Enumeration of all $(2k+1, k, k-1)$-NRBIBDs for $3 \leq k \leq 13$

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Abstract

A backtracking over near parallel classes with an early isomorph rejection is carried out to enumerate all the near resolvable $(2k+1, k, k-1)$ balanced incomplete block designs for $3 \leq k \leq 13$. We first prove some results which enable us to restrict the search space of near parallel classes. The number of nonisomorphic designs is equal to 1 for each $3 \leq k \leq 8$ and there are respectively 2, 0, 19, 8 and 374 nonisomorphic designs for $k = 9$, 10, 11, 12 and 13.

1 Introduction

An *incomplete block design* (IBD) is a pair $(V, \mathcal{D})$ where V is a v-set of symbols and $\mathcal{D}$ is a collection of k-subsets of V called *blocks* where $k < v$. A balanced incomplete block design (BIBD) is an IBD $(V, \mathcal{D})$ such that each pair of V is contained in exactly λ blocks. We denote such design as a (v, k, λ)-BIBD.

A *near parallel class* on an IBD $(V, \mathcal{D})$, with respect to a symbol s, is a set of blocks that partitions the set $V - \{s\}$ into $(v-1)/k$ blocks of that design. We call s the *missing* symbol of this class.

A *near resolvable incomplete block design* (NRIBD) is an IBD $(V, \mathcal{D})$ whose blocks can be partitioned into v near parallel classes, and every

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symbol of V is missing from exactly one class (see, for example, [1]). Two necessary conditions for the existence of a (v, k, λ)-NRBIBD are $v \equiv 1 \pmod k$ and $\lambda \equiv 0 \pmod{k-1}$.

Two (v, k, λ)-BIBDs are *isomorphic* if there exists a bijection between the symbol sets that maps blocks onto blocks; such a bijection is called an *isomorphism.* An *automorphism* of a design is an isomorphism of the design onto itself. The *(full) automorphism group* of a design consists of all its automorphisms with composition of permutations as the group operation. In this work, we use a backtracking over near parallel classes with partial isomorph rejection to enumerate all nonisomorphic near resolvable $(2k+1, k, k-1)$-BIBDs for $3 \leq k \leq 13$. Kaski and Östergård in [3] and [6] have enumerated some near resolvable incomplete block designs.

This paper is organized into six sections. Section 1 is this introduction. In Section 2 we define the intersection matrix of any two near parallel classes of a $(2k+1, k, k-1)$-NRBIBD and we determine the structure of these matrices. In Section 3 we define 2-concurrence (3)-designs (these designs are a generalization of the 2-concurrence designs defined in [2]). Then we prove that any $(2k+1, k, k-1)$-NRBIBD is also a 2-concurrence (3)-design. In Section 4 we investigate the structure of $(2k+1, k, k-1)$-NRBIBDs in order to restrict the search space. Section 5 outlines the backtracking over parallel classes, with partial isomorph rejection, that was used to enumerate such designs. The last section enumerates all $(2k+1, k, k-1)$-NRBIBDs for $3 \leq k \leq 13$.

2 Intersection Pattern

In this section terminology is similar to the one developed in [7, 5, 8].

Let $\{\mathcal{N}_{t_0}, \ldots, \mathcal{N}_{t_{2k}}\}$ be the near parallel classes of a $(2k+1, k, k-1)$-NRBIBD where t_i is the missing symbol of the near parallel class $\mathcal{N}_{t_i}$ with blocks $\mathcal{N}_{t_i} = (B_{t_i,1}, B_{t_i,2})$. So $V = \{t_0, \ldots, t_{2k}\}$. To simplify notation, $\mathcal{N}_{t_i}$ will be denoted as $\mathcal{N}_i$. For two different symbols, i and j, $B_{i,p(j)}$ denotes the block of $\mathcal{N}_i$ containing j (i.e $j \in B_{i,p(j)}$); the other block of $\mathcal{N}_i$ is denoted by $B_{i,n(j)}$. For two different near parallel classes $\mathcal{N}_i$ and $\mathcal{N}_j$ we define the *near parallel class intersection matrix* (NPCIM) as the 2×2 matrix $A(i,j) = ((a(i,j)_{s,t})$, where

$$a(i,j)_{s,t} = |B_{i,s} \cap B_{j,t}|. \tag{1}$$

Clearly, $0 \leq a(i,j)_{s,t} \leq k$.

Lemma 1. *Let $\mathcal{D}$ be a $(2k+1, k, k-1)$-NRBIBD. Let $B_{i,s}$ be a block of the near parallel class $\mathcal{N}_i$ of $\mathcal{D}$. Then*

(*i*) *For any near parallel class $\mathcal{N}_j$ $(j \neq i)$ of $\mathcal{D}$ we have*

$$a(i,j)_{s,1} + a(i,j)_{s,2} = \begin{cases} k-1, & j \in B_{i,s} \\ k, & j \notin B_{i,s} \end{cases}$$

(*ii*) *There are exactly k near parallel classes of $\mathcal{D}$ for each one of the two cases of statement (i).*

Proof. Statement (*i*) follows from the next simple observation

$$B_{i,s} \cap (B_{j,1} \cup B_{j,2}) = B_{i,s} \cap (V - \{j\}) = \begin{cases} B_{i,s} - \{j\}, & j \in B_{i,s} \\ B_{i,s}, & j \notin B_{i,s} \end{cases}$$

Let $B_{i,s} = \{x_1, \ldots, x_k\}$ and let $\{y_1, \ldots, y_k\}$ be the other block of $\mathcal{N}_i$. Hence $B_{i,s} \cap (V - \{x_\ell\}) = B_{i,s} - \{x_\ell\}$ and $B_{i,s} \cap (V - \{y_\ell\}) = B_{i,s}$ for $1 \leq \ell \leq k$. These equalities prove (*ii*). □

Lemma 2. *Let $\mathcal{N}_i$ and $\mathcal{N}_j$ be two different near parallel classes of a $(2k+1, k, k-1)$-NRBIBD. Then, the NPCIM $A(i,j)$ $(i \neq j)$ satisfies*

$$(k-2)\binom{k}{2} = \sum_{j \neq i} \sum_{t=1}^{2} \binom{a(i,j)_{s,t}}{2}. \tag{2}$$

Proof. Let $B_{i,s}$ be a block of the near parallel class $\mathcal{N}_i$. Since $\lambda = k-1$, the left hand side of (2) gives the number of occurrences of the pairs of symbols belonging to $B_{i,s}$ into the other blocks of the design. On the other hand, $\binom{a(i,j)_{s,t}}{2}$ is the number of pairs of symbols belonging to the set $B_{i,s} \cap B_{j,t}$, for each block $B_{j,t}$ $(j \neq i)$. Therefore,

$$\sum_{j \neq i} \sum_{t=1}^{2} \binom{a(i,j)_{s,t}}{2}$$

is also the number of occurrences of the pairs of symbols of $B_{i,s}$ into the other blocks of the design. This proves formula (2). □

Lemma 3. *For k even, we have that*

$$\binom{a}{2}+\binom{k-a}{2}\geq\binom{\frac{k}{2}}{2}+\binom{\frac{k}{2}}{2} \;=\; \frac{k^2}{4}-\frac{k}{2}, \quad \text{for } 0\leq a\leq k. \tag{3}$$

$$\binom{a}{2}+\binom{k-1-a}{2}\geq\binom{\frac{k-2}{2}}{2}+\binom{\frac{k}{2}}{2} \;=\; \frac{k^2}{4}-k+1, \quad \text{for } 0\leq a\leq k-1. \tag{4}$$

For k odd, we have

$$\binom{a}{2}+\binom{k-1-a}{2}\geq\binom{\frac{k-1}{2}}{2}+\binom{\frac{k-1}{2}}{2} \;=\; \frac{k^2}{4}-k+\frac{3}{4}, \quad \text{for } 0\leq a\leq k-1. \tag{5}$$

$$\binom{a}{2}+\binom{k-a}{2}\geq\binom{\frac{k-1}{2}}{2}+\binom{\frac{k+1}{2}}{2} \;=\; \frac{k^2}{2}-\frac{k}{2}+\frac{1}{4}, \quad \text{for } 0\leq a\leq k. \tag{6}$$

Proof. Let $f(a)=\binom{a}{2}+\binom{k-a}{2}$ and $g(a)=\binom{a}{2}+\binom{k-1-a}{2}$ be two functions defined on $[0,k]$ and $[0,k-1]$, respectively. Then, using a standard calculus argument, we can show that for k even, $f(a)\geq\frac{k^2}{4}-\frac{k}{2}$ for all $a\in[0,k]$ and $g(a)\geq\frac{k^2}{4}-k+1$ for all integer $a\in[0,k-1]$. The case k odd is analogous. □

Lemma 4. *Let $\mathcal{N}_i$ and $\mathcal{N}_j$ be two different near parallel classes of a $(2k+1,k,k-1)$-NRBIBD. Then, up to permutation of rows and columns, the NPCIM $A(i,j)$ has the form*

$$(i)\ \begin{bmatrix}\frac{k}{2} & \frac{k-2}{2}\\ \frac{k}{2} & \frac{k}{2}\end{bmatrix} \text{ for } k \text{ even}, \quad (ii)\ \begin{bmatrix}\frac{k-1}{2} & \frac{k-1}{2}\\ \frac{k-1}{2} & \frac{k+1}{2}\end{bmatrix} \text{ for } k \text{ odd.}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned}\frac{k^3}{2}-\frac{3k^2}{2}+k &= (k-2)\binom{k}{2}=\sum_{j\neq i}\sum_{t=1}^{2}\binom{a(i,j)_{s,t}}{2}\\ &= \sum_{j\neq i}\left[\binom{a(i,j)_{s,1}}{2}+\binom{a(i,j)_{s,2}}{2}\right].\end{aligned} \tag{7}$$

By Lemma 1 there exists a permutation $\ell_1,\dots \ell_k,\ell_{k+1},\dots \ell_{2k}$ of the set $\{0,\dots 2k\}-\{i\}$ such that $a(i,\ell_j)_{s,1}+a(i,\ell_j)_{s,2}=k$ for $j=1,\dots k$ and $a(i,\ell_j)_{s,1}+a(i,\ell_j)_{s,2}=k-1$ for $j=k+1,\dots 2k$. So by (7) we have that

$$\frac{k^3}{2}-\frac{3k^2}{2}+k=\sum_{j=1}^{k}\left[\binom{a(i,\ell_j)_{s,1}}{2}+\binom{k-a(i,\ell_j)_{s,1}}{2}\right]$$
$$+\sum_{j=k+1}^{2k}\left[\binom{a(i,\ell_j)_{s,1}}{2}+\binom{k-1-a(i,\ell_j)_{s,1}}{2}\right]. \quad (8)$$

However, for k even, by (4) and (5) we get

$$\sum_{j=1}^{k}\left[\binom{a(i,\ell_j)_{s,1}}{2}+\binom{k-a(i,\ell_j)_{s,1}}{2}\right]\geq k\left[\frac{k^2}{4}-\frac{k}{2}\right], \text{ and} \quad (9)$$

$$\sum_{j=k+1}^{2k}\left[\binom{a(i,\ell_j)_{s,1}}{2}+\binom{k-1-a(i,\ell_j)_{s,1}}{2}\right]\geq k\left[\frac{k^2}{4}-k+1\right]. \quad (10)$$

But

$$k\left[\frac{k^2}{4}-\frac{k}{2}\right]+k\left[\frac{k^2}{4}-k+1\right]=\frac{k^3}{2}-\frac{3k^2}{2}+k. \quad (11)$$

Then, by equalities (8) and (11) the inequalities (9) and (10) are actually equalities. It follows from (4) and equality (9) that

$$\binom{a(i,\ell_j)_{s,1}}{2}+\binom{k-a(i,\ell_j)_{s,1}}{2}=\binom{\frac{k}{2}}{2}+\binom{\frac{k}{2}}{2}, \quad \text{for} \quad j=1,\dots,k.$$

Solving this quadratic equation for $a(i,\ell_j)_{s,1}$, we get that $a(i,\ell_j)_{s,1}=\frac{k}{2}$.

Since $a(i,\ell_j)_{s,2}=k-a(i,\ell_j)_{s,1}$, we have $a(i,\ell_j)_{s,2}=\frac{k}{2}$. Now using (5) and equality (10), we have that either $a(i,\ell_j)_{s,1}=\frac{k}{2}$ and $a(i,\ell_j)_{s,2}=\frac{k-1}{2}$, or $a(i,\ell_j)_{s,1}=\frac{k-1}{2}$ and $a(i,\ell_j)_{s,2}=\frac{k}{2}$. Thus we have showed (i). The case k odd is proved similarly. □

3 2-concurrence (t)-Designs

An equi-replicate incomplete block (EIB) design is an IBD $(V,\mathcal{D})$ such that every symbol occurs in r blocks. We shall refer to the symbols $x_1,\dots,x_t$ as i-th associates if they occur together in λ_i blocks. Then, we define a *2-concurrence (t)-design* (see [2], for the case $t=2$) as an EIB design satisfying the conditions below:

(1) Any t symbols are either first or second associates.

(2) For each symbol x there are exactly n_i subsets $\{y_1, \ldots, y_{t-1}\}$ such that the symbols $x, y_1, \ldots, y_{t-1}$ are i-th associates, the number n_i being independent of x.

The numbers v, b, r, k, λ_1, λ_2, n_1 and n_2 are called the parameters of the design. Using a standard double counting argument we can prove that these parameters satisfy: $vr = bk$,

$$n_1 + n_2 = \binom{2k}{2}, \tag{12}$$

$$n_1\lambda_1 + n_2\lambda_2 = 2k\binom{k-1}{2}. \tag{13}$$

Theorem 5. *Any $(2k+1, k, k-1)$-NRBIBD is also a 2-concurrence (3)-design with parameters $v = 2k+1$, $b = 2v$, $r = 2k$ and*

$$\lambda_1 = \frac{k-2}{2},\ \lambda_2 = \frac{k-4}{2},\ n_1 = \frac{k(k-2)}{2},\ n_2 = \frac{3k^2}{2}\ \textit{for } k \textit{ even},$$

and

$$\lambda_1 = \frac{k-1}{2},\ \lambda_2 = \frac{k-3}{2},\ n_1 = \binom{k+1}{2}\ n_2 = 3\binom{k}{2}\ \textit{for } k \textit{ odd}.$$

Proof. We will prove that every 3-subset of V belongs to λ_1 or λ_2 blocks. Since each near parallel class of every $(2k+1, k, k-1)$-NRBIBD has two blocks, it follows that for any three symbols of V it holds one and only one of the following four possibilities with respect to each near parallel class $\mathcal{N}$:

(a) All three symbols belong to the same block of $\mathcal{N}$.

(b) Two of them belong to one block and the third symbol belongs to the other block of $\mathcal{N}$.

(c) Two of them belong to one block and the third symbol is the missing symbol of $\mathcal{N}$.

(d) Two symbols occur in different blocks and the third one is the missing symbol of $\mathcal{N}$.

Now, let x_1, x_2, x_3 and x_4 be the number of near parallel classes satisfying (a), (b), (c) and (d), respectively. Since our design is a $(2k+1, k, k-1)$-NRBIBD, we have

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2k+1, \\ 3x_1 + x_2 + x_3 &= 3(k-1), \\ x_3 + x_4 &= 3. \end{aligned}$$

The above system has the following solution sets

$$x_1 = \frac{k-2}{2},\ x_2 = \frac{3k-2}{2},\ x_3 = 1,\ x_4 = 2, \tag{14}$$

$$x_1 = \frac{k-4}{2},\ x_2 = \frac{3k}{2},\ x_3 = 3,\ x_4 = 0, \tag{15}$$

for k even, and for k odd

$$x_1 = \frac{k-1}{2},\ x_2 = \frac{3k-3}{2},\ x_3 = 0,\ x_4 = 3, \tag{16}$$

$$x_1 = \frac{k-3}{2},\ x_2 = \frac{3k-3}{2},\ x_3 = 2,\ x_4 = 1. \tag{17}$$

Hence, $\lambda_1 = \frac{k-2}{2}$ and $\lambda_2 = \frac{k-4}{2}$ for k even, and for k odd, $\lambda_1 = \frac{k-1}{2}$ and $\lambda_2 = \frac{k-3}{2}$.

Solving the system of linear equations (12)-(13) with $v = 2k+1$ and $r = 2k$, we get the desired values for n_1 and n_2. □

Corollary 6. *For any $(2k+1, k, k-1)$-NRBIBD we have, for k even,*

(jss) Any triple $\{a, b, c\} \subset V$ occurs λ_1 times iff

(a) $B_{a,p(b)} = B_{a,p(c)}$, $B_{b,p(a)} \neq B_{b,p(c)}$ and $B_{c,p(a)} \neq B_{c,p(b)}$,

(b) $B_{a,p(b)} \neq B_{a,p(c)}$, $B_{b,p(a)} \neq B_{b,p(c)}$ and $B_{c,p(a)} = B_{c,p(b)}$ or

(c) $B_{a,p(b)} \neq B_{a,p(c)}$, $B_{b,p(a)} = B_{b,p(c)}$ and $B_{c,p(a)} \neq B_{c,p(b)}$.

(jjj) Any triple $\{a, b, c\} \subset V$ occurs λ_2 times iff $B_{a,p(b)} = B_{a,p(c)}$, $B_{b,p(a)} = B_{b,p(c)}$ and $B_{c,p(a)} = B_{c,p(b)}$.

And, for k odd,

(sss) Any triple $\{a, b, c\} \subset V$ occurs λ_1 times iff $B_{a,p(b)} \neq B_{a,p(c)}$, $B_{b,p(a)} \neq B_{b,p(c)}$ and $B_{c,p(a)} \neq B_{c,p(b)}$.

(jjs) Any triple $\{a, b, c\} \subset V$ occurs λ_2 times iff

(a) $B_{a,p(b)} \neq B_{a,p(c)}$, $B_{b,p(a)} = B_{b,p(c)}$ *and* $B_{c,p(a)} = B_{c,p(b)}$,

(b) $B_{a,p(b)} = B_{a,p(c)}$ $B_{b,p(a)} \neq B_{b,p(c)}$ *and* $B_{c,p(a)} = B_{c,p(b)}$, *or*

(c) $B_{a,p(b)} = B_{a,p(c)}$, $B_{b,p(a)} = B_{b,p(c)}$ *and* $B_{c,p(a)} \neq B_{c,p(b)}$.

Proof. Suppose that k is even. Assume that the triple $\{a, b, c\}$ occurs exactly $\lambda_1 = \frac{k-2}{2}$ times in the design. Then by (14) we have $x_3 = 1$ and $x_4 = 2$ (here we use the notation of the previous lemma). However, $x_3 = 1$ means that there is a near parallel class, say $\mathcal{N}_a$, such that the other two symbols, namely b and c, belong to the same block of $\mathcal{N}_a$. That is, $B_{a,p(b)} = B_{a,p(c)}$. Therefore, $x_4 = 2$ means now that $B_{c,p(a)} \neq B_{c,p(b)}$ and $B_{b,p(a)} \neq B_{b,p(c)}$. Thus we have proved $(jss)(a)$. The other cases are proved similarly. □

4 Initial Structures

In this section we give the initial structure for any $(2k+1, k, k-1)$-NRBIBD and some results that will restrict the search space.

Theorem 7. *For any* $(2k+1, k, k-1)$*-NRBIBD, up to isomorphism, we have*

(i) For k even (resp. k odd) the near parallel classes $\mathcal{N}_0$ and $\mathcal{N}_1$ are given in the first and the second rows of Table 1 (resp. 2).

(ii) For the near parallel class $\mathcal{N}_i$ $(2 \leq i \leq 2k)$, for k even (resp. k odd), $\mathcal{N}_i \cap \{0, 1, 2\}$ is given in the corresponding row of Table 1 (resp. Table 2).

(This structure is called the initial structure for the design. It is denoted by $\mathcal{E}$ for k even and $\mathcal{F}$ for k odd).

Proof. Assume that k is even. ¿From Theorem 5, we assume without loss of generality that the triple $\{0, 1, 2\}$ occurs exactly in λ_1 blocks. Then by Lemma 4 and the statement (jss) of Corollary 6, we can assume the first and second near parallel classes are given in the first and the second rows of Table 1, respectively. Also by Corollary $6(jss)$, we can suppose that the missing symbol of the third near parallel class is 2 and, moreover, the symbols 0 and 1 appear in different blocks of this near parallel class (see the third row of Table 1).

The triple $\{0, 1, 2\}$ occurs exactly in λ_1 blocks and each one of the pairs $\{0, 1\}$, $\{0, 2\}$ and $\{1, 2\}$ occurs exactly in $\lambda = k - 1$ blocks. Therefore, since

$\lambda_1 = \frac{k-2}{2}$, $\lambda = k-1$ and $\lambda = \lambda_1 + (\lambda_1 + 1)$, a simple counting argument proves (ii). The case k odd is proved similarly. □

Table 1: Initial structure $\mathcal{E}$ for k even.

		First block	Second block
	0	$1\ 2 \cdots \frac{k}{2}\ \ \frac{k}{2}+1\ \frac{k}{2}+2 \cdots k$	$k+1\ k+2 \cdots \frac{3k}{2}\ \frac{3k}{2}+1 \cdots 2k$
	1	$2\ 3 \cdots \frac{k}{2}+1\ k+1\ k+2 \cdots \frac{3k}{2}$	$0\ \ \frac{k}{2}+2 \cdots k\ \ \frac{3k}{2}+1 \cdots 2k$
	2	0	1
λ_1	3	0 1 2	
	$\vdots$	$\vdots$ $\vdots$ $\vdots$	
	$\frac{k}{2}+1$	0 1 2	
λ_1+1	$\frac{k}{2}+2$	0 1	2
	$\vdots$	$\vdots$ $\vdots$	$\vdots$
	$k+1$	0 1	2
λ_1	$k+2$	0	1 2
	$\vdots$	$\vdots$	$\vdots$ $\vdots$
	$\frac{3k}{2}$	0	1 2
λ_1+1	$\frac{3k}{2}+1$	0 2	1
	$\vdots$	$\vdots$ $\vdots$	$\vdots$
	$2k$	0 2	1

Suppose that $\mathcal{N}_{t_0}, \ldots, \mathcal{N}_{t_{2k}}$ are the near parallel classes of a $(2k+1, k, k-1)$-NRBIBD with initial structure $\mathcal{E}$ for k even ($\mathcal{F}$ for k odd). It follows from Theorem 7 that $t_i = i$ for $i = 0, 1, 2$. However, to determine the missing element of the other near parallel classes we need some definitions. For two different symbols i and j, define the sets:

$$\begin{aligned} J(i,j) &= (B_{i,p(j)} \cap B_{j,p(i)}) \cup (B_{i,n(j)} \cap B_{j,n(i)}), \\ S(i,j) &= (B_{i,p(j)} \cap B_{j,n(i)}) \cup (B_{i,n(j)} \cap B_{j,p(i)}) \end{aligned}$$

for k even. For k odd, we define as well

$$\begin{aligned} S'(i,j) &= (B_{i,p(j)} \cap B_{j,p(i)}) \cup (B_{i,n(j)} \cap B_{j,n(i)}), \\ J'(i,j) &= (B_{i,p(j)} \cap B_{j,n(i)}) \cup (B_{i,n(j)} \cap B_{j,p(i)}). \end{aligned}$$

Table 2: Initial structure $\mathcal{F}$ for k odd.

		First block	Second block
	0	$2\ 3 \cdots \frac{k+3}{2}\ \frac{k+5}{2} \cdots\ k+1$	$1\ k+2 \cdots \frac{3k+1}{2}\ \frac{3k+3}{2} \cdots 2k$
	1	$2\ 3 \cdots \frac{k+3}{2}\ k+2 \cdots \frac{3k+1}{2}$	$0\ \frac{k+5}{2} \cdots\ k+1\ \frac{3k+3}{2} \cdots 2k$
	2	0	1
λ_1	3	$0\ 1\ 2$	
	$\vdots$	$\vdots\ \vdots\ \vdots$	
	$\frac{k+3}{2}$	$0\ 1\ 2$	
λ_1	$\frac{k+5}{2}$	$0\ 1$	2
	$\vdots$	$\vdots\ \vdots$	$\vdots$
	$k+1$	$0\ 1$	2
λ_1	$k+2$	0	$1\ 2$
	$\vdots$	$\vdots$	$\vdots\ \vdots$
	$\frac{3k+1}{2}$	0	$1\ 2$
λ_1	$\frac{3k+3}{2}$	$0\ 2$	1
	$\vdots$	$\vdots\ \vdots$	$\vdots$
	$2k$	$0\ 2$	1

The next lemma determines the missing element of the near parallel class $\mathcal{N}_{t_i}$ $(3 \leq i \leq 2k)$ for k even.

Lemma 8. *In any $(2k+1, k, k-1)$-NRBIBD with initial structure $\mathcal{E}$ (k even) we have*

$$\begin{aligned} \{t_3, \ldots, t_{\frac{k}{2}+1}\} &= J(0,1) \cap J(1,2), & (18)\\ \{t_{\frac{k}{2}+2}, \ldots, t_{k+1}\} &= J(0,1) \cap S(1,2), & (19)\\ \{t_{k+2}, \ldots, t_{\frac{3k}{2}}\} &= S(0,1) \cap J(1,2), & (20)\\ \{t_{\frac{3k}{2}+1}, \ldots, t_{2k}\} &= S(0,1) \cap S(1,2). & (21) \end{aligned}$$

Proof. It follows from Table 1 that the symbols 0, 1 and 2 appear in the same block of each near parallel class $\mathcal{N}_{t_i}$ $(3 \leq i \leq \frac{k}{2}+1)$. This means that $B_{t_i,p(0)} = B_{t_i,p(1)} = B_{t_i,p(2)}$ for $3 \leq i \leq \frac{k}{2}+1$. This proves (18). However, for $\frac{k}{2}+2 \leq i \leq k+1$ we have $B_{t_i,p(0)} = B_{t_i,p(1)} \neq B_{t_i,p(2)}$. This proves (19). Similarly, we can show (20) and (21). □

Lemma 9. *In any $(2k+1, k, k-1)$-NRBIBD with initial structure $\mathcal{F}$ (k odd) we have*

$$\begin{aligned}
\{t_3, \ldots, t_{\frac{k+3}{2}}\} &= J'(0,1) \cap J'(1,2), \\
\{t_{\frac{k+5}{2}}, \ldots, t_{k+1} &= J'(0,1) \cap S'(1,2), \\
\{t_{k+2}, \ldots, t_{\frac{3k+1}{2}}\} &= S'(0,1) \cap J'(1,2), \\
\{t_{\frac{3k+3}{2}}, \ldots, t_{2k}\} &= S'(0,1) \cap S'(1,2).
\end{aligned}$$

Proof. The proof is similar to that of Lemma 8. □

5 Backtrack Algorithm

Let $X = \{0,1,2\}$ and $Y = \{3, \ldots, 2k\}$. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, ..., b_{3-n}\}$ $(1 \le n \le 3)$ be two disjoint subsets of X. Then for each $y \in Y$, we define

$$\begin{aligned}
C_y(A, B) = \{ & (A \cup \{y_1, ..., y_{k-n}\}, B \cup \{y_{k-n+1}, \ldots, y_{2k-3}\}) \\
& \mid \{y_1, \ldots, y_{2k-3}\} = Y - \{y\} \}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
C_2(\{0\}, \{1\}) = \{ & \{0, y_1, ..., y_{k-1}\}, \{1, y_k, \ldots, y_{2k-2}\}) \\
& \mid \{y_1, \ldots, y_{2k-2}\} = Y \}.
\end{aligned}$$

A simple calculation shows that $|C_y(A, B)| = \binom{2k-1-|A|}{k-|A|}$.

In order to generate all possible $(2k+1, k, k-1)$-NRBIBDs with the initial structure either $\mathcal{E}$ or $\mathcal{F}$, we used a backtrack algorithm over near parallel classes. Our initial solution is the partial design formed by the first two classes of the initial structure $\mathcal{E}$, for k even. The third near parallel class will be chosen in the set $C_2(\{0\}, \{1\})$. Note that 0, 1 and 2 are the missing symbols of the first, second and third near parallel classes of the design, respectively.

For each choice of the third class we found the sets $J(0,1), J(1,2)$, $S(0,1)$ and $S(1,2)$. Then, it follows from Lemma 8 that

(1) The near parallel class $\mathcal{N}_\ell$ $(3 \le \ell \le \frac{k+1}{2})$ must be chosen in the set $C_t(\{0,1,2\}, \emptyset)$ with $t \in J(0,1) \cap J(1,2)$.

(2) The near parallel class $\mathcal{N}_\ell$ ($\frac{k}{2}+2 \le \ell \le k+1$) must be chosen in the set $C_t(\{0,1\},\{2\})$ with $t \in J(0,1) \cap S(1,2)$.

(3) The near parallel class $\mathcal{N}_\ell$ ($k+2 \le \ell \le \frac{3k}{2}$) must be chosen in the set $C_t(\{0\},\{1,2\})$ with $t \in S(0,1) \cap J(1,2)$.

(4) The near parallel class $\mathcal{N}_\ell$ ($\frac{3k}{2}+1 \le \ell \le 2k$) must be chosen in the set $C_t(\{0,2\},\{1\})$ with $t \in S(0,1) \cap S(1,2)$.

For each choice of these classes, we check that each partial design is simultaneously a partial $(2k+1, k, k-1)$-NRBIBD and a partial 2-concurrence (3)-design.

A partial isomorph rejection scheme [10] is employed to avoid processing isomorphic subproblems in the backtrack tree. The scheme described here is closely related to that in [8]. However, here we perform isomorph rejection at every level of the search unlike the authors of [8], who performed isomorph rejection only at certain levels. For each partial solution $\mathcal{D} = (\mathcal{N}_0, \ldots, \mathcal{N}_j)$ $(3 \le j \le 2k)$ with initial structure $\mathcal{E}$, let $A(\mathcal{D})$ be the design formed by the classes $\mathcal{N}_0, \ldots, \mathcal{N}_j$ and $\{A_{j+1}, B_{j+1}\}, \ldots, \{A_{2k}, B_{2k}\}$ (where $\{A_{j+1}, B_{j+1}\}, \ldots, \{A_{2k}, B_{2k}\}$ are classes of $\mathcal{E}$). Define

$$\mathcal{A}(j) = \{A(\mathcal{N}_0, \ldots, \mathcal{N}_j) | (\mathcal{N}_0, \ldots, \mathcal{N}_j) \text{ is a partial solution at level } j\}.$$

A partial isomorph rejection at the level j means that only one partial solution $(\mathcal{N}_0, \ldots, \mathcal{N}_j)$ from each isomorphism class of $\mathcal{A}(j)$, which is called a *certificate*, will be extended in the search tree. Informally, we say that the subtree rooted at $\mathcal{D} = (\mathcal{N}_0, \ldots, \mathcal{N}_j)$ is the same as the one rooted at $\mathcal{Q} = (\mathcal{Q}_0, \ldots, \mathcal{Q}_j)$ if the designs $A(\mathcal{D}_0, \ldots, \mathcal{D}_j)$ and $A(\mathcal{Q}_1, \ldots, \mathcal{Q}_j)$ are isomorphic. Clearly, our partial isomorph rejection allows the possibility of generating partial designs which are isomorphic. However, the extensions of these designs are removed at the leaves of the search tree.

In order to determine the certificates in every level $3 \le j \le 2k$ with the initial structure $\mathcal{E}$ (k even), we use the package *nauty* due to McKay [9] as follows. When a partial solution $\mathcal{D} = (\mathcal{N}_0, \ldots, \mathcal{N}_j)$ is generated, we construct the bipartite point-block incidence graph $G(\mathcal{D})$ of $A(\mathcal{D})$ and then call nauty to get the canonical form of $G(\mathcal{D})$. Hence the partial design $\mathcal{D} = (\mathcal{N}_0, \ldots, \mathcal{N}_j)$ is a certificate in this testing level if the canonical form of $G(\mathcal{D})$ was not generated before. Note that a certificate, in our context, is a partial design $(\mathcal{N}_0, \ldots, \mathcal{N}_j)$ generated by our backtracking algorithm. It is not the canonical form obtained by McKay's program.

The next theorem proves that the proposed algorithm works.

Theorem 10. *Our isomorph rejection algorithm generates, without repetition, all nonisomorphic* $(2k+1, k, k-1)$*-NRBIBDs.*

Proof. The proof is similar to that of [8, Theorem 6]. □

For the rest of this section, n denotes the number of certificates at the third testing level and $\mathcal{D}(1), \ldots, \mathcal{D}(n)$ denote such certificates for an initial structure $\mathcal{E}$ for k even ($\mathcal{F}$ for k odd). Note that for each $1 \leq j \leq n$, $\mathcal{D}(j) = (\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2)$ is a partial design, $\mathcal{N}_0$ and $\mathcal{N}_1$ are the first and the second near parallel classes of the initial structure, and $\mathcal{N}_2$ is a near parallel class chosen in $\mathcal{C}_2(\{0\}, \{1\})$.

Lemma 11. *Suppose that* a_i *is the missing element of the near parallel class* $\mathcal{N}_{a_i}$ $(1 \leq i \leq 3)$. *Then if the triple* $\{a_1, a_2, a_3\}$ *occurs exactly* λ_1 *times, the partial design* $(\mathcal{N}_{a_1}, \mathcal{N}_{a_2}, \mathcal{N}_{a_3})$ *is isomorphic to* $\mathcal{D}(j)$ *for some* $1 \leq j \leq n$.

Proof. The proof is similar to that of Theorem 7. □

This result can be used in order to reduce even more the search space.

The generation of all $(2k+1, k, k-1)$-NRBIBDs with initial structure $\mathcal{E}$ for k even ($\mathcal{F}$ for k odd) can be attained in n steps. At the i-th step $(1 \leq i \leq n)$, we generate all designs $\mathcal{D}$ such that any partial design $(\mathcal{N}_{a_1}, \mathcal{N}_{a_2}, \mathcal{N}_{a_3})$ with a_1, a_2, a_3 first associates is isomorphic to $\mathcal{D}(j)$, for some $i \leq j \leq n$. Note that at the n-th step, our algorithm generates only designs such that any partial design $(\mathcal{N}_{a_1}, \mathcal{N}_{a_2}, \mathcal{N}_{a_3})$ with a_1, a_2, a_3 first associates is isomorphic to $\mathcal{D}(n)$. Moreover, designs generated at different steps cannot be isomorphic. It follows from Lemma 11 that our algorithm generates all nonisomorphic $(2k+1, k, k-1)$-NRBIBDs with initial structure $\mathcal{E}$ for k even ($\mathcal{F}$ for k odd). A similar procedure was used in [8] to reduce the search space to find all (10,5,16)-RBIBDs.

6 Computational Results

The backtracking algorithm described in this work was implemented in C and ran on an 1.7 GHz PC machine. Using the algorithm we obtained the following main theorem in this work.

Theorem 12. *Table 3 gives the number of all nonisomorphic* $(2k+1, k, k-1)$*-NRBIBDs and the sizes of their automorphism groups for* $3 \leq k \leq 13$.

Kaski [4] has obtained these results for $k \leq 11$. The second row of Table 3 gives the parameter k of the design. The first column gives the size of the (full) automorphism group. Columns labeled by $3, 6 \ldots, 13$ give the number of nonisomorphic $(2k+1, k, k-1)$-NRBIBDs for $k = 3, 4, \ldots, 13$, respectively.

Table 3: The nonisomorphic $(2k+1, k, k-1)$-NRBIBDs for $5 \leq k \leq 13$

	k										
$\|GA\|$	3	4	5	6	7	8	9	10	11	12	13
1									16	1	312
2										2	9
3										1	35
5									1		
6							1			2	15
21					1						
39											1
42	1										1
54											1
55									1		
72										1	
110			1								
144		1									
156				1							
272						1					
342							1				
506									1		
1200										1	
2106											1
Total	1	1	1	1	1	1	2	0	19	8	374
CPU (min)	0*	0*	0*	0*	0*	0.005	0.033	0.15	3.72	23.6	2904.2

* CPU time less than 0.0015 minutes

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