Stratified Domination in Oriented Graphs

Ralucca Gera and Ping Zhang
Department of Mathematics
Western Michigan University
Kalamozoo, MI 49008, USA

ABSTRACT

An oriented graph is 2-stratified if its vertex set is partitioned into two classes, where the vertices in one class are colored red and those in the other class are colored blue. Let H be a 2-stratified oriented graph rooted at some blue vertex. An H-coloring of an oriented graph D is a red-blue coloring of the vertices of D in which every blue vertex v belongs to a copy of H rooted at v in D. The H-domination number $\gamma_H(D)$ is the minimum number of red vertices in an H-coloring of D. We investigate H-colorings in oriented graphs where H is the red-red-blue directed path of order 3. Relationships between the H-domination number γ_H and both the domination number γ and open domination γ_o in oriented graphs are studied. It is shown that $\gamma(D) \leq \gamma_H(D) \leq \gamma_o(D) \leq \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor$ for every oriented graph D. All pairs of positive integers that can be realized as (1) domination number and H-domination number and (2) the H-domination number and open domination number of some oriented graph are determined. Sharp bounds are established for the H-domination number of an r-regular oriented graph in terms of r and its order.

Key Words: stratified oriented graph, H-domination, domination, open domination.

AMS Subject Classification: 05C15, 05C20, 05C69.

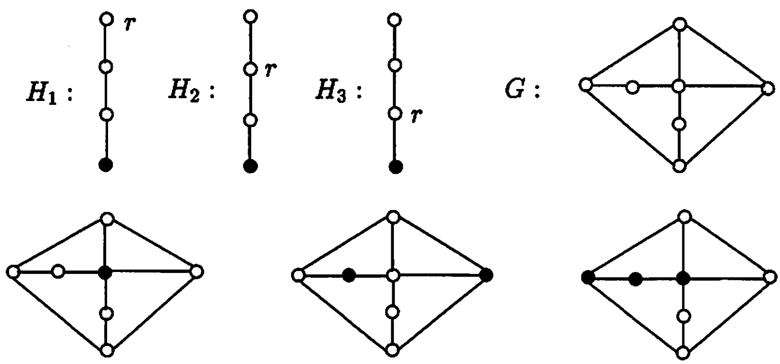
1 Introduction

We refer to the books [3, 5, 6] for graph theory notation and terminology not described in this paper. An area of graph theory that has received considerable attention in recent decades is domination. Although initiated by Berge [1] and Ore [7] in 1958 and 1962, respectively, it was a paper by Cockayne and Hedetniemi [4] in 1977 that began the popularity of the subject and has led to a theory. This subject is based on a very simple definition: A vertex v *dominates* a vertex u in a graph G if either $u = v$ or u is adjacent to v. Over the years a large number of variations of domination have surfaced. Each type of domination is based on a condition under which a vertex v dominates a vertex u in a graph G. As with standard domination, many definitions of domination state that a vertex v dominates a vertex u in a graph G if either $u = v$ or u satisfies some condition involving v. There are also those definitions of domination that state a vertex v dominates a vertex u not if $u = v$ but if u satisfies some condition involving v. The simplest example of this is *total* or *open domination* where v dominates u if u is adjacent to v. An advantage of the former type of domination is that every graph G contains a set of vertices (called a *dominating set*) such that every vertex of G is dominated by some vertex of S; while this is not necessarily the case for the latter type of domination. For example, graphs with isolated vertices contain no *open dominating sets.*

In 1999 a new way of looking at domination was introduced in [2] that encompassed several of the best known domination parameters defined earlier (including standard domination and open domination). This new view of domination was based on a simple but fundamental idea introduced by Rashidi [8] in 1994. A graph whose vertex set $V(G)$ is partitioned is a *stratified graph.* If $V(G)$ is partitioned into k subsets, then G is *k-stratified.* In particular, the vertex set of a 2-stratified graph is partitioned into two subsets. Typically, the vertices of one subset in a 2-stratified graph are considered to be colored red and those in the other subset are colored blue. A *red-blue coloring* of a graph G is an assignment of colors to the vertices of G, where each vertex is colored either red or blue.

We now describe how domination was defined in [2] with the aid of stratification. Let F be a 2-stratified graph in which some blue vertex r is designated as the "root" of F. Thus F is said to *be rooted at* r. By an *F-coloring* of a graph G, we mean a red-blue coloring of G such that for every blue vertex u of G, there is a copy of F in G with r at u. Therefore, every blue vertex u of G belongs to a copy of F rooted at u. A red vertex v in G is said to *F-dominate* a vertex u if $u = v$ or there exists a copy of F rooted at u and containing the red vertex v. The set S of red vertices in a red-blue coloring of G is an *F-dominating set* of G if every vertex of G is F-dominated by some vertex of S, that is, this red-blue coloring of G is an F-coloring. The minimum number of red vertices in an F-dominating set is called the *F-domination number* $\gamma_F(G)$ of G. An F-dominating set with $\gamma_F(G)$ vertices is a *minimum F-dominating set.* The F-domination number of every graph G is defined since $V(G)$ is an F-dominating set.

To illustrate these concepts, consider the three 2-stratified graphs H_1, H_2, and H_3 and the graph G of Figure 1, where solid vertices denote red vertices and open vertices denote blue vertices. Each of the 2-stratified graphs H_1, H_2, and H_3 has the same 2-stratification of the path P_4 of order 4 but is rooted at a different blue vertex. A minimum H_i-dominating set of G with exactly i red vertices is also shown in that figure for $i = 1, 2, 3$. Therefore, $\gamma_{H_i}(G) = i$ for $i = 1, 2, 3$.



$\gamma_{H_1}(G) = 1$ $\gamma_{H_2}(G) = 2$ $\gamma_{H_3}(G) = 3$

Figure 1: A minimum H_i-dominating set ($i = 1, 2, 3$) for a graph G

As described in [2], the most studied types of domination in graphs can be defined in terms of an appropriately chosen rooted 2-stratified graph, in fact, in terms of an appropriately selected rooted 2-stratified path of order 3. This gives rise to an infinite class of domination parameters, each of which is defined for every graph. In this work, we extend the concept of stratified domination to oriented graphs.

2 Stratification and Domination in Oriented Graphs

If a digraph D has the property that for each pair u, v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D, then D is an *oriented graph.* For a vertex v in an oriented graph, the number of vertices to which a vertex v is adjacent is the *outdegree* of v and is denoted by $\operatorname{od} v$ and the number of vertices from which v is adjacent is the *indegree* of v and is denoted by $\operatorname{id} v$. Thus the *degree* of v is $\deg v = \operatorname{od} v + \operatorname{id} v$.

An oriented graph whose vertex set is partitioned into two subsets is called a 2-*stratified oriented graph,* where the vertices of one subset are considered to be colored red and those in the other subset are colored blue. For a oriented graph D, a *red-blue coloring* of D is a coloring in which every vertex is colored red or blue. It is acceptable if all vertices of D are colored the same. If there is at least one vertex of each color, then the red-blue coloring of D produces a 2-stratification of D. Let F be a (connected) 2-stratified oriented graph rooted at some blue vertex. An *F-coloring* of an oriented graph D is a red-blue coloring of the vertices of D in which every blue vertex v belongs to a copy F' of F rooted at v in D. In this case, v is said to be *F-dominated* by some red vertex in F'. A red vertex is F-dominated by itself. The *F-domination number* $\gamma_F(D)$ is the minimum number of red vertices in an F-coloring of D. The set of red vertices in an F-coloring c of D is also called an *F-dominating set* of D and is denoted by R_c. If $|R_c| = \gamma_F(D)$, then c is a *minimum F-coloring* of D and R_c is a *minimum F-dominating set* of D. As with graphs, the F-domination number of every oriented graph D is defined since $V(D)$ is an F-dominating set. Therefore, if F has r red vertices, then

$$r \leq \gamma_F(D) \leq n \tag{1}$$

for every oriented graph D of order $n \geq r$. Furthermore, if D has no subdigraph isomorphic to F, then $\gamma_F(D) = n$.

If F is a connected 2-stratified oriented graph of order 2, then F is one of the 2-stratifications of $\vec{P_2}$ in Figure 2. In each case, the F-domination number is a well-known domination parameter, as we show next.

Figure 2: Two 2-stratified oriented graphs of $\vec{P_2}$

Let D be an oriented graph. A vertex v is said to *dominate* (or *out-dominate*) itself together with all vertices adjacent from v. A set $S \subseteq V(D)$ is a *dominating set* for D if every vertex in D is dominated by some vertex in S. The *domination number* $\gamma(D)$ is the minimum cardinality of a dominating set in D. A dominating set of cardinality $\gamma(D)$ is called a *minimum dominating set* of D. We first establish a result that is not unexpected.

Proposition 2.1 *For the 2-stratified oriented graph F_1 of $\vec{P_2}$,*

$$\gamma(D) = \gamma_{F_1}(D)$$

for every oriented graph D.

Proof. Let D be an oriented graph and let S be a minimum dominating set of D. Then the red-blue coloring c of D defined by coloring each vertex in S red and the remaining vertices blue is an F_1-coloring using $|S|$ red vertices. Therefore, $\gamma_{F_1}(D) \leq |S| = \gamma(D)$. To show that $\gamma(D) \leq \gamma_{F_1}(D)$, let there be given a minimum F_1-coloring c' and let $R_{c'}$ be the set of red vertices of D assigned by c'. Then $R_{c'}$ is a dominating set of D and so $\gamma(D) \leq |R_{c'}| = \gamma_{F_1}(D)$. ∎

The *converse* D^* of an oriented graph D has the same vertex set as D and the arc (u, v) is in D^* if and only if the arc (v, u) is in D. It is straightforward to verify the following.

Proposition 2.2 *For the 2-stratified oriented graph F_2 of $\vec{P_2}$,*

$$\gamma_{F_2}(D) = \gamma_{F_1}(D^*)$$

for every oriented graph D.

By Proposition 2.2, we need not be concerned with studying the parameter γ_{F_2} since the F_2-domination number of an oriented graph always equals the F_1-domination number of its converse. Also, since the F_1-domination number of an oriented graph equals the well-studied out-domination number, we proceed to consider F-domination for 2-stratified oriented graphs of higher order. This leads us to 2-stratified paths of order 3. There are six of these, as shown in Figure 3.

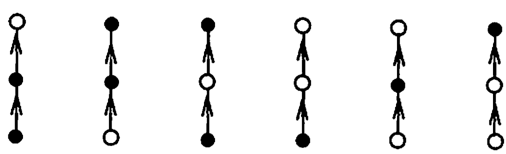


Figure 3: Six 2-stratifications of $\vec{P_3}$

As with graphs, each of these 2-stratified oriented graphs in Figure 3 gives rise to a domination parameter in digraphs. However, we are especially interested in the 2-stratified oriented graph H_1. Not only is the H_1-domination number a new domination parameter, it also possesses several interesting features as we will see in this work. To simplify the notation, we write $H = H_1$. Necessarily, the only blue vertex in H is the root of H. Since H has two red vertices, it follows by (1) that if D is a digraph of order $n \geq 2$, then

$$2 \leq \gamma_H(D) \leq n. \tag{2}$$

To illustrate H-domination, consider the tournament T in Figure 4. Since the red-blue coloring of T shown in Figure 4 is an H-coloring of T with three red vertices and so $\gamma_H(T) \leq 3$. Therefore, either $\gamma_H(T) = 2$ or $\gamma_H(T) = 3$. We claim that $\gamma_H(T) = 3$. Assume, to the contrary, that $\gamma_H(T) = 2$. Let c be a minimum H-coloring of T, where x and y are the two red vertices of T. Necessarily, x and y are adjacent vertices in T. Assume, without loss of generality, that (x, y) is an arc of T. Since c is an H-coloring of T, for every blue vertex z, the digraph T must contain the directed path x, y, z and so $\operatorname{od} y \geq 3$. Since every vertex of T has outdegree 2, this is impossible. Therefore, as claimed, $\gamma_H(T) = 3$.

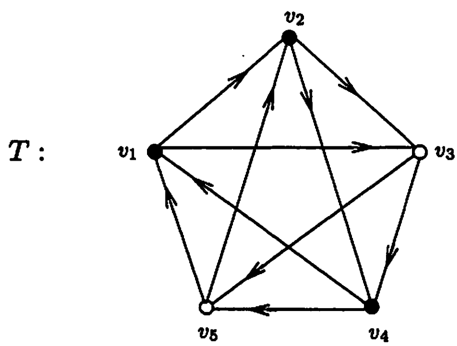


Figure 4: An H-coloring of an oriented graph

The argument just used to show that $\gamma_H(T) \neq 2$ for the digraph T of Figure 4 gives us the following result.

Proposition 2.3 *An oriented graph D of order $n \geq 3$ has $\gamma_H(D) = 2$ if and only if D contains a vertex v with* $\operatorname{id} v = 1$ *and* $\operatorname{od} v = n - 2$.

We now introduce some additional definitions that will be useful in what follows. Let c be an H-coloring of an oriented graph D and let w be a blue vertex of D. Necessarily, w is the terminal vertex of a directed red-red-blue path $\vec{P}_3$, say $\vec{P}_3 : u, v, w$. In this case, we say that w is *H-dominated* by v or that v *H-dominates* w. That is, in this context, every blue vertex w of D is H-dominated by some red vertex v and so w is adjacent from v, which, in turn, is adjacent from another red vertex. Consequently, if a red vertex v H-dominates a blue vertex w, then v is adjacent to w and adjacent from another red vertex. The following three observations are useful.

Observation 2.4 *Let v be a vertex in an oriented graph D.*

(*a*) *If* $\operatorname{od} v = 0$, *then v cannot dominate or H-dominate any other vertex in D.*

(*b*) *If* $\operatorname{id} v = 0$, *then v can neither H-dominate nor be H-dominated by any other vertex in D.*

Observation 2.5 *Let v be a vertex in an oriented graph D with* $\operatorname{id} v = 1$ *and let c be an H-coloring of D.*

(a) *If (u,v) is an arc of D, then at least one of u and v must be colored red by c.*

(b) *If (u,v) is an arc of D, $\mathrm{id}\, u = 1$, and (w,u) is an arc of D, then at least two of u, v, and w must be colored red by c.*

Observation 2.6 *Let D be a connected oriented graph of order n and let Δ be the maximum outdegree among all vertices of D with positive indegree. Then $\gamma_H(D) \leq n - \Delta$.*

The following is an immediate consequence of Observation 2.4.

Corollary 2.7 *Let I be the set of all vertices of an oriented graph D with indegree 0. Then*

(a) *I belongs to every dominating set of D, and*

(b) *$I \subseteq R_c$ for every H-coloring c of D.*

Recall that if D is a nontrivial oriented graph of order n and $\gamma_H(D) = k$, then $2 \leq k \leq n$. Next, we show that every pair k, n of integers with $2 \leq k \leq n$ is realizable as the H-domination number and the order of some connected oriented graph, respectively.

Proposition 2.8 *For each pair k, n of integers with $2 \leq k \leq n$, there exists a connected oriented graph D of order n with $\gamma_H(D) = k$.*

Proof. Let $\vec{K}_{1,k-1}$ be the orientation of the star $K_{1,k-1}$ whose vertex set is $\{u, v_1, v_2, \ldots, v_{k-1}\}$, where u is the central vertex of $K_{1,k-1}$ and $(u, v_i) \in E(\vec{K}_{1,k-1})$ for $1 \leq i \leq k-1$. Construct the oriented graph D from $\vec{K}_{1,k-1}$ by adding $n-k$ new vertices $w_1, w_2, \ldots, w_{n-k}$ together with the $n-k$ new arcs (v_1, w_i) for $1 \leq i \leq n-k$. Then the order of D is n.

We show that $\gamma_H(D) = k$. Define a red-blue coloring c of D by assigning red to every vertex of $V(\vec{K}_{1,k-1})$, and blue to the remaining vertices of D. Then c is an H-coloring of D with k red vertices, so $\gamma_H(D) \leq k$. Next, we show that $\gamma_H(D) \geq k$. Let there be given a minimum H-coloring c of D and let R_c be the set of red vertices of c. Since $\mathrm{id}\, u = 0$, it follows that $u \in R_c$. Also, since every vertex v_i $(1 \leq i \leq k-1)$ is adjacent from only a vertex with indegree 0, it follows that v_i is not H-dominated by any vertex distinct from v_i. Therefore, $v_i \in R_c$ for $1 \leq i \leq k-1$. Thus $V(\vec{K}_{1,k-1}) \subseteq R_c$ and so $\gamma_H(D) = |R_c| \geq k$. Therefore, $\gamma_H(D) = k$. ∎

3 Comparing H-Domination with Standard Dominations

In this section, we compare H-domination number with two well-known domination parameters in digraphs, namely domination and open domination. A vertex v is said to *openly dominate* (or *openly out-dominate*) all vertices adjacent from v. A set $S \subseteq V(D)$ is an *open dominating set* for D if every vertex in D is openly dominated by some vertex in S. The *open domination number* $\gamma_o(D)$ is the minimum cardinality of an open dominating set in D. An open dominating set of cardinality $\gamma_o(D)$ is called a *minimum open dominating set* of D. The following three observations are useful.

Observation 3.1 *Let D be an oriented graph. Then the open domination number $\gamma_o(D)$ is defined for an oriented graph D if and only if* $\operatorname{id} x \geq 1$ *for every vertex x in D.*

Observation 3.2 *If S is an open dominating set of an oriented graph D, then $|S| \geq 3$ and the subdigraph $\langle S \rangle$ induced by S contains a directed cycle. In particular, for every oriented graph D for which $\gamma_o(D)$ is defined, $\gamma_o(D) \geq 3$.*

Observation 3.3 *If D is an oriented graph with $\gamma(D) = 1$, then $\gamma_o(D)$ is not defined.*

We now show that if D is any oriented graph for which $\gamma_o(D)$ is defined, then $\gamma_H(D)$ is bounded above by $\gamma_o(D)$ and bounded below by $\gamma(D)$.

Theorem 3.4 *Let D be an oriented graph such that* $\operatorname{id} x \geq 1$ *for every $x \in V(D)$. For the 2-stratification H of $\vec{P}_3$,*

$$\gamma(D) \leq \gamma_H(D) \leq \gamma_o(D).$$

Proof. We first show $\gamma(D) \leq \gamma_H(D)$. Let there be given a minimum H-coloring c of D and let R_c be the set of the red vertices of D assigned by c. We show that R_c is a dominating set of D. If $v \in V(D) - R_c$, then v is a blue vertex and v belongs to a copy of H rooted at v. This implies that v is adjacent from a red vertex in R_c, and so v is dominated by some vertex in R_c. Therefore, R_c is a dominating set of D and so $\gamma(D) \leq |R_c| = \gamma_H(D)$.

Next, we show that $\gamma_H(D) \leq \gamma_o(D)$. Let S be a minimum open dominating set of D. Define a coloring c by assigning red to each vertex in S and blue to each vertex in $V(D) - S$. We show that c is an H-coloring. Let $v \in V(D) - S$ be a blue vertex. Since S is an open dominating set of D, it follows that v is openly dominated by some vertex $u \in S$ and so v is adjacent from the red vertex u. Moreover, u is also openly dominated by some vertex $w \in S$ and so u is adjacent from some red vertex w. Hence v belongs to a copy of H with vertex set $\{u, v, w\}$. Therefore, c is an H-coloring of D and so $\gamma_H(D) \leq |S| = \gamma_o(D)$. ∎

The two inequalities in Theorem 3.4 can both be strict. For example, for the directed cycle $\vec{C}_4$ of order 4, we have $\gamma(\vec{C}_4) = 2$, $\gamma_H(\vec{C}_4) = 3$, and $\gamma_o(\vec{C}_4) = 4$. On the other hand, both inequalities in Theorem 3.4 can be equalities. For example, for each integer $k \geq 3$, let $\vec{C}_k : u_1, u_2, \ldots, u_k, u_1$ be the directed k-cycle and let the *out-corona* $Cor(\vec{C}_k)$ of $\vec{C}_k$ be the oriented graph obtained from $\vec{C}_k$ by adding k new vertices $v_1, v_2, \ldots, v_k$ and k new arcs (u_i, v_i) for $1 \leq i \leq k$. It is straightforward to verify that

$$\gamma(Cor(\vec{C}_k)) = \gamma_H(Cor(\vec{C}_k)) = \gamma_o(Cor(\vec{C}_k)) = k. \tag{3}$$

for each integer $k \geq 3$.

By Theorem 3.4, if D is a connected oriented graph with $\gamma(D) = a$ and $\gamma_H(D) = b$, then $a \leq b$ and $b \geq 2$. Next, we show that every pair a, b of positive integers with $a \leq b$ and $b \geq 2$ is realizable as the domination number and H-domination number of some connected oriented graph D.

Proposition 3.5 *For every pair a, b of positive integers with $a \leq b$ and $b \geq 2$, there exists a connected oriented graph D such that $\gamma(D) = a$ and $\gamma_H(D) = b$.*

Proof. If $a = b = 2$, then let $D = \vec{C}_3$ and so $\gamma(D) = \gamma_H(D) = 2$; while if $a = b \geq 3$, then $D = Cor(\vec{C}_a)$ and the result follows by (3). Thus, we may assume that $1 \leq a < b$. Let $D = \vec{K}_{a,b-a}$ be the oriented graph obtained from the complete bipartite graph with partite sets $U = \{u_1, u_2, \ldots, u_a\}$ and $V = \{v_1, v_2, \ldots, v_{a-b}\}$ such that $E(\vec{K}_{a,b-a}) = \{(u_i, v_j) : 1 \leq i \leq a, 1 \leq j \leq b-a\}$. Observe that $\text{id}\, u = 0$ for each vertex $u \in U$ and so each vertex in U is dominated only by itself. Thus U belongs to every dominating set of D. Since U is a dominating set of D, it follows that $\gamma(D) = a$. Furthermore, D does not contain the directed $\vec{P}_3$ as a subdigraph and so $\gamma_H(D) = |V(D)| = b$. ∎

Although every pair a, b of positive integers with $a \leq b$ and $b \geq 2$ is realizable as the domination number and H-domination number of some connected oriented graph, this is not the case for the H-domination number γ_H and open domination number γ_o. Next we determine all pairs a, b of integers with $2 \leq a \leq b$ that are realizable as the H-domination number and open domination number of some connected oriented graph, beginning with those pairs a, b, where $a = 2$.

Proposition 3.6 *Let D be a connected oriented graph of order $n \geq 3$ for which $\gamma_o(D)$ exists. If $\gamma_H(D) = 2$, then $\gamma_o(D) = 3$.*

Proof. Let D be a connected oriented graph of order $n \geq 3$ with $\gamma_H(D) = 2$ and let c be a minimum H-coloring of D such that $R_c = \{x, y\}$. Thus either $(x, y) \in E(D)$ or $(y, x) \in E(D)$, say the former. This implies that every blue vertex $z \in V(D) - \{x, y\}$ is H-dominated by y. Hence $(y, z) \in E(D)$ for each $z \in V(D) - \{x, y\}$. Since $\gamma_o(D)$ exists, it follows by Observation 3.1 that $\operatorname{id} v \geq 1$ for every vertex v of D. Since $(x, y) \in E(D)$, it follows that $(z', x) \in E(D)$ for some $z' \in V(D) - \{x, y\}$. Then $\{x, y, z'\}$ is an open dominating set of D and so $\gamma_o(D) \leq 3$. It then follows by Observation 3.2 that $\gamma_o(D) = 3$. ■

We saw in Theorem 3.4 that if D is a connected oriented graph for which $\gamma_o(D)$ exists, then $\gamma_o(D)$ is bounded below by $\gamma_H(D)$. We now show that $\gamma_o(D)$ is bounded above by $\left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor$.

Theorem 3.7 *If D is a connected oriented graph for which $\gamma_o(D)$ exists, then*

$$\gamma_o(D) \leq \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor.$$

Proof. Let c be a minimum H-coloring of D and let R_c be the set of red vertices in D. Furthermore, let R_1 be the subset of R_c consisting of all vertices that H-dominate at least one blue vertex in D and let $R_2 = R_c - R_1$. (Note that R_2 may be empty.) Thus $\gamma_H(D) = |R_c| = |R_1| + |R_2|$. We consider two cases.

Case 1. $|R_2| \leq |R_1|$. Then $|R_2| \leq \frac{1}{2}\gamma_H(D)$. Observe that every blue vertex in D is adjacent from some red vertex in R_1 and so is openly dominated by some vertex in R_1. Also, every red vertex in R_1 is adjacent from some red vertex in R_c and so is openly dominated by some vertex in R_c. Thus every vertex in $V(D) - R_2$ is openly dominated by some vertex in R_c. Since $\gamma_o(D)$ exists, every vertex in D has positive indegree. For each vertex $x \in R_2$, let $y_x \in V(D)$ such that y_x is adjacent to x, that is, x is openly dominated by y_x. Let

$$Y = \{y_x : x \in R_2\}.$$

Then $|Y| \leq |R_2|$ and every vertex in R_2 is openly dominated by some vertex in Y. Hence $R_c \cup Y$ is an open dominating set of D and so

$$\begin{aligned}\gamma_o(D) &\leq |R_c \cup Y| \leq |R_c| + |Y| \leq |R_c| + |R_2| \\ &\leq \gamma_H(D) + \frac{1}{2}\gamma_H(D) = \frac{3\gamma_H(D)}{2}.\end{aligned}$$

Case 2. $|R_1| \leq |R_2|$. Then $|R_1| \leq \frac{1}{2}\gamma_H(D)$. Observe that every blue vertex in D is openly dominated by some vertex in R_1. For each vertex

$v \in R_1$, let $w_v \in V(D)$ such that w_v is adjacent to v and let $W = \{w_v : v \in R_1\}$. Then $|W| \le |R_1|$ and every vertex in R_1 is openly dominated by some vertex in W. Again, for each vertex $x \in R_2$, let $y_x \in V(D)$ such that y_x is adjacent to x, that is, x is openly dominated by y_x. Let $Y = \{y_x : x \in R_2\}$. Then $|Y| \le |R_2|$ and every vertex in R_2 is openly dominated by some vertex in Y. Hence $R_1 \cup W \cup Y$ is an open dominating set of D and so

$$\begin{aligned}\gamma_o(D) &\le |R_1 \cup W \cup Y| \le |R_1| + |W| + |Y| \\ &\le |R_1| + |R_1| + |R_2| = |R_1| + |R_c| \\ &\le \frac{1}{2}\gamma_H(D) + \gamma_H(D) = \frac{3\gamma_H(D)}{2}.\end{aligned}$$

Therefore, $\gamma_o(D) \le \left\lfloor \frac{3\gamma_H(D)}{2} \right\rfloor$, as desired. ■

By Proposition 3.6, if D is a connected oriented graph of order $n \ge 3$ for which $\gamma_o(D)$ exists and $\gamma_H(D) = 2$, then $\gamma_o(D) = 3$. Thus, there is no connected oriented graph with H-domination number 2 and open domination number 4 or more. On the other hand, we show that every pair a, b of integers with $3 \le a \le b \le \lfloor \frac{3a}{2} \rfloor$ is realizable as the H-domination number and open domination number of some connected oriented graph.

Theorem 3.8 *For every pair a, b of integers with $3 \le a \le b \le \lfloor \frac{3a}{2} \rfloor$, there exists a connected oriented graph D such that $\gamma_H(D) = a$ and $\gamma_o(D) = b$.*

Proof. If $a = b \ge 3$, then let $D = Cor(\vec{C}_a)$ and the result follows by (3). Thus we may assume that $a < b$. We consider two cases.

Case 1. $b = a + 1$. Let $k \ge 2$ be an integer and for each integer i with $1 \le i \le a - 1$, let $G_i : u_i, v_i, w_i, u_i$ be a directed 3-cycle. Let D be the oriented graph obtained from the digraphs G_i $(1 \le i \le a-1)$ by identifying the vertices w_i for $1 \le i \le a - 1$ and labeling the identified vertex by w. Then $\gamma_H(D) = a$ and $\gamma_o(D) = a + 1$.

Case 2. $b \ge a + 2$. Let

$$k = b - a - 1 \text{ and } \ell = 3a - 2b + 1.$$

Since $b \ge a + 2$ and $b \le \lfloor \frac{3a}{2} \rfloor$, it follows that $k \ge 1$ and $\ell \ge 1$.

For each i with $1 \le i \le k + \ell$, let G_i be a copy of $\vec{C}_3 : v_{i,1}, v_{i,2}, v_{i,3}, v_{i,1}$. First, we construct an oriented graph D' obtained from the first ℓ copies of of G_i $(1 \le i \le \ell)$ by identifying all the vertices $v_{i,1}$ $(1 \le i \le \ell)$ and labeling the identified vertex by v_1. Then we construct the oriented graph D from D' and the k copies of G_j $(\ell + 1 \le j \le \ell + k)$ by adding a new vertex v and the $k + 1$ new arcs (v_1, v) and $(v_{j,1}, v)$ for $\ell + 1 \le j \le \ell + k$.

We show that $\gamma_H(D) = a$ and $\gamma_o(D) = b$. Let

$$\begin{aligned} V_1 &= \{v_1\} \cup \{v_{j,1} : \ell + 1 \le j \le \ell + k\}, \\ V_2 &= \{v_{t,2} : 1 \le t \le \ell + k\}, \\ V_3 &= \{v_{i,3} : 1 \le i \le \ell - 1\}, \\ V_3' &= \{v_{\ell,3}\} \cup \{v_{j,3} : \ell + 1 \le j \le \ell + k\}. \end{aligned}$$

We first show that $\gamma_H(D) = a$. Since the set $V_1 \cup V_3 \cup V_3'$ is an H-dominating set,

$$\begin{aligned} \gamma_H(D) &\le |V_1 \cup V_3 \cup V_3'| = 2k + \ell + 1 \\ &= 2(b - a - 1) + (3a - 2b + 1) + 1 = a. \end{aligned}$$

To show that $\gamma_H(D) \ge a$, let c be a minimum H-coloring of D and let R_c be the set of red vertices of D. By Observation 2.5, (1) R_c contains at least one vertex in $\{v_{i,2}, v_{i,3}\}$ for $1 \le i \le \ell$ and (2) R_c contains at least two vertices in $\{v_{j,1}, v_{j,2}, v_{j,3}\}$ for $\ell + 1 \le j \le \ell + k$. Thus $|R_c| \ge 2k + \ell$. We claim that $|R_c| > 2k + \ell$. Assume, to the contrary, that $|R_c| = 2k + \ell$. Then R_c contains exactly one vertex in $\{v_{i,2}, v_{i,3}\}$ for $1 \le i \le \ell$ and exactly two vertices in $\{v_{j,1}, v_{j,2}, v_{j,3}\}$ for $\ell + 1 \le j \le \ell + k$. In particular, $v_1 \notin R_c$. Observe that the only directed path $\vec{P}_3$ in D having v_1 as a terminal vertex is $v_{i,2}, v_{i,3}, v_1$ for some i with $1 \le i \le \ell$. Since R_c contains exactly one vertex in $\{v_{i,2}, v_{i,3}\}$ for all i with $1 \le i \le \ell$, it follows that v_1 is not H-dominated by R_c, which is a contradiction. Therefore,

$$\gamma_H(D) = |R_c| \ge 2k + \ell + 1 = a$$

and so $\gamma_H(D) = a$.

Now we show that $\gamma_o(D) = b$. First we show that $\gamma_o(D) \le b$. Since the set $V_1 \cup V_2 \cup V_3'$ is an open dominating of D, it follows that

$$\begin{aligned} \gamma_o(D) &\le |V_1 \cup V_2 \cup V_3'| = 3k + (\ell + 2) \\ &= 3(b - a - 1) + (3a - 2b + 1) + 2 = b. \end{aligned}$$

To show $\gamma_o(D) \ge b$, let S_o be a minimum open dominating set. Observe that (i) for each i with $1 \le i \le \ell$, the vertex $v_{i,3}$ is only openly dominated by $v_{i,2}$. Thus $v_{i,2} \in S_o$ for $1 \le i \le \ell$, (ii) for each i with $1 \le i \le \ell$, the vertex $v_{i,2}$ is only openly dominated by v_1 and so $v_1 \in S_o$, and (iii) the vertex v_1 is only openly dominated by some vertex $v_{i,3}$, where $1 \le i \le \ell$. It follows by (i)–(iii) that S_o must contain at least $\ell + 2$ vertices in the set

$$\{v_1\} \cup \left(\bigcup_{i=1}^{\ell} V(G_i) \right).$$

Furthermore, for each j with $\ell+1 \leq j \leq \ell+k$, (1) the vertex $v_{j,1}$ can only be openly dominated by $v_{j,3}$, (2) the vertex $v_{j,2}$ can only be openly dominated by $v_{j,1}$, and (3) the vertex $v_{j,3}$ can only be openly dominated by $v_{j,2}$. This implies that $\{v_{i,1}, v_{i,2}, v_{i,3}\} \subseteq S_o$ for $\ell+1 \leq i \leq \ell+k$. Therefore,

$$\begin{aligned} \gamma_o(D) &= |S_o| \geq 3k + (\ell+2) \\ &= 3(b-a-1) + (3a-2b+3) = b. \end{aligned}$$

This completes the proof. ∎

Combining Proposition 3.6 and Theorems 3.7 and 3.8, we have the following characterization of those pairs a, b of integers with $2 \leq a \leq b$ that are realizable as the H-domination number and open domination number of some connected oriented graph.

Corollary 3.9 *Let a and b be integers with $2 \leq a \leq b$. Then there exists a connected oriented graph D such that $\gamma_H(D) = a$ and $\gamma_o(D) = b$ if and only if*

$$(a, b) = (2, 3) \quad \text{or} \quad 3 \leq a \leq b \leq \lfloor \tfrac{3a}{2} \rfloor.$$

4 H-Domination in Regular Oriented Graphs

A connected oriented graph D is said to be *r-regular* if

$$\operatorname{id} v = \operatorname{od} v = r$$

for some nonnegative integer r and for every $v \in V(D)$. If D is a connected r-regular oriented graph of order n, then the underlying graph of D is $2r$-regular. Thus $n \geq 2r+1$ and so $r \leq \lfloor \frac{n-1}{2} \rfloor$. In this section, we investigate H-domination in r-regular oriented graphs.

We have seen in (2) that if D is a connected oriented graph of order $n \geq 2$, then $2 \leq \gamma_H(D) \leq n$. Moreover, by Proposition 2.8, for each pair k, n of integers with $2 \leq k \leq n$, there exists a connected oriented graph D of order n with $\gamma_H(D) = k$. However, this is not the case for r-regular connected oriented graphs; that is, for fixed positive integers r and n with $n \geq 2r+1$, there are pairs k, n of integers with $2 \leq k \leq n$ such that there is no connected r-regular oriented graph of order n with $\gamma_H(D) = k$, as we will see in this section.

If $r = 1$, then the directed n-cycle $\vec{C}_n$ is the only connected 1-regular oriented graph of order $n \geq 3$. We now determine the H-domination number of $\vec{C}_n$ for every integer $n \geq 3$.

Theorem 4.1 *For each integer $n \geq 3$,*

$$\gamma_H(\vec{C}_n) = \left\lceil \frac{2n}{3} \right\rceil .$$

Proof. We first show that $\gamma_H(\vec{C}_n) \leq \lceil 2n/3 \rceil$. Let $\vec{C}_n : v_1, v_2, \ldots, v_n, v_1$. Then $n = 3k + t$, where $k \geq 1$ and $0 \leq t \leq 2$. Define a red-blue coloring c' of $\vec{C}_n$ by assigning blue to each vertex in the set $\{v_{3i} : 1 \leq i \leq k\}$ and red to the remaining vertices of $\vec{C}_n$. Since each blue vertex v_{3i} belongs to a copy of H rooted at v_{3i} in $\vec{C}_n$, namely the directed path: $v_{3i-2}, v_{3i-1}, v_{3i}$, it follows that c' is an H-coloring of $\vec{C}_n$ with exactly $2k + t$ red vertices. Hence $\gamma_H(\vec{C}_n) \leq 2k + t$. Observe that

$$\left\lceil \frac{2n}{3} \right\rceil = \left\lceil \frac{6k+2t}{3} \right\rceil = 2k + \left\lceil \frac{2t}{3} \right\rceil = \begin{cases} 2k & \text{if } t = 0 \\ 2k+1 & \text{if } t = 1 \\ 2k+2 & \text{if } t = 2 \end{cases}$$

and so $\lceil 2n/3 \rceil = 2k + t$. Therefore, $\gamma_H(\vec{C}_n) \leq \lceil 2n/3 \rceil$.

Next, we show that $\gamma_H(\vec{C}_n) \geq \lceil 2n/3 \rceil$. Certainly, at least one vertex of $\vec{C}_n$ is colored blue in a minimum H-coloring of $\vec{C}_n$. Assume, without loss of generality, that v_{3k} is colored blue. By Observation 2.5, at least two vertices in each set $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$, $1 \leq i \leq k$, are colored red. Furthermore, since v_{3k} is blue, it follows by Observation 2.5 that v_{3k+1} and v_{3k+2} are red (where the addition in $3k+1$ and $3k+2$ is done modulo n). Hence at least $2k + t$ vertices of $\vec{C}_n$ are colored red. Thus $\gamma_H(\vec{C}_n) \geq 2k + t = \lceil 2n/3 \rceil$. Therefore, $\gamma_H(\vec{C}_n) = \lceil 2n/3 \rceil$. ■

We now turn to r-regular connected oriented graphs for $r \geq 2$. First, we present a lower bound for H-domination number of a connected r-regular oriented graph of order n in terms of r and n.

Theorem 4.2 *Let $r \geq 2$ be an integer. If D is a connected r-regular oriented graph of order $n \geq 2r + 1$, then*

$$\gamma_H(D) \geq \min\left\{ \frac{n+r}{1+r}, \frac{n}{r} \right\}.$$

Proof. Let D be a connected r-regular oriented graph of order $n \geq 2r+1$ with $\gamma_H(D) = k$. By Observation 2.6 , we know that $\gamma_H(D) \leq n - r$ and so we may assume that $k < n$. Let c be a minimum H-coloring of D and let R_c be the set of the red vertices of c in D. Furthermore, let R_1 be the subset of R_c consisting of all vertices that H-dominate at least one blue vertex in D and let $R_2 = R_c - R_1$. Thus $k = |R_c| = |R_1| + |R_2|$. Let

$$B = V(D) - R_c$$

be the set of the blue vertices of c in D. We consider two cases.

Case 1. $R_2 = \emptyset$. Since D is r-regular, each vertex in R_1 can H-dominate at most r blue vertices. Furthermore, since each vertex v in R_1 H-dominates at least one blue vertex and $R_2 = \emptyset$, it follows that v must also be adjacent from some red vertex u in R_1. Hence there are at least k arcs in D, both of whose incident vertices are red. This implies that R_1 can H-dominate at most $kr - k$ blue vertices and so $|B| \le kr - k$. Therefore,

$$n = |B| + |R_c| \le (kr - k) + k = kr$$

and so $k \ge n/r$.

Case 2. $R_2 \ne \emptyset$. Since each blue vertex can only be H-dominated by some red vertex in R_1 and each vertex in R_1 can H-dominate at most r blue vertices,

$$|B| \le |R_1|r = (|R_c| - |R_2|)r.$$

Since $|R_2| \ge 1$, it follows that

$$\begin{aligned} n &= |B| + |R_c| \le (|R_c| - |R_2|)r + |R_c| \\ &\le (|R_c| - 1)r + |R_c| = (k-1)r + k \\ &= k(r+1) - r. \end{aligned}$$

Hence $k \ge (n + r)/(r + 1)$. ■

Since $(n + r)/(1 + r) \le n/r$ if $n \ge r^2$, the following is a consequence of Theorem 4.2.

Corollary 4.3 *Let $r \ge 2$ be an integer. If D is a connected r-regular oriented graph of order $n \ge r^2$, then*

$$\gamma_H(D) \ge \frac{n+r}{1+r}.$$

The lower bound in Theorem 4.2 (or in Corollary 4.3) is sharp. In fact, more can be said.

Proposition 4.4 *For each integer $r \ge 2$, there exist an integer $n \ge r^2$ and a connected r-regular oriented graph D of order n such that*

$$\gamma_H(D) = \frac{n+r}{1+r} = \min\left\{\frac{n+r}{1+r}, \frac{n}{r}\right\}.$$

Proof. Let $K_{1,r} : u, v_1, v_2, \ldots, v_r$ be the star of order $r + 1$ centered at u, that is, $\deg u = r$. Recall that $\vec{K}_{1,r}$ is the orientation of $K_{1,r}$ such that (u, v_i) is an arc in $\vec{K}_{1,r}$ for $1 \le i \le r$. For each j with $1 \le j \le r$, let D_j be a copy of $\vec{K}_{1,r}$, where

$$V(D_j) = \{v_j, w_{j,1} w_{j,2} \ldots, w_{j,r}\} \text{ and od } v_j = r.$$

Then the digraph D is obtained from the digraphs D_j $(1 \le j \le r)$ by adding (1) the arcs $(w_{j,1}, u)$ for $1 \le j \le r$, (2) the arcs $(w_{j,1}, v_i)$ for $1 \le i, j \le r$ and $i \ne j$, (3) the arcs $(w_{j,t}, w_{j,1})$ for $1 \le j \le r$ and $2 \le t \le r$, and (4) the arcs $(w_{r,s}, w_{1,t})$ and $(w_{j,s}, w_{j+1,t})$ for $1 \le j \le r-1$, $2 \le s, t \le r$. (See Figure 5.)

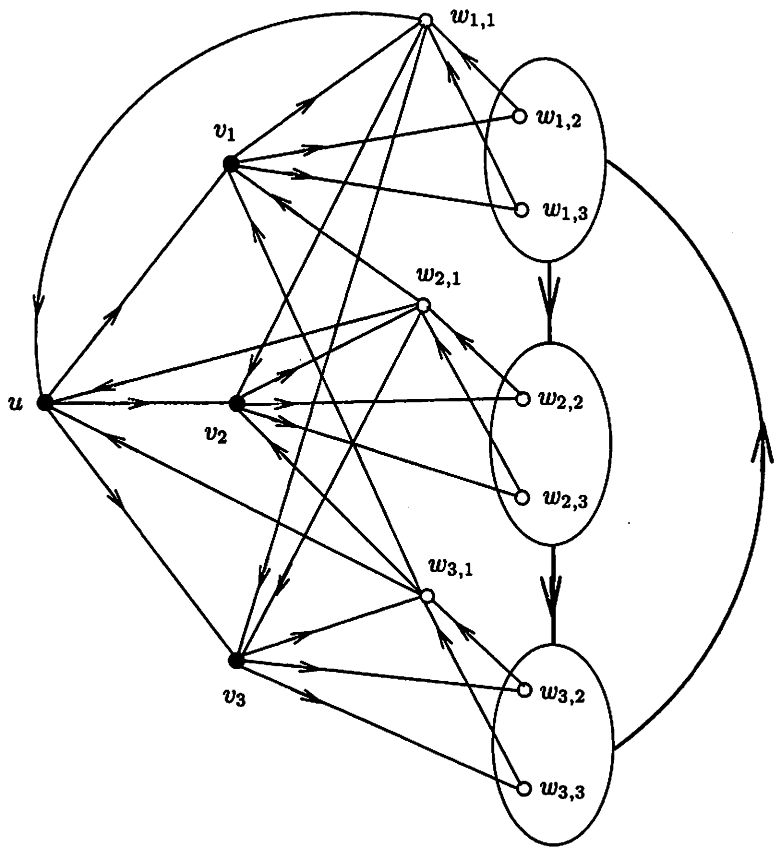


Figure 5: The r-regular oriented graph D for $r = 3$

Observe that D is a connected r-regular oriented graph of order $n = r^2 + r + 1$. Since

$$\frac{n+r}{r+1} = \frac{(r^2+r+1)+r}{r+1} = r+1$$

and $n \ge r^2$, it follows by Corollary 4.3 that $\gamma_H(D) \ge r + 1$. On the other hand, let c be the red-blue coloring of D with

$$R_c = \{u\} \cup \{v_j : 1 \le j \le r\}.$$

Since each blue vertex $w_{j,t}$ $(1 \le j,t \le r)$ belongs to the red-red-blue path $u, v_j, w_{j,t}$, it follows that c is an H-coloring of D and so $\gamma_H(D) \le |R_c| = r+1$. Therefore,

$$\gamma_H(D) = r + 1 = \frac{n+r}{1+r},$$

as desired. ∎

Proposition 4.5 *For each integer $r \ge 2$, there exist an integer n with $2r+1 \le n \le r^2$ and a connected r-regular oriented graph D of order n for which*

$$\gamma_H(D) = \frac{n}{r} = \min\left\{\frac{n+r}{1+r}, \frac{n}{r}\right\}.$$

Proof. Let k be an integer with $3 \le k \le r$ and let $\vec{C}:\ v_1, v_2, \ldots, v_k, v_1$ be a directed k-cycle. For each i with $1 \le i \le k$, let $D_i \cong \overline{K}_{r-1}$ be the empty digraph of order $r-1$ with

$$V(D_i) = W_i = \{w_{i,j} :\ 1 \le j \le r-1\}.$$

Then the oriented graph D is constructed from $\vec{C}$ and D_i $(1 \le i \le k)$ by adding (1) the arcs $(w_{1,s}, w_{k,t})$ for $1 \le s,t \le r-1$ and $(w_{i,s}, w_{i-1,t})$ for $2 \le i \le k$ and $1 \le s,t \le r-1$, (2) the arcs $(v_1, w_{1,j})$ and $(w_{1,j}, v_k)$ for $1 \le j \le r-1$ and $(v_i, w_{i,j})$ and $(w_{i,j}, v_{i-1})$ for $2 \le i \le k$ and $1 \le j \le r-1$. (See Figure 6.)

Then D is a connected r-regular oriented graph of order $n = kr$. Thus $k = n/r$. Since $n = kr \le r^2$, it follows by Theorem 4.2 that

$$\gamma_H(D) \ge \frac{n}{r} = k.$$

Therefore, it suffices to show that $\gamma_H(D) \le k$. Let c be the red-blue coloring that assigns red to the vertices of $\vec{C}$ and blue to the remaining vertices of D. For each i and j $(1 \le i \le k, 1 \le j \le r-1)$, where i is expressed as an integer modulo k, since each blue vertex $w_{i,j} \in W_i$ belongs to the red-red-blue path $v_{i-1}, v_i, w_{i,j}$, it follows that c is an H-coloring of D with exactly k red vertices. Thus $\gamma_H(D) \le k$ and so $\gamma_H(D) = k$. ∎

As a final note, we mention that when $k = r$ in the proof of Proposition 4.5, the digraph D constructed has order $n = r^2$ and therefore,

$$\gamma_H(D) = r = \frac{n}{r} = \frac{n+r}{1+r}.$$

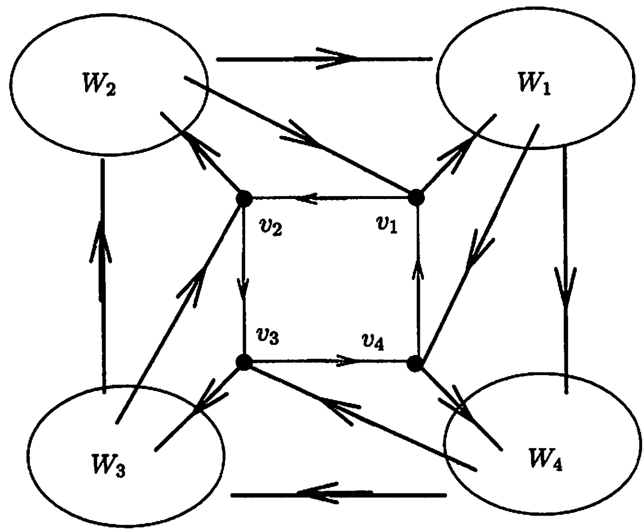


Figure 6: The r-regular oriented graph D for $k = 4$

By Observation 2.6 , if D is a connected r-regular oriented graph of order n, then $\gamma_H(D)$ is bounded above by $n - r$. This upper bound can be improved, as we show next. We first establish some definitions. For a vertex u of an oriented graph D, the *out-neighborhood* $N^+(u)$ and *in-neighborhood* $N^-(u)$ of u are defined by

$$N^+(u) = \{x : (u,x) \in E(D)\} \text{ and } N^-(u) = \{x : (x,u) \in E(D)\}.$$

Proposition 4.6 *Let $r \geq 2$ be an integer. If D is a connected r-regular oriented graph of order n, then*

$$\gamma_H(D) \leq n - \left\lceil \frac{3r}{2} \right\rceil + 1. \tag{4}$$

Proof. Let D be a connected r-regular oriented graph and $x, y \in V(D)$ such that $(x,y) \in E(D)$. Let

$$N^+(y) = \{y_1, y_2, \ldots, y_r\} \text{ and } N^+[y] = N^+(y) \cup \{y\}.$$

Furthermore, let

$$D_{x,y} = \langle N^+(y) \cup \{x\} \rangle$$

be the subdigraph induced by $N^+(y) \cup \{x\}$. Since the order of $D_{x,y}$ is $r+1$, the size of $D_{x,y}$ is at most $\binom{r+1}{2} = \frac{r(r+1)}{2}$ and so

$$\sum_{v \in V(D_{x,y})} \text{od}_{D_{x,y}} v \leq \frac{r(r+1)}{2}, \tag{5}$$

where $\mathrm{od}_{D_{x,y}}v$ is the outdegree of v in $D_{x,y}$. Let

$$k = \min\{\mathrm{od}_{D_{x,y}}v : \ v \in V(D_{x,y})\}.$$

By (5), $k \le r/2$ and so $k \le \lfloor r/2 \rfloor$. We consider two cases.

Case 1. *$k = \mathrm{od}_{D_{x,y}}y_i$ for some i with $1 \le i \le r$, say $k = \mathrm{od}_{D_{x,y}}y_1$.* Since $\mathrm{od}_D y_1 = r$, it follows that y_1 is adjacent to at least $\lceil r/2 \rceil$ vertices that are not in $D_{x,y}$. Let Z be the set of vertices of maximum cardinality such that

$$Z \subseteq N^+(y_1) \text{ and } Z \cap V(D_{x,y}) = \emptyset.$$

Then $|Z| \ge \lceil r/2 \rceil$. Define the red-blue coloring c of D that assigns blue to each vertex in

$$(N^+(y) - \{y_1\}) \cup Z = \{y_2, y_3, \cdots, y_r\} \cup Z$$

and red to the remaining vertices of D. Since (i) each blue vertex y_i $(2 \le i \le r)$ is H-dominated by y, and (ii) each blue vertex in Z is H-dominated by y_1, it follows that c is an H-coloring of D with

$$R_c = V(D) - (\{y_2, y_3, \cdots, y_r\} \cup Z).$$

Therefore,

$$\begin{aligned} \gamma_H(D) &\le |R_c| = n - [(r-1) + |Z|] \\ &\le n - \left(r - 1 + \left\lceil \frac{r}{2} \right\rceil\right) = n - \left\lceil \frac{3r}{2} \right\rceil + 1. \end{aligned}$$

Case 2. $k = \mathrm{od}_{D_{x,y}}x$ and $\mathrm{od}_{D_{x,y}}y_i > k$ for all i with $1 \le i \le r$. Thus x is adjacent to at least $\lceil r/2 \rceil$ vertices that are not in $D_{x,y}$. Since $(x, y) \in E(D)$, it follows that x is adjacent to at least $\lceil r/2 \rceil - 1$ vertices that are not in $V(D_{x,y}) \cup \{y\}$. Let X be the set of vertices of maximum cardinality such that

$$X \subseteq N^+(x) \text{ and } X \cap (V(D_{x,y}) \cup \{y\}) = \emptyset.$$

Then $|X| \ge \lceil r/2 \rceil - 1$. We consider two subcases.

Subcase 2.1. *There exists $z \notin N^+(y)$ such that $(z, x) \in E(D)$.* Define a red-blue coloring c by assigning blue to each vertex in

$$N^+(y) \cup X = \{y_1, y_2, \cdots, y_r\} \cup X$$

and red to the remaining vertices of D. Since (i) each blue vertex in $N^+(y)$ is H-dominated by y, and (ii) each blue vertex in X is H-dominated by x, it follows that c is an H-coloring of D with $R_c = V(D) - (N^+(y) \cup X)$. Therefore,

$$\begin{aligned} \gamma_H(D) &\le |R_c| = n - (r + |X|) \\ &\le n - \left(r + \left\lceil \frac{r}{2} \right\rceil - 1\right) = n - \left\lceil \frac{3r}{2} \right\rceil + 1. \end{aligned}$$

Subcase 2.2. Subcase 2.1 does not occur. This implies that $N^-(x) \subseteq N^+(y)$. Since $\mathrm{id}\, x = r$ and $|N^+(y)| = r$, it follows that $N^-(x) = N^+(y)$ and so $|X| = r - 1$. Define the red-blue coloring c of D that assigns blue to each vertex in

$$(N^+(y) - \{y_1\}) \cup X = \{y_2, y_3, \cdots, y_r\} \cup X$$

and red to the remaining vertices of D. Since (i) each blue vertex y_i ($2 \le i \le r$) is H-dominated by y, and (ii) each blue vertex in X is H-dominated by x, it follows that c is an H-coloring of D with

$$R_c = V(D) - (\{y_2, y_3, \cdots, y_r\} \cup X).$$

Since $r \ge 2$, it follows that

$$\begin{aligned} \gamma_H(D) &\le |R_c| = n - [(r-1) + |X|] \\ &= n - 2(r-1) \le n - \left\lceil \frac{3r}{2} \right\rceil + 1, \end{aligned}$$

as desired. ■

Both equality and strict inequality in (4) are possible. For example, let K_{2r+1} be the complete graph of order $2r + 1 \ge 5$ with $V(K_{2r+1}) = \{v_0, v_1, \ldots, v_{2r}\}$. Define the orientation $\vec{K}_{2r+1}$ of K_{2r+1} by $E(\vec{K}_{2r+1}) = \{(v_i, v_{i+j}) : 0 \le i \le 2r, 1 \le j \le r\}$, where the subscripts are expressed as the integers $0, 1, 2, \ldots, 2r$ modulo $2r + 1$. Observe that $\vec{K}_{2r+1}$ is r-regular. Then it can be shown that $\gamma_H(\vec{K}_{2r+1}) = 3$ for each $r \ge 2$. Observe that the 2-regular tournament $\vec{K}_5$ has order $n = 5$ and the 3-regular tournament $\vec{K}_7$ has order $n = 7$. In each case, $n - \lceil 3r/2 \rceil + 1 = 3$ and $\gamma_H(\vec{K}_5) = \gamma_H(\vec{K}_7) = 3$. Therefore, the equality in (4) holds for $\vec{K}_5$ and $\vec{K}_7$. On the other hand, the 4-regular tournament $\vec{K}_9$ has order $n = 9$. In this case, $n - \lceil 3r/2 \rceil + 1 = 4$; while $\gamma_H(\vec{K}_9) = 3$. Hence the strict inequality in (4) holds for $\vec{K}_9$.

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