# On the Summation of a New Class of Infinite Series

Peter J. Larcombe

Derbyshire Business School
University of Derby, Kedleston Road, Derby DE22 1GB, U.K.
{P.J.Larcombe@derby.ac.uk}

**Abstract**

Previous work on a certain class of non-terminating expansions of the sine function leads directly to a new result for associated infinite series by straightforward integration. A general identity is established, particular cases verified and two proofs of its hypergeometric form given.

## Introduction

Consider, for $m$ (even) $\geq 2$, the expansion of $\sin(m\alpha)$ (in odd powers of $\sin(\alpha)$)

$$\sin(m\alpha) = \sum_{n=0}^{\infty} S_n^{(m)} \sin^{2n+1}(\alpha), \tag{1}$$

which is valid for $|\alpha| < \frac{\pi}{2}$. In [1] a closed form for the general coefficient of the r.h.s. of (1) was derived, having been developed analytically from a result associated with Euler. Let $c_n$ denote the $(n+1)$th term

$$c_n = \frac{1}{n+1}\binom{2n}{n} \tag{2}$$

of the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{1, 1, 2, 5, 14, \ldots\}$. Then, defining

$$Q(n;m) = \begin{cases} 1 & m = 2 \\ \frac{(2n+3)(2n+5)\cdots[2n+(m-1)]}{(2n-3)(2n-5)\cdots[2n-(m-1)]} & m = 4, 6, 8, \ldots, \end{cases} \tag{3}$$

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and introducing the additional Catalan number $c_{-1} = -\frac{1}{2}$, the following result was established [1, Theorem 2, p.11].

Theorem 1 For integer $m$ (even) $\geq 2$, $n \geq 0$,

$$S_n^{(m)} = (-1)^{\frac{m}{2}} 2^{1-2n} m Q(n;m) c_{n-1}.$$

Remark 1 The function $Q(n;m)$ can be expressed in factorial form

$$\begin{aligned} Q(n;m) &= \frac{(n-1)!n![2n-(m+1)]!(2n+m)!}{[n-(\frac{1}{2}m+1)]!(n+\frac{1}{2}m)![2(n-1)]!(2n+1)!} \\ &= \frac{[2n-(m+1)]!(2n+m)!}{[n-(\frac{1}{2}m+1)]!(n+\frac{1}{2}m)!(2n+1)!}\frac{1}{c_{n-1}}, \end{aligned} \tag{4}$$

which holds for $m = 2, 4, 6$, *etc.*, and gives a modified version of $S_n^{(m)}$ (with Catalan element absent); we, however, work with (3) for convenience.

# New Series Results

## Formulation

Noting that $S_0^{(m)} = m$, then re-arranging (1) as

$$\sin(m\alpha) - m\sin(\alpha) = \sum_{n=1}^{\infty} S_n^{(m)} \sin^{2n+1}(\alpha) \tag{5}$$

and integrating both sides w.r.t. $\alpha$ from 0 to $\frac{1}{2}\pi$ (or $-\frac{1}{2}\pi$ to 0) we have[1]

$$\frac{1}{m}\left[1-(-1)^{\frac{m}{2}}\right] - m = \sum_{n=1}^{\infty} S_n^{(m)} \int_0^{\frac{\pi}{2}} \sin^{2n+1}(\alpha)\ d\alpha. \tag{6}$$

It is straightforward to show that for $n \geq 1$,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^{2n+1}(\alpha)\ d\alpha &= \frac{2\cdot 4\cdot 6\cdot \ \cdots \ \cdot(2n)}{1\cdot 3\cdot 5\cdot \ \cdots \ \cdot(2n+1)} \\ &= \frac{4^n}{(2n+1)}\frac{n!^2}{(2n)!} \\ &= \frac{4^n}{(n+1)(2n+1)c_n}, \end{aligned} \tag{7}$$

[1]Either range of integration is admissible because the convergence criteria of the series in (1) can be extended to $|\alpha| \leq \frac{\pi}{2}$ (including *both* the interval endpoints $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$); as a point of completeness, the reader is referred to Appendix A for a formal argument.

so that, from the well known relation

$$c_{n-1} = \frac{(n+1)}{2(2n-1)} c_n \tag{8}$$

between adjacent Catalan numbers (valid for $n \geq 1$, given $c_0 = 1$), (6) gives rise to an interesting result using Theorem 1; it is duly found that

$$\sum_{n=1}^{\infty} \frac{Q(n;m)}{(2n-1)(2n+1)} = \frac{1}{m^2}\left[(-1)^{\frac{m}{2}} - 1\right] - (-1)^{\frac{m}{2}}. \tag{9}$$

Clearly the r.h.s. simplifies, according to the precise value of $m$, and we have a final form of the result thus.

<u>**Theorem 2**</u> For integer $m$ (even) $\geq 2$,

$$\sum_{n=1}^{\infty} \frac{Q(n;m)}{(2n-1)(2n+1)} = \begin{cases} 1 - \frac{2}{m^2} & m = 2, 6, 10, 14, \ldots \\ -1 & m = 4, 8, 12, 16, \ldots \end{cases}$$

## Verification

We now verify (informally) a few cases. Consider first $m = 2$. Writing $\sum_{n=1}^{\infty} 1/(2n+1) = \sum_{n=0}^{\infty} 1/(2n+1) - 1 = \sum_{n=1}^{\infty} 1/(2n-1) - 1$, the l.h.s. of Theorem 2 becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2}\left\{\sum_{n=1}^{\infty} \frac{1}{(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{(2n+1)}\right\} \\ &= \frac{1}{2}\left\{\sum_{n=1}^{\infty} \frac{1}{(2n-1)} - \left[\sum_{n=1}^{\infty} \frac{1}{(2n-1)} - 1\right]\right\} \\ &= \frac{1}{2} \\ &= 1 - \frac{2}{2^2}, \end{aligned} \tag{10}$$

concurring with the r.h.s. For $m = 4$, the partial fraction decomposition

$$\frac{(2n+3)}{(2n-1)(2n-3)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n-3)} + \frac{C}{(2n+1)} \tag{11}$$

has solution constants $A = -1$, $B = \frac{3}{4}$, $C = \frac{1}{4}$, whereupon, again in agreement with Theorem 2, we see by a similar procedure that

$$\sum_{n=1}^{\infty} \frac{(2n+3)}{(2n-1)(2n-3)(2n+1)}$$

$$
\begin{aligned}
&= \frac{1}{4}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n+1)}+3\sum_{n=1}^{\infty}\frac{1}{(2n-3)}-4\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\right\} \\
&= \frac{1}{4}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)}-1+3\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)}-1\right]-4\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\right\} \\
&= -1. \qquad (12)
\end{aligned}
$$

Finally, $m = 6$ leads to

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(2n+3)(2n+5)}{(2n-1)(2n-3)(2n-5)(2n+1)} \\
&= \sum_{n=1}^{\infty}\frac{\frac{3}{2}}{(2n-1)}-\sum_{n=1}^{\infty}\frac{3}{(2n-3)}+\sum_{n=1}^{\infty}\frac{\frac{5}{3}}{(2n-5)}-\sum_{n=1}^{\infty}\frac{\frac{1}{6}}{(2n+1)} \\
&= \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}-3\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)}-1\right] \\
&\qquad +\frac{5}{3}\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)}-\frac{4}{3}\right]-\frac{1}{6}\left[\sum_{n=1}^{\infty}\frac{1}{(2n-1)}-1\right] \\
&= 3-\frac{20}{9}+\frac{1}{6} \\
&= \frac{17}{18} \\
&= 1-\frac{2}{6^2}, \qquad (13)
\end{aligned}
$$

being once more the anticipated value. Other examples can be dealt with in the same manner, however see the below caveat.

Remark 2 The above illustrations, whilst useful checks, are non-rigorous in the sense that for correctness a limiting argument is required on each occasion. When $m = 2$, for instance, we should strictly write (because both $\sum_{n=1}^{\infty} 1/(2n-1)$ and $\sum_{n=1}^{\infty} 1/(2n+1)$ are unbounded, see Appendix B)

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} &= \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{(2n-1)(2n+1)}\right\} \\
&= \cdots \\
&= \frac{1}{2}\lim_{N\to\infty}\left\{1-\frac{1}{(2N+1)}\right\} \\
&= \frac{1}{2}, \qquad (14)
\end{aligned}
$$

and likewise in treating other cases.

# Hypergeometric Form

Since, at $n = 0$, (7) holds and (8) is consistent with the definition of $c_{-1}$, then if (1) (rather than (5) as before) is integrated directly it is found that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{Q(n;m)}{(2n-1)(2n+1)} &= \frac{1}{m^2}\left[(-1)^{\frac{m}{2}} - 1\right] \\ &= \begin{cases} -\frac{2}{m^2} & m = 2, 6, 10, 14, \ldots \\ 0 & m = 4, 8, 12, 16, \ldots, \end{cases} \end{aligned} \qquad (15)$$

which is equally available by a trivial modification of (9) noting that from (3) $Q(0;m) = (-1)^{\frac{m}{2}-1}\ \forall m \geq 2$. Equation (15) can, of course, be validated independently by means of a computer-based hypergeometric approach, for writing $a_n = a(n;m) = Q(n;m)/(2n-1)(2n+1)$, with ratio

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2n-1)}{(2n+3)}\frac{Q(n+1;m)}{Q(n;m)} \\ &= \frac{[n+\frac{1}{2}(1-m)][n+\frac{1}{2}(1+m)]}{(n+\frac{3}{2})^2}\end{aligned} \qquad (16)$$

by (3), and first term $a_0 = -Q(0;m) = (-1)^{\frac{m}{2}}$, then hypergeometric theory (see any of several textbooks on the subject) gives, in standard notation,

$$\sum_{n=0}^{\infty} \frac{Q(n;m)}{(2n-1)(2n+1)} = (-1)^{\frac{m}{2}}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}(1-m), \frac{1}{2}(1+m), 1 \\ \frac{3}{2}, \frac{3}{2}\end{matrix}\ \right|\ 1\right), \qquad (17)$$

whose r.h.s. evaluation to the correct closed form may, as a check, be accomplished easily through the application of Gosper's algorithm by symbolic computation (see [2], the author of which has executed appropriate code to do precisely this and to whom we are grateful). We can do better still, however, and to finish we detail two different hypergeometric proofs of this identity:

<u>Theorem 3</u>

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2}(1-m), \frac{1}{2}(1+m), 1 \\ \frac{3}{2}, \frac{3}{2}\end{matrix}\ \right|\ 1\right) = \frac{1}{m^2}\left[1 - (-1)^{\frac{m}{2}}\right].$$

<u>Proof I</u> Consider the equation [3, (3.8.2), p.21]

$${}_3F_2\left(\left.\begin{matrix}a, b, e+f-a-b-1 \\ e, f\end{matrix}\ \right|\ 1\right)$$

$$= \frac{\Gamma(e)\Gamma(f)\Gamma(e-a-b)\Gamma(f-a-b)}{\Gamma(e-a)\Gamma(e-b)\Gamma(f-a)\Gamma(f-b)} + \frac{1}{(a+b-e)}\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(e+f-a-b)} \times {}_3F_2\left(\left.\begin{matrix} e-a, e-b, 1 \\ e-a-b+1, e+f-a-b \end{matrix}\right| 1\right), \qquad \text{(I1)}$$

which according to Bailey dates from a paper of 1891 by Saalschütz. If either $a$ or $b$ is a negative integer, it reduces to the well known Pfaff-Saalschütz identity giving the sum of a terminating ${}_3F_2$ hypergeometric series with unity argument. Setting $a = \frac{1}{2}(1-m)$, $b = 1$ and $e = f = \frac{3}{2}$, and cancelling a (non-negative integer) term $1 + \frac{1}{2}m$ that appears as both an upper and lower parameter of the r.h.s. hypergeometric function, the above yields

$${}_3F_2\left(\left.\begin{matrix} \frac{1}{2}(1-m), \frac{1}{2}(1+m), 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix}\right| 1\right) = \left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}\frac{\Gamma(\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}m)}\right]^2 - \frac{2}{m}\frac{\Gamma^2(\frac{3}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m)\Gamma(\frac{3}{2}+\frac{1}{2}m)}\,{}_2F_1\left(\left.\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2}+\frac{1}{2}m \end{matrix}\right| 1\right). \qquad \text{(I2)}$$

Gauss' Theorem states that

$${}_2F_1\left(\left.\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}\right| 1\right) = \frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)} \qquad \text{(I3)}$$

iff $\mathrm{Re}\{b_1-(a_1+a_2)\} > 0$, whence (since $\mathrm{Re}\{\frac{3}{2}+\frac{1}{2}m-(\frac{1}{2}+1)\} = \frac{1}{2}m \geq 1 > 0$)

$${}_2F_1\left(\left.\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2}+\frac{1}{2}m \end{matrix}\right| 1\right) = \frac{\Gamma(\frac{3}{2}+\frac{1}{2}m)\Gamma(\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}m)\Gamma(\frac{1}{2}+\frac{1}{2}m)} \qquad \text{(I4)}$$

and the transformation (I2) becomes the evaluation

$${}_3F_2\left(\left.\begin{matrix} \frac{1}{2}(1-m), \frac{1}{2}(1+m), 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix}\right| 1\right) = \left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})}\frac{\Gamma(\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}m)}\right]^2 - \frac{2}{m}\frac{\Gamma^2(\frac{3}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m)\Gamma(\frac{1}{2}+\frac{1}{2}m)}\frac{\Gamma(\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}m)}. \qquad \text{(I5)}$$

Now, the property (for $s \neq 0, -1, -2, -3, \ldots$)

$$s\Gamma(s) = \Gamma(s+1) \tag{I6}$$

of the Gamma function gives, noting that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, ratios

$$\begin{aligned} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} &= \frac{1}{2}, \\ \frac{\Gamma(\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}m)} &= \frac{2}{m}, \end{aligned} \tag{I7}$$

in addition to which

$$\Gamma\left(\frac{1}{2} - \frac{m}{2}\right)\Gamma\left(\frac{1}{2} + \frac{m}{2}\right) = (-1)^{\frac{m}{2}}\pi \tag{I8}$$

is a special case of the general result

$$\Gamma\left(\frac{1}{2} - s\right)\Gamma\left(\frac{1}{2} + s\right) = (-1)^{s}\pi, \qquad s = 0, 1, 2, \ldots \tag{I9}$$

All of this reduces (I5) to read simply

$$\begin{aligned} {}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), \frac{1}{2}(1+m), 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \, 1\right) &= \left[\frac{1}{2}\frac{2}{m}\right]^2 - \frac{2}{m}\frac{\frac{1}{4}\pi}{(-1)^{\frac{m}{2}}\pi}\frac{2}{m} \\ &= \frac{1}{m^2} - \frac{1}{(-1)^{\frac{m}{2}}m^2} \\ &= \frac{1}{m^2}\left[1 - (-1)^{\frac{m}{2}}\right], \end{aligned} \tag{I10}$$

as required.□

<u>Proof II</u> Consider [4, Theorem 3.5.5(ii), p.149] (derived from a combination of Theorem 3.5.5(i) therein, and a result attributed to Thomae), which states that for $a + b = 1$, $e + f = 2c + 1$,

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a, b, c \\ e, f \end{matrix} \middle| \, 1\right) \\ &= \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma[\frac{1}{2}(a+e)]\Gamma[\frac{1}{2}(a+f)]\Gamma[\frac{1}{2}(b+e)]\Gamma[\frac{1}{2}(b+f)]}. \end{aligned} \tag{II1}$$

Choosing $a = \frac{1}{2}(1-m)$, $b = \frac{1}{2}(1+m)$, $c = 1$ and $e = f = \frac{3}{2}$ we have

$${}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), \frac{1}{2}(1+m), 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \, 1\right) = f(m), \tag{II2}$$

say, where the r.h.s of (II1) contracts, using (I6), to

$$f(m)=\frac{\pi^2}{8}\frac{1}{\Gamma^2(1-\frac{1}{4}m)\Gamma^2(1+\frac{1}{4}m)}. \tag{II3}$$

We evidently have two cases to consider.

Case (a) In the first instance we see that for those values of $m=4,8,12,16,\ldots$, then $\Gamma(1+\frac{1}{4}m)=\Gamma(2)=1!,\Gamma(3)=2!,\Gamma(4)=3!,\Gamma(5)=4!,\ldots$, whilst $\Gamma(1-\frac{1}{4}m)=\Gamma(0),\Gamma(-1),\Gamma(-2),\Gamma(-3),\ldots$, which are each unbounded so that $f(m)=0.\Box$

Case (b) Set $p=0,1,2,3,\ldots$, and $m=4p+2=2,6,10,14,\ldots$ Then, noting that $\Gamma(1-\frac{1}{4}m)\Gamma(1+\frac{1}{4}m)=\Gamma(\frac{1}{2}-p)\Gamma(\frac{3}{2}+p)$, which by (I6) and in turn (I9) $=(\frac{1}{2}+p)\Gamma(\frac{1}{2}-p)\Gamma(\frac{1}{2}+p)=\pi(-1)^p(\frac{1}{2}+p)$, (II3) simplifies to

$$f(m(p))=\frac{1}{2(2p+1)^2}, \tag{II4}$$

whence $f(m)=\frac{2}{m^2}$ follows trivially upon writing $p=p(m).\Box$

# Summary

A previous presentation concerning a class of expansions of the sine function allows, via integration, a general closed form to be found for sums of a new group of associated infinite series; the corresponding hypergeometric identity has been proven in two different ways.[2]

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# Appendix A

As is seen in [5], the convergence condition for the r.h.s. series of (1) follows from consideration of particular cases of $m$ beginning with $m=2$. Accordingly, it is but required to show that the expansion of $\sin(2\alpha)$ holds

[2]Note that, using the same approach as here, Q.Q. Liu and M.R. Xinrong have produced two slight generalisations of Theorem 3 ("Two New ${}_3F_2$ Summation Formulas", *J. Comb. Math. Comb. Comp.*, to appear).

for $|\alpha| \leq \frac{\pi}{2}$ in order that (1) does also.

To do so, we refer in the first instance to an earlier paper by Larcombe and Fennessey [6] in which it was proven (in four different ways) that the series form of the function $\sqrt{1-4z}$ converges (to zero) at $z = \frac{1}{4}$ (in addition to the known points on the open interval $(-\frac{1}{4}, \frac{1}{4})$). We now establish the following result.

<u>Lemma</u> The infinite series expansion of $\sqrt{1-4z}$ is a convergent one at $z = -\frac{1}{4}$, and converges to $\sqrt{2}$.

<u>Proof</u> Consider the expansion

$$\sqrt{1-4z} = 1 - [2z + 2z^2 + 4z^3 + 10z^4 + 28z^5 + \cdots].$$

At $z = \frac{1}{4}$ the series, as stated above, is known to converge to zero, and we have (where $\beta = \frac{1}{4}$)

$$2\beta + 2\beta^2 + 4\beta^3 + 10\beta^4 + 28\beta^5 + \cdots = 1.$$

At $z = -\frac{1}{4}$, the expansion of $\sqrt{1-4z}$ is

$$1 + 2\beta - 2\beta^2 + 4\beta^3 - 10\beta^4 + 28\beta^5 - \cdots,$$

which converges absolutely since the series of absolute values is

$$1 + 2\beta + 2\beta^2 + 4\beta^3 + 10\beta^4 + 28\beta^5 + \cdots = 1 + 1 = 2.$$

The result is proven by virtue of the fact that (i) absolute convergence is sufficient for convergence of a series, and (ii) the convergent value is, employing Abel's Theorem (see, for example, [7, Chapter 3, p.149]), $\sqrt{1-4(-1/4)} = \sqrt{2}$ .□

We now finish the argument. The series form of the function $\sqrt{1-4z}$ has been shown to be convergent for $|z| \leq \frac{1}{4}$. Hence, so is that of $\sqrt{1+4z}$, and thus $\sqrt{1 \pm z}$ converges for $|z| \leq 1$. Since convergence of the series representation of $\sin(2\alpha)$ is based on the convergence of $\sqrt{1 \pm z}$ with $z = \mp\sin^2(\alpha)$ (see the proof of Result I in [5, p.41]), it follows that the associated condition is $|\alpha| \leq \frac{\pi}{2}$ as required.

# Appendix B

With reference to Remark 2, we show here that the series $\sum_{n=1}^{\infty} 1/(2n-1)$ and $\sum_{n=1}^{\infty} 1/(2n+1)$ are unbounded.

Consider, first,

$$\begin{aligned}
\sum_{n=1}^{r}\frac{1}{(2n-1)} &= 1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2r-3}+\frac{1}{2r-1}\\
&= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\\
&\quad \cdots+\frac{1}{2r-3}+\frac{1}{2r-2}+\frac{1}{2r-1}+\frac{1}{2r}\\
&\quad -\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2r-2}+\frac{1}{2r}\right]\\
&= \sum_{n=1}^{2r}\frac{1}{n}-\frac{1}{2}\sum_{n=1}^{r}\frac{1}{n}\\
&= \sum_{n=1}^{2r}\frac{1}{n}-\ln(2r)-\frac{1}{2}\left[\sum_{n=1}^{r}\frac{1}{n}-\ln(r)\right]+\ln(2\sqrt{r}). \qquad \text{(B1)}
\end{aligned}$$

The r.h.s. is written in this way for it is known that $\sum_{n=1}^{\mu}\frac{1}{n}-\ln(\mu)\rightarrow C_0$ (*i.e.*, the Euler-Mascheroni constant $C_0 = 0.5772\ldots$) as $\mu\rightarrow\infty$, giving

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{(2n-1)} &= \lim_{r\rightarrow\infty}\left\{\sum_{n=1}^{r}\frac{1}{(2n-1)}\right\}\\
&= C_0-\frac{1}{2}C_0+\lim_{r\rightarrow\infty}\{\ln(2\sqrt{r})\}\\
&= \frac{1}{2}C_0+\lim_{r\rightarrow\infty}\{\ln(2\sqrt{r})\}, \qquad \text{(B2)}
\end{aligned}$$

which is unbounded; it follows similarly that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{(2n+1)} &= \lim_{r\rightarrow\infty}\left\{\sum_{n=1}^{r}\frac{1}{(2n+1)}\right\}\\
&= \cdots\\
&= \lim_{r\rightarrow\infty}\left\{\sum_{n=1}^{r}\frac{1}{(2n-1)}+\frac{1}{(2r+1)}-1\right\}\\
&= \frac{1}{2}C_0-1+\lim_{r\rightarrow\infty}\{\ln(2\sqrt{r})\} \qquad \text{(B3)}
\end{aligned}$$

is also unbounded.

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