Figure 6: Let $v_0 \in V(G)$ be a diametrical vertex, and let V_i be the set of vertices at distance i from v_0. Unless otherwise noted, we will follow the convention that $v_i \in V_i$.

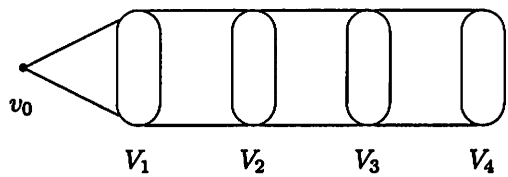


Figure 6: A graph with diameter 4 and diametrical vertex v_0

Let $\mathcal{H}$ be the family of diameter 4 graphs with a diametrical vertex v_0, having the following characteristics:

1. $\langle v_0 \cup V_1 \rangle$ is complete.
2. $\langle V_1 \cup V_2 \rangle$ is complete.
3. $\langle V_4 \rangle$ is complete.
4. $V_3 = A \cup B$ where
   (a) $A = \{a \in V_3 : \ a \text{ dominates } V_4\}$.
   (b) $B = \{b \in V_3 : \ b \text{ dominates } |V_4| - 1 \text{ vertices in } V_4\}$.
   (c) No edge from a vertex in V_2 to a vertex in A dominates V_3.
   (d) If $B \neq \emptyset$, then every $v_4 \in V_4$ is adjacent to some $b \in B$.
   (e) For every $a \in A$, there exists either
      i. $x \in V_3 \cup V_4$ such that $ax \succ_t V_2 \cup V_3 \cup V_4$, or
      ii. v_2, and $x \in V_2 \cup V_3 \cup V_4$ such that $v_2ax \mapsto v_0$.
   (f) For every $a \in A$, there exists either
      i. v_2 such that $\{v_2, a\} \succ V_1 \cup V_2 \cup V_3 \cup V_4$, or
      ii. $x \in V_3 \cup V_4$ such that $ax \succ_t V_3 \cup V_4$, or
      iii. a', v_1, v_2 such that $v_1v_2a' \mapsto a$.
   (g) If $v_3v_3' \notin E(G)$, then without loss of generality $v_1v_2v_3 \mapsto v_3'$.
   (h) If $v_2b \notin E(G)$, then there is an $a \in A$ such that $v_1v_2a \mapsto b$.

(i) For every $b \in B$, (and corresponding $v_4 \in V_4$ such that $bv_4 \notin E(G)$) there exists v_1, v_2 such that $v_1v_2b \mapsto v_4$.

(j) If $v_2a \notin E(G)$, then either $[v_1v_2, a] \succ G$ or $v_1v_2a' \mapsto a$.

(k) For every v_2, v_4, either $[v_1v_2, v_4] \succ G$ or $v_1v_2b \mapsto v_4$ for some $b \in B$.

Note that $\mathcal{G} \subset \mathcal{H}$. We make the observation that the complexity of this characterization resides in the structure of $\langle V_3 \rangle$, and the edges from V_2 and V_4 to V_3. Also, condition (4) leads to several immediate conclusions:

- No vertex in A dominates V_3. (4c)
- $|V_3| \geq 2$. (4c)
- $\langle B \rangle$ is complete. (Referring to (4g), since no vertex in B dominates V_4, we must have $v_3 \notin B$.)
- If $A = \emptyset$, $\langle V_2 \cup V_3 \rangle$ is complete. (4h)
- If $B = \emptyset$, then A is not complete and $|V_2| > 1$. (4c)
- If $|V_4| = 1$, then $B = \emptyset$. (Take (4b) together with (4d).)

The following conclusion is not quite so immediate, so we state it as a lemma.

Lemma 13 *For $b \in B$ and $v_4 \in V_4$, if $v_4 \succ B - b$, then $b \succ V_2$.*

Proof: Suppose there exists $b \in B$ and $v_4 \in V_4$ such that $v_4 \succ B - b$, and suppose also that $v_2b \notin E(G)$ for some $v_2 \in V_2$. By condition (4k) the only possibility is that $v_1v_2b' \mapsto v_4$, for some $b' \in B$. Then $b' \neq b$, because b is not adjacent to v_2. But then $b' \succ V_4 - v_4$, contradicting the fact that $v_4 \succ B - b$. □

The following lemma places restrictions on the sizes of A and B. It also leads us to the conclusion that, if $|B| = 2$, then $|V_2| > 1$ and $\langle V_2 \cup B \rangle$ is complete.

Lemma 14 *For any graph $G \in \mathcal{H}$, either $B = \emptyset$ or $|B| > 1$. Also, if $|A| = 1$, then $|B| > 2$.*

Proof: If $B \neq \emptyset$, then by (4b) and (4d), $|B| > 1$. Now suppose $A = \{a\}$ and $B = \{b_1, b_2\}$. Since b_1 is not adjacent to exactly one vertex in V_4, say v_4, we must have v_4 adjacent to b_2. But then $v_4 \succ B - b_1$, and by Lemma 13, $b_1 \succ V_2$. Likewise, $b_2 \succ V_2$. Then considering that there is a vertex v_2 which is adjacent to a, we have $v_2 a \succ V_3$, contradicting condition (4c). □

In Figure 7 we see two examples of 4_t-critical graphs in $\mathcal{H}$ with $|V_3| = 4$. It is possible to show that these are the only structures for $\langle V_3 \rangle$ when $|V_3| = 4$ and both A and B are nonempty.

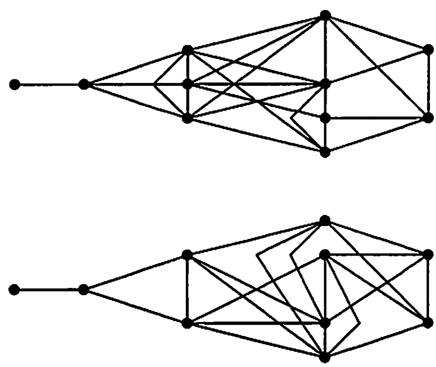


Figure 7: Two 4_t-critical graphs in $\mathcal{H}$ with $|V_3| = 4$. Top: $|A| = 1$, $|B| = 3$. Bottom: $|A| = |B| = 2$.

Theorem 15 *Let G be a graph with diameter 4. If G is 4_t-critical, then $G \in \mathcal{H}$.*

Proof: Consider $st \in E(\overline{G})$. First assume that s and t are in V_1. By Proposition 2 we must have $sv_2v_3 \mapsto t$, with $v_2 \in V_2$ and $v_3 \in V_3$. Now consider the (nonadjacent) vertices v_2 and $v_4 \in V_4$. Again by Proposition 2, with $v_1 \in V_1$, either $[v_1v_2, v_4] \succ G$, or $v_1v_2y \mapsto v_4$. If $[v_1v_2, v_4] \succ G$, then using the fact that $sv_2v_3 \mapsto t$, we must have $\{v_1, v_2, v_3\} \succ_t G$, contradicting that fact that $\gamma_t(G) = 4$. If $v_1v_2y \mapsto v_4$, we can again combine this with the fact that $sv_2v_3 \mapsto t$, again contradicting that $\gamma_t(G) = 4$. Thus $\langle v_0 \cup V_1 \rangle$ is complete.

Next assume that s and t are in V_2. Then $v_1sv_3 \mapsto t$ for some $v_1 \in V_1$ and $v_3 \in V_3$. Considering $\{s, v_4\}$, we see that, with $x \in V_1$, either $[xs, v_4] \succ G$, or $xsy \mapsto v_4$. In either case $\{x, s, v_3\} \succ_t G$. Thus $\langle V_2 \rangle$ is complete. To see that $\langle V_1 \cup V_2 \rangle$ is complete, consider $s \in V_1$ and $t \in V_2$. The only possibility here is $[s, tv_3] \succ G$. But then since t is adjacent to some x in V_1, $\{x, t, v_3\} \succ_t G$, contradicting that $\gamma_t(G) = 4$.

It is clear from Proposition 2 that $\langle V_4 \rangle$ is complete. It is also clear, since G is 4_t-critical, that no edge from a vertex in V_2 to a vertex in A can dominate V_3, and (4c) is proved. To show that $V_3 = A \cup B$, suppose there is a vertex $v \in V_3$, such that v dominates fewer than $|V_4| - 1$ vertices in V_4. Let s and t be vertices in V_4 that are not dominated by v. Then there is an x, y such that $xyv \mapsto s$. But this is a contradiction since this set does not dominate t.

For condition (4d), consider $v_0v_4 \in E(\overline{G})$. Applying Proposition 2, we note that, if v_4 is not adjacent to any $b \in B$, $[wv_4, v_0] > G$ implies $w \in A$, contradicting (4c). Similarly, $v_0v_1v_2 \mapsto G - v_4$ and $v_4v_3v_2 \mapsto G - v_0$ both contradict (4c).

For condition (4e), consider $v_0v_4 \in E(\overline{G})$, and apply Proposition 2. For condition (4f), consider $v_1v_4 \in E(\overline{G})$. Similarly, conditions (4g) through (4k) are immediate consequences of Proposition 2 and the construction of A and B. □

Now consider the families $\mathcal{H}_1 = \{H \in \mathcal{H} : A = \emptyset\}$ and $\mathcal{H}_2 = \{H \in \mathcal{H} : B = \emptyset\}$. Figure 8 shows 4_t-critical graphs in $\mathcal{H}_1$, and Figure 9 shows 4_t-critical graphs in $\mathcal{H}_2$.

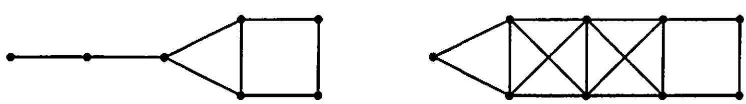


Figure 8: 4_t-critical graphs in $\mathcal{H}_1$.

Figure 9: 4_t-critical graphs in $\mathcal{H}_2$.

Lemma 16 *If $G \in \mathcal{H}_1$, then G is 4_t-critical.*

Proof: It is clear by the construction of $V_3 = B$ that $\gamma_t(G) = 4$. To show that G is 4_t-critical is simply a matter of checking that addition of any of the edges v_0v_2, v_0v_3, v_0v_4, v_1v_3, v_1v_4, or v_2v_4 reduces $\gamma_t(G)$. □

Lemma 17 *If $G \in \mathcal{H}_2$, then G is 4_t-critical.*

Proof: Since any v_1 and v_2 together with any v_3 and v_4 totally dominate G, and no edge from a vertex in V_2 to a vertex in $A = V_3$ dominates V_3, $\gamma_t(G) = 4$. As in the previous proof, it is easy to show that addition of edges v_0v_2, v_0v_3, v_0v_4, v_1v_3, or v_1v_4 reduces $\gamma_t(G)$. By condition (4j), if $v_2a \notin G$ then either $\{v_1, v_2, a\} \succ_t G + v_2a$ or $\{v_1, v_2, a'\} \succ_t G + v_2a$. If $v_3v_3' \notin G$, then by (4g) (without loss of generality) $\{v_1, v_2, v_3\} \succ_t G + v_3v_3'$. And finally, (4k) shows that $G + v_2v_4$ is totally dominated by either $\{v_1, v_2, v_4\}$ or $\{v_1, v_2, b\}$, for some $b \in B$. □

Theorem 18 *If $G \in \mathcal{H}$, then G is 4_t-critical.*

Proof: We assume that $A \neq \emptyset$ and $B \neq \emptyset$. Then any connected set $\{v_1, v_2, b, v_4\}$ totally dominates G. Also, since no b dominates V_4 and no edge v_2a dominates V_3, there is no set $\{v_1, v_2, v_3\}$ that totally dominates G. Hence $\gamma_t(G) = 4$.

Consider addition of each of the following edges:

v_0v_2: If v_2 is adjacent to some $b \in B$, then $\{v_2, b, v_4\} \succ_t G + v_0v_2$ Otherwise, select some $b \in B$, and by condition (4h), there exists $a \in A$ such that $v_1v_2a \mapsto b$. Then choose v_4 such that $\{v_2, a, v_4\} \succ_t G + v_0v_2$.

v_0v_3: If $v_3 = b \in B$, then $\{v_2, v_3, v_4\}$ totally dominates $G + v_0v_3$. If $v_3 = a \in A$, then by condition (4e), there exists $x \in V_2 \cup V_3 \cup V_4$, v_2 such that $\{v_2, v_3, x\} \succ_t G + v_0v_3$.

v_0v_4: Since $\langle B \rangle$ is complete, $\{v_4, b, v_2\} \succ_t G + v_0v_4$.

v_1v_3: If $v_3 = b$, then $\{v_1, v_3, v_4\} \succ_t G + v_1v_3$. If $v_3 = a$, then by condition (4f), there exists v_2 such that $\{v_2, v_1, a\} \succ_t G + v_0v_3$ (case i), or there exists $x \in V_3 \cup V_4$ such that $\{v_1, a, x\} \succ_t G + v_0v_3$ (case ii), or there exists a' such that $\{v_1, v_2, a'\} \succ_t G + v_0v_3$ (case iii).

v_1v_4: Then there is a $b \in B$ such that $\{v_1, v_4, b\} \succ_t G + v_1v_4$.

v_2v_3: If $v_3 = b \in B$, then by condition (4h) there is an $a \in A$ such that $\{v_1, v_2, a\} \succ_t G + v_2v_3$. If $v_3 = a \in A$, then by condition (4j), either $\{v_1, v_2, a\} \succ_t G + v_2v_3$, or $\{v_1, v_2, a'\} \succ_t G + v_2v_3$, for some $a' \in A$.

v_2v_4: By condition (4k), either $\{v_1, v_2, v_4\} \succ_t G + v_2v_4$, or $\{v_1, v_2, b\} \succ_t G + v_2v_4$ for some $b \in B$.

v_3v_3': By condition (4g), without loss of generality, $\{v_1, v_2, v_3\} \succ_t G + v_3v_3'$.

v_3v_4: This implies that $v_3 \in B$. Then by condition (4i), there exists v_1 and v_2 such that $\{v_1, v_2, v_3\} \succ_t G + v_3v_4$. □

One may have noticed on examination of Figure 7 that in each of these graphs, there is an edge from A to B. It is then natural to ask if this is always the case: if A and B are both nonempty, is an edge from A to B required? The answer is no, as evidenced by Figure 10. It looks complex, but this is a very structured graph.

Note that since $\gamma_t(G) = 4$ and there are no edges from A to B, we must have a vertex in V_4 which dominates B. In addition, there can be no vertices v_4 such that $v_4 \succ B - b$. (If so, considering $G + ab$, we see that G is not 4_t-critical .) This in turn implies that $|B| \geq 4$. (If $B = \{b_1, b_2, b_3\}$ then there must be at least one vertex v_4 that is not adjacent to two vertices in B, say b_1 and b_2. But then v_4 is adjacent to b_3, so there exists $v_4' \neq v_4$ which is not adjacent to b_3. But v_4' is adjacent to both b_1 and b_2. Hence $v_4' \succ B - b_3$.)

In Figure 10, $|A| = 1$, and $\langle V_2 \cup A \rangle$ is complete. In fact A can be replaced by any complete graph as long as $\langle V_2 \cup A \rangle$ is complete. This in turn implies that each v_2 is adjacent to all but one vertex $b \in B$, otherwise consider $G + ab$. Thus $|V_2| \geq |B| \geq 4$, and if $|V_2| = |B|$, $\langle V_2 \cup B \rangle$ is complete minus a perfect matching.

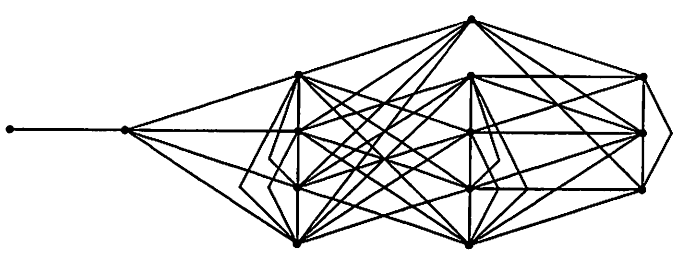


Figure 10: A 4_t-critical graph in $\mathcal{H}$ with no edges between A and B.

Acknowledgements. The authors thank the referees, whose suggestions contributed much to the improvement of this paper.

References

[1] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York (1998).

[2] T. W. Haynes, M. A. Henning, and L. C. van der Merwe, Domination and total domination critical trees with respect to relative complements, *Ars Combin.* 59 (2001) 117–127.

[3] T. W. Haynes, C. M. Mynhardt, and L. C. van der Merwe, Criticality index of total domination, *Congr. Numer.* 131 (1998) 67–73.

[4] L. C. van der Merwe, C. M. Mynhardt, and T. W. Haynes Total domination edge critical graphs, *Utilitas Math.* 54 (1998) 229–240.

[5] L. C. van der Merwe, C. M. Mynhardt, and T. W. Haynes Total domination edge critical graphs with maximum diameter, *Discuss. Math; Graph Theory* 21 (2001) 187–205.